SEARCHING FOR NEW NONTRIVIAL CHOREOGRAPHIES FOR THE PLANAR THREE-BODY PROBLEM

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GRID'2023 conference, 3-7 July, MLIT, JINR, Dubna, Russia

Celest Mech Dyn Astr (2016) 124:155–162
DOI 10.1007/s10569-015-9657-9

ORIGINAL ARTICLE

Three topologically nontrivial choreographic motions of three bodies

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Received: 16 June 2015 / Revised: 20 September 2015 / Accepted: 23 September 2015 /
Published online: 7 October 2015

Abstract Choreographies are periodic orbits in which all bodies move on the same trajectory with equal time delay. The best known three-body choreography is figure-eight orbit. Here we introduce a search method specialized for choreographies and present three new orbits with vanishing angular momentum that are the first clear examples of choreographies that cannot be described as k-th powers of the figure-eight solution, according to the topological classification of orbits. We have also found 17 new "powers of the eight" choreographies. According to our numerical computation, one of two distinct $k = 7$ choreographies is linearly stable.

Keywords Three-body problem · Choreography · Stability · Homotopy classes

Differential equations

The differential equations for the three-body problem are derived from Newton's second law and Newton's law of gravity:

$$m_i \ddot{r}_i = \sum_{j=1, j\neq i}^{3} G m_i m_j \frac{(r_j - r_i)}{\|r_i - r_j\|^3}, i = 1, 2, 3.$$

We consider normalization $G = m_1 = m_2 = m_3 = 1$ and planar motion. We solve the system numerically in the following first order form:

$$\dot{x}_i = vx_i, \ \dot{y}_i = vy_i$$

$$\dot{vx}_i = \sum_{j=1, j\neq i}^{3} \frac{(x_j - x_i)}{\|r_i - r_j\|^3}, \ \dot{vy}_i = \sum_{j=1, j\neq i}^{3} \frac{(y_j - y_i)}{\|r_i - r_j\|^3}, \ i = 1, 2, 3$$

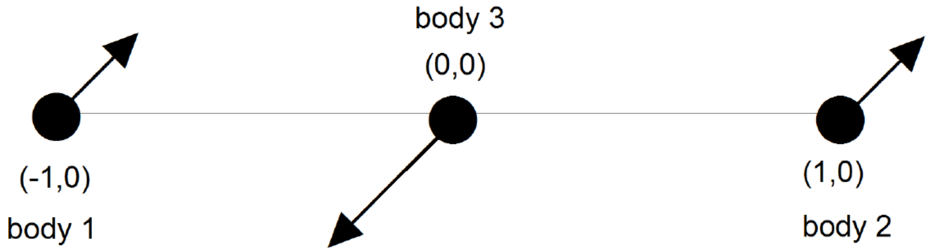
So we have a vector of 12 unknown functions:

$$X(t) = (x_1, y_1, x_2, y_2, x_3, y_3, vx_1, vy_1, vx_2, vy_2, vx_3, vy_3)^\top$$

. The model treats the bodies as mass points.

Initial configuration (Euler initial configuration)

We search for periodic planar orbits as Suvakov and Shibayama - with zero angular momentum and symmetric initial configuration with parallel velocities (Euler configuration)



$$(vx_1(0), vy_1(0)) = (vx_2(0), vy_2(0)) = (v_x, v_y)$$
$$(vx_3(0), vy_3(0)) = (-2v_x, -2v_y)$$
$v_x \in [0, 0.8], v_y \in [0, 0.8]$ are parameters.

Let us denote the periods of the orbits with T. Our goal is to find triplets (v_x, v_y, T) for which the solution is a choreography.

WHAT IS A CHOREOGRAPHY?

