The ϕ^4 oscillons in a ball: numerical approach and parallel implementation

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Introduction

We consider the ϕ^4 equation

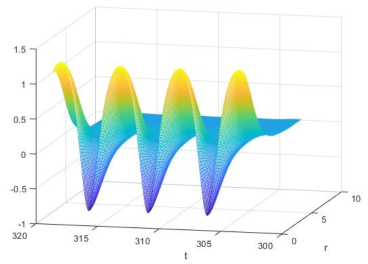
$$\Phi_{tt} - \Delta\Phi - \Phi + \Phi^3 = 0, \quad \Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}. \qquad (1)$$

Localized long-lived pulsating states (oscillons) in the three-dimensional ϕ^4 theory are of interest in a number of physical and mathematical applications including cosmological and high-energy physics contexts.

Computer simulations revealed the formation of long-lived pulsating structures of large amplitude and nearly unchanging width.
Bogolyubskii & Makhankov, JETP Lett **24** 12 (1976)
Bogolyubskii & Makhankov, JETP Lett **25** 107 (1977)

Aim of the study

Example of numerical simulations of pulsating solution of Eq.(1):



Our aim is a study of structure and resonant properties of the oscillon by examining the periodic standing wave in a ball of a large but finite radius. Numerical approach is based on numerical continuation of solutions of a boundary value problem for the respective nonlinear PDE on the rectangle $[0,T]\times[0,R]$ where T – period, R – radius. Stability analysis is based on the Floquet theory.

Boundary value problem

Let $\Phi(r,t)$ be a spherically-symmetric solution of equation (1) approaching $\Phi_0 = -1$ (one of two vacuum solutions) as $r \to \infty$. The difference

$$\phi = \Phi - \Phi_0$$

obeys

$$\phi_{tt} - \phi_{rr} - \frac{2}{r}\phi_r + 2\phi - 3\phi^2 + \phi^3 = 0. \tag{2a}$$

Instead of searching for solutions of the equation (2a) vanishing at infinity, we consider solutions satisfying the boundary conditions

$$\phi_r(0,t) = \phi(R,t) = 0 \tag{2b}$$

with a large R. (The first condition in (2b) ensures the regularity of the Laplacian at the origin.) One more boundary condition stems from the requirement of periodicity with some T:

$$\phi(r,T) = \phi(r,0). \tag{2c}$$

Energy and frequency

The periodic standing waves are characterised by their energy

$$E = 4\pi \int_0^R \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} + \phi^2 - \phi^3 + \frac{\phi^4}{4} \right) r^2 dr \qquad (3)$$

and frequency

$$\omega = \frac{2\pi}{T}. \qquad (4)$$

If the solution with frequency ω does not change appreciably as R is increased — in particular, if the energy (3) does not change — this standing wave provides a fairly accurate approximation for the periodic solution in an infinite space.

In what follows, we present results of analysis of the boundary-value problem (2), including the $E(\omega/\omega_0)$ dependence where $\omega_0 = \sqrt{2}$

Numerical approach

We introduce $\tau = t/T$, $\psi(r,\tau) = \phi(r,t)$. Hence,

$$\psi_\tau(r,\tau) = T\phi_t(r,t), \quad \psi_{\tau\tau}(r,\tau) = T^2\phi_{tt}(r,t),$$

and our boundary value problem takes a form:

$$\psi_{tt} + T^2 \cdot [-\psi_{rr} - \frac{2}{r}\psi_r + 2\psi - 3\psi^2 + \psi^3] = 0, \tag{5a}$$

$$\psi_r(0,t) = \psi(R,t) = 0, \quad \psi(r,1) = \psi(r,0). \tag{5b}$$

- Newtonian iteration
- Predictor-corrector numerical continuation with the crossing through the turning points
- 2nd order accuracy finite difference approximation of derivatives.

Stability analysis

To classify the stability of the resulting standing waves against spherically-symmetric perturbations we considered the linearised equation

$$y_{tt} - y_{rr} - \frac{2}{r} y_r - y + 3(\phi - 1)^2 y = 0 \qquad (6)$$

with the boundary conditions $y_r(0,t) = y(R,t) = 0$. We expand $y(r,t)$ in the sine Fourier series, substitute the expansion to Eq. (6) and, after transformations, finally obtain a system of $2N$ ODEs wrt unknown time-dependent Fourier coefficients:

$$\dot{u}_m = v_m, \qquad \dot{v}_m + \mathcal{F} = 0, \qquad (7)$$

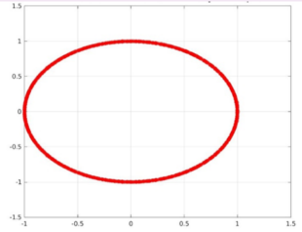
$$\mathcal{F} = (2+k_m^2)u_m - 3\sum_{n=1}^{N}(A_{m-n} - A_{m+n})u_n + \frac{3}{2}\sum_{n=1}^{N}(A_{m-n} - A_{m+n})u_n,$$

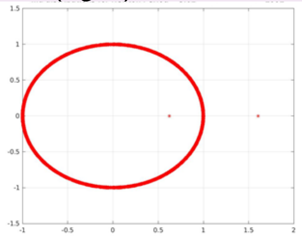
A_n, B_n are periodic functions of t, with period T:

$$A_n(t) = \frac{2}{R}\int_0^R \phi(r,t)\cos(k_n r)dr, \quad B_n(t) = \frac{2}{R}\int_0^R \phi^2(r,t)\cos(k_n r)dr$$

Calculation of Floquet multiplyers

The system (7) is solved, numerically, $2N$ times with series of varied initial conditions at the time-interval $[0, T]$ in order to form a matrix M_T. Eigenvalues $\mu = \exp(\lambda T)$ of this matrix are the Floquet multipliers. The solution $\phi(r, t)$ is deemed stable if all its Floquet multipliers lie on the unit circle $|\zeta| = 1$ (*left*) and unstable if there are multipliers outside the circle (*right*).





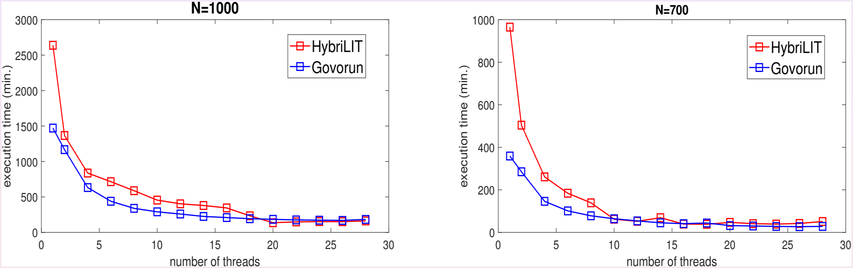
Floquet multipliers at the (Reμ,Imμ) plane. Stability: T=4.7206, ω/ω_0=0.94117. Instability: T=5.025, ω/ω_0=0.88416.

Numerical approach, parallel implementation

- MATLAB implementation.
- Procedure **ode45** with the tolerance parameter value 10^{-7}.
- Cubic spline interpolation for $A_{m\pm n}$ and $B_{m\pm n}$ terms.
- Standardly, N=1000, i.e. we have to numerically solve 2000 independent Cauchy problems to form matrix M_T.
- Long calculation – each run takes about 2 days at the HybriLIT cluster and about one day at the supercomputer "Govorun".
- We need massive calculations in wide range of period T.
- Parallel implementation is based on the using the **parfor** operation which provides automatic splitting the calculations of $2N$ Cauchy problems to available parallel threads ("workers").
- The speedup is about 20 times at the Hybrilit cluster and about 10 times at the supercomputer "Govorun".

Effect of parallel implementation

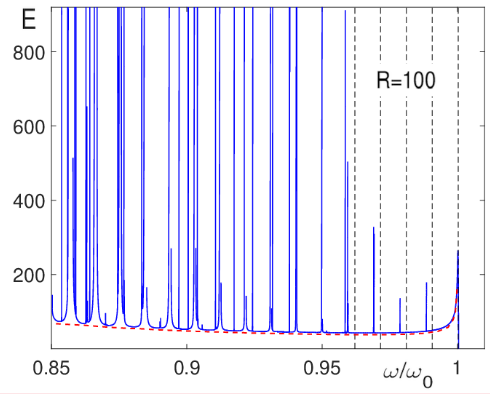
Execution time depending on a number of MATLAB-threads at the Hybrilit cluster and at the supercomputer "Govorun" in case N=1000 (left panel) and N=700 (right panel)

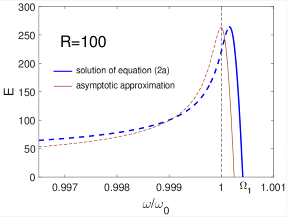


When the MATLAB-threads number is between 15 and 30, the execution time becomes the same at the Hybrilit and Govorun machines.