Let $q_i(t) = (x_i(t), y_i(t))$, $i = 1, 2, 3$ are the positions of the three bodies. A T-periodic path $q(t) = (q_1(t), q_2(t), q_3(t))$ is a choreography, if

$$q_i(t) = q_{i+1}(t + T/3)$$

This means that the three bodies move along one and the same trajectory with a time delay of $T/3$. The condition can be regarded as a cyclic periodicity condition (a periodicity condition with respect to a cyclic permutation of the indexes of the bodies) at $T/3$ which is satisfied in addition to the standard periodic condition:

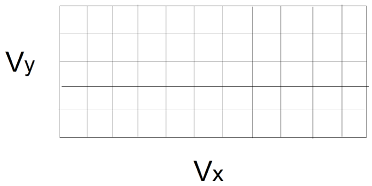
$$q_i(t) = q_i(t + T)$$

A choreography is called trivial if it is a topological power of the famous figure-eight orbit, otherwise it is called nontrivial.

Four stages of the numerical procedure

The numerical procedure consists of four stages:

Stage (I) - Computing initial approximations for the correction method (grid-search algorithm on the rectangular window $(\boldsymbol{v_x}, \boldsymbol{v_y}) \in [\mathbf{0}, \mathbf{0.8}] \times [\mathbf{0}, \mathbf{0.8}]$ with step size = 1/4096)



Candidates for correction are the triplets $(\boldsymbol{v_x}, \boldsymbol{v_y}, \overline{\boldsymbol{T}})$, such that the cyclically permuted return proximity $\boldsymbol{R}(\overline{\boldsymbol{T}})$ has local minima on the grid for $\boldsymbol{v_x}, \boldsymbol{v_y}$ and $\boldsymbol{R}(\overline{\boldsymbol{T}})$ is less than 0.1, $\overline{\boldsymbol{T}}$ is an approximation of $\boldsymbol{T}/\mathbf{3}$:

$$\boldsymbol{R}(\boldsymbol{t}) = \|\hat{\boldsymbol{P}}\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{X}(\mathbf{0})\|_{\mathbf{2}}$$

$$\boldsymbol{R}(\overline{\boldsymbol{T}}) = \min_{\mathbf{1} < \boldsymbol{t} \leq \boldsymbol{T_0}} \boldsymbol{R}(\boldsymbol{t}) < \mathbf{0.1}$$

$\hat{\boldsymbol{P}}$ is a cyclic permutation of the body indices.

Four stages of the numerical procedure

Stage (II) - Applying the modified Newton's method with the cyclic perturbed periodic condition at $\boldsymbol{T/3}$, which can converge or diverge. Convergence means that we catch a choreographic periodic solution.

Stage (III) Checking the results from stage (II) by applying the classic Newton's method with the standard periodic condition at $\boldsymbol{T}$.

Stage (IV) - Applying the classic Newton's method with increased order of method and precision for computing the solution with many correct digits, in this work - 180 correct digits. This stage can be regarded as a verification for the existence of the solutions.

The form of the linear system at each step of Newton's or modified Newton's method for the standard p.c.

Let $\boldsymbol{v_x}, \boldsymbol{v_y}, \boldsymbol{T}$ are approximations of the initial velocities and the period for some periodic solution: $\boldsymbol{X(T) \approx X(0)}$. These approximations are improved with corrections $\boldsymbol{\Delta v_x, \Delta v_y, \Delta T}$ by expanding the periodicity condition in a multivariable Taylor series up to the first order. We obtain the following linear system with 12x3 matrix:

$$
\begin{pmatrix} x_1(T) \\ y_1(T) \\ x_2(T) \\ y_2(T) \\ x_3(T) \\ y_3(T) \\ vx_1(T) \\ vy_1(T) \\ vx_2(T) \\ vy_2(T) \\ vx_3(T) \\ vy_3(T) \end{pmatrix}
+
\begin{pmatrix}
\frac{\partial x_1}{\partial v_x}(T) & \frac{\partial x_1}{\partial v_y}(T) & \dot{x}_1(T) \\
\frac{\partial y_1}{\partial v_x}(T) & \frac{\partial y_1}{\partial v_y}(T) & \dot{y}_1(T) \\
\frac{\partial x_2}{\partial v_x}(T) & \frac{\partial x_2}{\partial v_y}(T) & \dot{x}_2(T) \\
\frac{\partial y_2}{\partial v_x}(T) & \frac{\partial y_2}{\partial v_y}(T) & \dot{y}_2(T) \\
\frac{\partial x_3}{\partial v_x}(T) & \frac{\partial x_3}{\partial v_y}(T) & \dot{x}_3(T) \\
\frac{\partial y_3}{\partial v_x}(T) & \frac{\partial y_3}{\partial v_y}(T) & \dot{y}_3(T) \\
\frac{\partial vx_1}{\partial v_x}(T) & \frac{\partial vx_1}{\partial v_y}(T) & \dot{vx}_1(T) \\
\frac{\partial vy_1}{\partial v_x}(T) & \frac{\partial vy_1}{\partial v_y}(T) & \dot{vy}_1(T) \\
\frac{\partial vx_2}{\partial v_x}(T) & \frac{\partial vx_2}{\partial v_y}(T) & \dot{vx}_2(T) \\
\frac{\partial vy_2}{\partial v_x}(T) & \frac{\partial vy_2}{\partial v_y}(T) & \dot{vy}_2(T) \\
\frac{\partial vx_3}{\partial v_x}(T) & \frac{\partial vx_3}{\partial v_y}(T) & \dot{vx}_3(T) \\
\frac{\partial vy_3}{\partial v_x}(T) & \frac{\partial vy_3}{\partial v_y}(T) & \dot{vy}_3(T)
\end{pmatrix}
\begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta T \end{pmatrix}
=
\begin{pmatrix} x_1(0) \\ y_1(0) \\ x_2(0) \\ y_2(0) \\ x_3(0) \\ y_3(0) \\ vx_1(0) + \Delta v_x \\ vy_1(0) + \Delta v_y \\ vx_2(0) + \Delta v_x \\ vy_2(0) + \Delta v_y \\ vx_3(0) - 2\Delta v_x \\ vy_3(0) - 2\Delta v_y \end{pmatrix}
$$

The form of the linear system for Newton's or modified Newton's method for the cyclically permuted p.c.

Let $\boldsymbol{v_x}, \boldsymbol{v_y}, \overline{\boldsymbol{T}}$ are approximations of the initial velocities and $\boldsymbol{T/3}$. These approximations are improved with corrections $\boldsymbol{\Delta v_x}, \boldsymbol{\Delta v_y}, \boldsymbol{\Delta \overline{T}}$ by solving the following linear system with 12x3 matrix:

$$
\begin{pmatrix} x_1(\overline{T}) \\ y_1(\overline{T}) \\ x_2(\overline{T}) \\ y_2(\overline{T}) \\ x_3(\overline{T}) \\ y_3(\overline{T}) \\ vx_1(\overline{T}) \\ vy_1(\overline{T}) \\ vx_2(\overline{T}) \\ vy_2(\overline{T}) \\ vx_3(\overline{T}) \\ vy_3(\overline{T}) \end{pmatrix}
+
\begin{pmatrix}
\frac{\partial x_1}{\partial v_x}(\overline{T}) & \frac{\partial x_1}{\partial v_y}(\overline{T}) & \dot{x}_1(\overline{T}) \\
\frac{\partial y_1}{\partial v_x}(\overline{T}) & \frac{\partial y_1}{\partial v_y}(\overline{T}) & \dot{y}_1(\overline{T}) \\
\frac{\partial x_2}{\partial v_x}(\overline{T}) & \frac{\partial x_2}{\partial v_y}(\overline{T}) & \dot{x}_2(\overline{T}) \\
\frac{\partial y_2}{\partial v_x}(\overline{T}) & \frac{\partial y_2}{\partial v_y}(\overline{T}) & \dot{y}_2(\overline{T}) \\
\frac{\partial x_3}{\partial v_x}(\overline{T}) & \frac{\partial x_3}{\partial v_y}(\overline{T}) & \dot{x}_3(\overline{T}) \\
\frac{\partial y_3}{\partial v_x}(\overline{T}) & \frac{\partial y_3}{\partial v_y}(\overline{T}) & \dot{y}_3(\overline{T}) \\
\frac{\partial vx_1}{\partial v_x}(\overline{T}) & \frac{\partial vx_1}{\partial v_y}(\overline{T}) & \dot{vx}_1(\overline{T}) \\
\frac{\partial vy_1}{\partial v_x}(\overline{T}) & \frac{\partial vy_1}{\partial v_y}(\overline{T}) & \dot{vy}_1(\overline{T}) \\
\frac{\partial vx_2}{\partial v_x}(\overline{T}) & \frac{\partial vx_2}{\partial v_y}(\overline{T}) & \dot{vx}_2(\overline{T}) \\
\frac{\partial vy_2}{\partial v_x}(\overline{T}) & \frac{\partial vy_2}{\partial v_y}(\overline{T}) & \dot{vy}_2(\overline{T}) \\
\frac{\partial vx_3}{\partial v_x}(\overline{T}) & \frac{\partial vx_3}{\partial v_y}(\overline{T}) & \dot{vx}_3(\overline{T}) \\
\frac{\partial vy_3}{\partial v_x}(\overline{T}) & \frac{\partial vy_3}{\partial v_y}(\overline{T}) & \dot{vy}_3(\overline{T})
\end{pmatrix}
\begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta \overline{T} \end{pmatrix}
=
\begin{pmatrix} x_2(0) \\ y_2(0) \\ x_3(0) \\ y_3(0) \\ x_1(0) \\ y_1(0) \\ vx_2(0)+\Delta v_x \\ vy_2(0)+\Delta v_y \\ vx_3(0)-2\Delta v_x \\ vy_3(0)-2\Delta v_y \\ vx_1(0)+\Delta v_x \\ vy_1(0)+\Delta v_y \end{pmatrix},
\begin{pmatrix} x_3(0) \\ y_3(0) \\ x_1(0) \\ y_1(0) \\ x_2(0) \\ y_2(0) \\ vx_3(0)-2\Delta v_x \\ vy_3(0)-2\Delta v_y \\ vx_1(0)+\Delta v_x \\ vy_1(0)+\Delta v_y \\ vx_2(0)+\Delta v_x \\ vy_2(0)+\Delta v_y \end{pmatrix}
$$