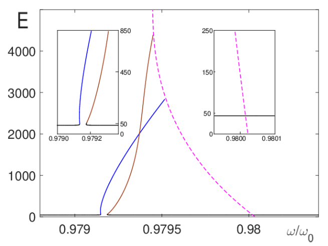
Results: Energy-frequency diagram

The branch of ϕ comes from E=0 at Ω_1. Continuation produces curve $E(\omega)$ with a sequence of spikes. Number and positions of spikes are R-sensitive. In contrast, the U-shaped envelope (red line) does not depend on R and has a single minimum, $\omega/\omega_0 = 0.967$.

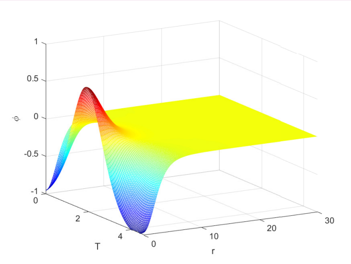


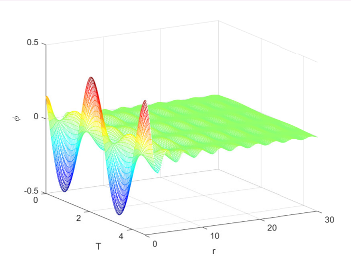


Results: bifurcation in the region of resonant spikes



magenta dashed: Bessel-like solution

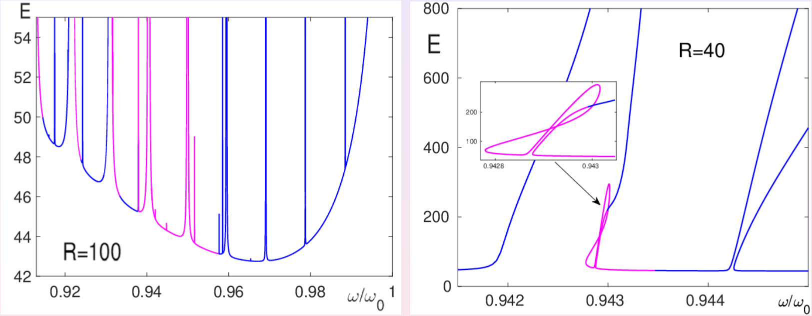




Results: stability domains

blue: unstable

magenta: stable



The Bessel waves (not shown here) are found to be stable from E=0 to the bifurcation point

Summary (1/2)

- Existence of two sets of spherically symmetric standing waves of Eq.(1) in a ball of finite radius is demonstrated.
- *First*, the Bessel-like waves without explicitly localized core, which are branching off the zero solution and decaying in proportion to r^{-1} as $r \to R$.
- The *second* type of nonlinear standing wave in a ball is characterised by an exponentially localised pulsating core and a small-amplitude slowly decaying second-harmonic tail. It comes of the nodeless $n = 1$ Bessel wave.
- Numerical continuation of this solution in frequency produces an $E(\omega)$ curve with a sequence of spikes near the undertone points $\omega = \Omega^{(n)}/2$ with some large n. The left and right slope of the spike adjacent to $\frac{1}{2}\Omega^{(n)}$ result from a period-doubling bifurcation of the n-th Bessel wave.

Summary (2/2)

- Away from the neighbourhoods of the spikes, the $E(\omega)$ curve follows a U-shaped arc with a single minimum at $\omega_{\min} = 0.967\omega_0$; the arc bounds all spikes from below. The arc is unaffected by the ball radius variations, as long as R remains large enough. This envelope curve describes the energy-frequency dependence of the nearly-periodic oscillons in the infinite space.
- We have classified stability of these solutions against spherically-symmetric perturbations. Specifically, we focused on the interval $0.91\omega_0 < \omega < \Omega^{(1)}$ and considered two values of R: $R = 40$ and $R = 100$. The ball of radius$R = 40$ has only short stability intervals, located at the base of two spikes in its $E(\omega)$ diagram. By contrast, the standing waves in the ball of $R = 100$ have long stretches of stable frequencies.

Open questions and Acknowledgments

What next?

- What was obtained at this stage (numerics & analytics) has been published in Phys Rev **D 107** (2023) 076023
- What about stability in case of the ball radius between 40 and 100?
- Now the calculations with R=70 are in process
- Accuracy should be improved

Acknowledgments

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Thank you for your attention!