We solve this system as a linear least square problem using $\boldsymbol{QR}$ decomposition based on Householder reflections.

Classic Newton's method vs Modified Newton's method

With classic Newton's method we correct this way:

$$v_x := v_x + \Delta v_x, \; v_y := v_y + \Delta v_y, \; T := T + \Delta T$$

For the modification of Newton's method based on continuous analog of Newton's method, we introduce a parameter $0 < \tau_k <= 1$, where k is the number of the iteration. Now we correct this way:

$$v_x := v_x + \tau_k \Delta v_x, \; v_y := v_y + \tau_k \Delta v_y, \; T := T + \tau_k \Delta T$$

Let R_k be the value of the return proximity $R(T)$ at the k-th iteration. With given τ_0 the next $\tau_k, k = 1, 2, \ldots$ is computed with the following adaptive algorithm:

$$\tau_k = \begin{cases} \min(1, \; \tau_{k-1} R_{k-1}/R_k), & R_k \le R_{k-1}, \\ \max(\tau_0, \; \tau_{k-1} R_{k-1}/R_k), & R_k > R_{k-1}, \end{cases}$$

The modified Newton's method has a larger domain of convergence!

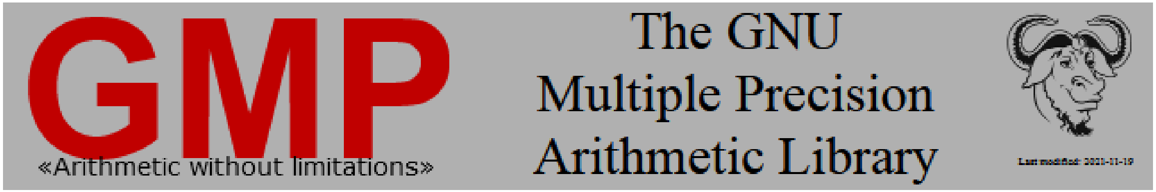
For simulation the system and computing the partial derivatives we use high precision Taylor Series Method

For stage (I) of the numerical procedure we use 44-th order of Taylor series and precision of 38 decimal digits.

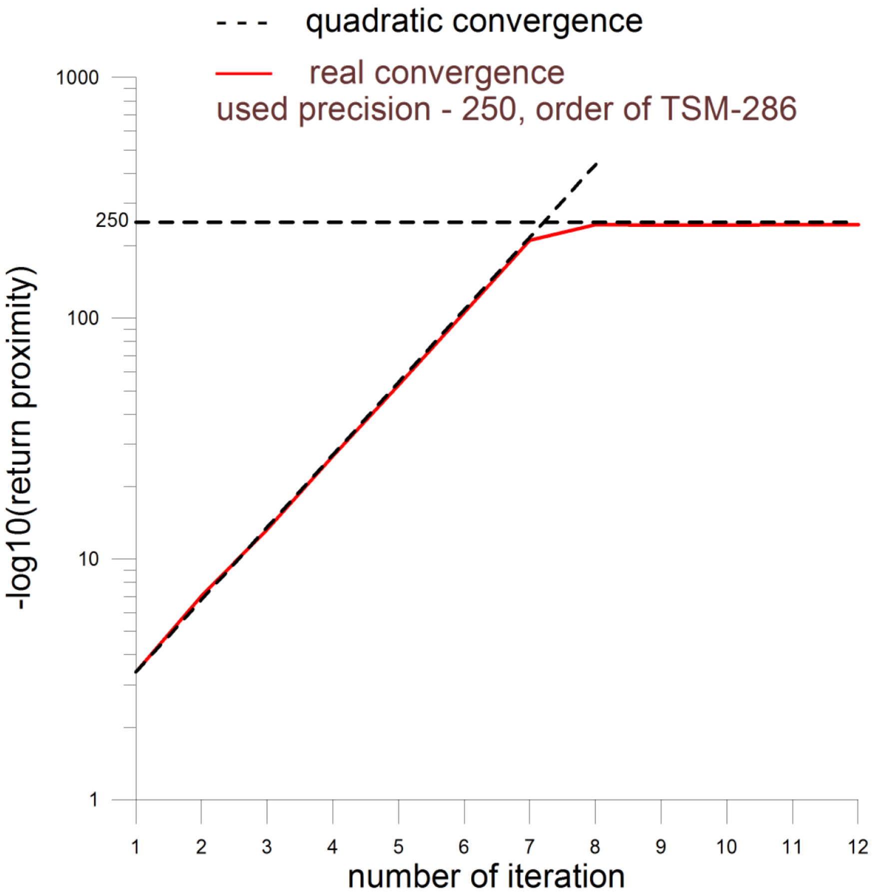
For stages (II) and (III) we use 154-th order of Taylor series and precision of 134 decimal digits.

For stage (IV) (computing the solutions with 180 correct digits) we made two computations. First computation with 242-th order method and 211 decimal digits of precision and the second computation for verification - with 286-th order method and 250 digits of precision.

For multiple precision floating point arithmetic we use GMP-library



Checking the convergence of Newton's method



Numerical results

As a result of the specialized search for $T \leq 300$ we found 259 trivial and 164 nontrivial choreographies (161 new ones).

A topological method is applied to classify the periodic orbits into families. Each family corresponds to a different conjugacy class of the free group on two letters (a,b). Trivial choreographies correspond to free word elements $(abAB)^n$ for some power n called topological power. Nontrivial choreographies are with a different free word elements.

For each found solution we compute the free group element and the four numbers (v_x, v_y, T, T^*) with 180 correct digits, where T^* is the scale-invariant period. The scale-invariant period is defined as $T^* = T|E|^{\frac{3}{2}}$, where E is the energy of our initial configuration.

The linear stability of all found orbits is investigated by a high precision computing of the eigenvalues of the monodromy matrices. All nontrivial choreographies are unstable. 13 of the trivial are linearly stable.

ACKNOWLEDGMENTS

We thank for the opportunity to use the computational resources of the HybriLIT platform at the Meshcheryakov Laboratory of Information Technologies of JINR, Dubna, Russia!

We would also like to thank Veljko Dmitrashinovich from Institute of Physics, Belgrade University, Serbia, for a valuable e-mail discussion and advice, and his encouragement to continue our numerical search for new choreographies.

THANK YOU FOR YOUR ATTENTION!