Modelling of basal melt of Antarctic ice sheet based on a one-dimensional Stefan problem approach

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GRID-2023

Dubna, 04 July 2023

Introduction

- The study of the life cycle of subglacial lakes provides an understanding of the processes of formation of subglacial relief, which is a great interest for subglacial hydrology
- Subglacial lakes are difficult to access, so computer modeling is very useful
- Existing mathematical models do not explicitly take into account the ice-water phase transition

Our model of Antarctic basal melting

- Stefan problem is a boundary value problem that describes the evolution of the boundary between phases of a material undergoing a phase change
- Our model: a set of one-dimensional Stefan problems that are solved along vertical axis
- For each geographical point of Antarctica, corresponding Stefan problem allows us to obtain the law of motion of phase boundaries
- The set of laws of motion of phase boundaries gives the law of motion of the phase surfaces

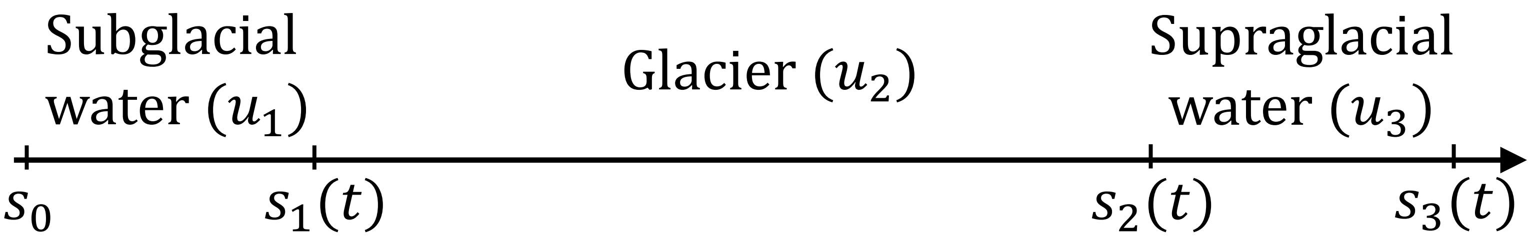
Model takes into account:

- Precipitation accumulation
- Surface temperature change
- Vertical movement of ice

Model does not take into account:

- Horizontal heat transfer
- Melt water movement
- Heat exchange with the ocean
- Horizontal movement of ice

Mathematical formulation of the three-phase Stefan problem



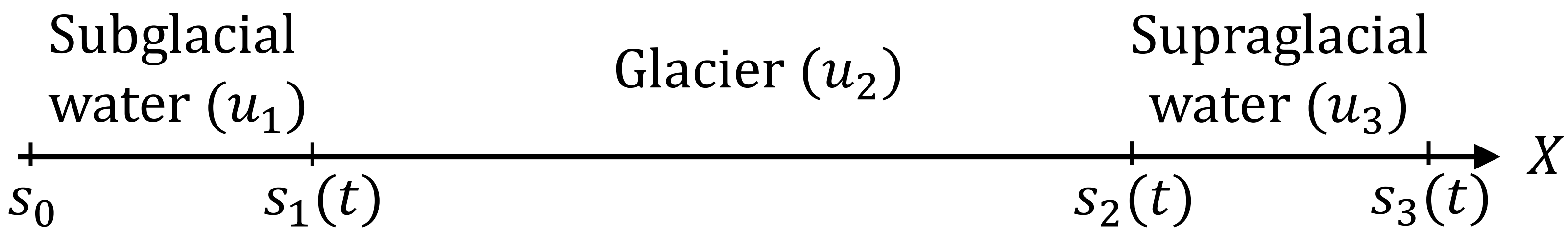
Heat equation:

$$\begin{cases} c_j \rho_j \dfrac{\partial u_j}{\partial t} = \lambda_j \dfrac{\partial^2 u_j}{\partial x^2}, & s_{j-1}(t) \le x \le s_j(t), \\ \alpha_{00} u_j + \alpha_{01} \dfrac{\partial u_j}{\partial x} = g_j^b(t), & x = s_{j-1}(t), \\ \alpha_{10} u_j + \alpha_{11} \dfrac{\partial u_j}{\partial x} = g_j^t(t), & x = s_j(t), \\ u_j = f(x), & t = 0 \end{cases}$$

Boundary conditions:

- s_0: $\partial u_j / \partial x = g_0 = \mathrm{const}$
- $s_{1,2}(t)$: $u_j = U_F$
- $s_3(t)$: $u_j = U(t)$

Stefan condition



- Lower edge of the glacier:

$$q\rho \frac{ds_1}{dt} = \lambda_i \frac{\partial u_2}{\partial x}\bigg|_{x=s_1(t)} - \lambda_w \frac{\partial u_1}{\partial x}\bigg|_{x=s_1(t)}$$

- The upper edge of the glacier (excluding subsidence):

$$v_2 = \frac{\lambda_i}{q\rho}\frac{\partial u_2}{\partial x}\bigg|_{x=s_1(t)} - \frac{\lambda_w}{q\rho}\frac{\partial u_1}{\partial x}\bigg|_{x=s_1(t)}$$

- The upper edge of the glacier (including subsidence):

$$\frac{ds_2}{dt} = \frac{ds_1}{dt}\left(1 - \frac{\rho_w}{\rho_i}\right) + v_2 \frac{\rho_w}{\rho_i} + \zeta$$

- Surface of supraglacial water (including subsidence):

$$\frac{ds_3}{dt} = \left(\frac{ds_1}{dt} - v_2\right)\left(1 - \frac{\rho_1}{\rho_2}\right) + \zeta$$

Boundary immobilization method

- The phases of matter are regions with free boundaries. The solution is to scale the interval:

$$\xi(x) \colon \left[s_{j-1}(t), s_j(t)\right] \to [0,1]$$

- Boundary immobilization method [Furzeland, 1980]:

$$\xi(x) = \frac{x - s_{j-1}(t)}{s_j(t) - s_{j-1}(t)}$$

Discretization

Finite difference method:

- Implicit difference scheme on an non-uniform grid
- Order of accuracy $O(h)$ for space and $O(t)$ for time

Non-uniform grid:

- Increased number of grid nodes near the phase boundaries
- Regular spacing between grid nodes
- Formula for non-uniform grid generation:

$$x = \sigma(y) = a + \frac{b-a}{1+\exp(-\alpha y)}$$

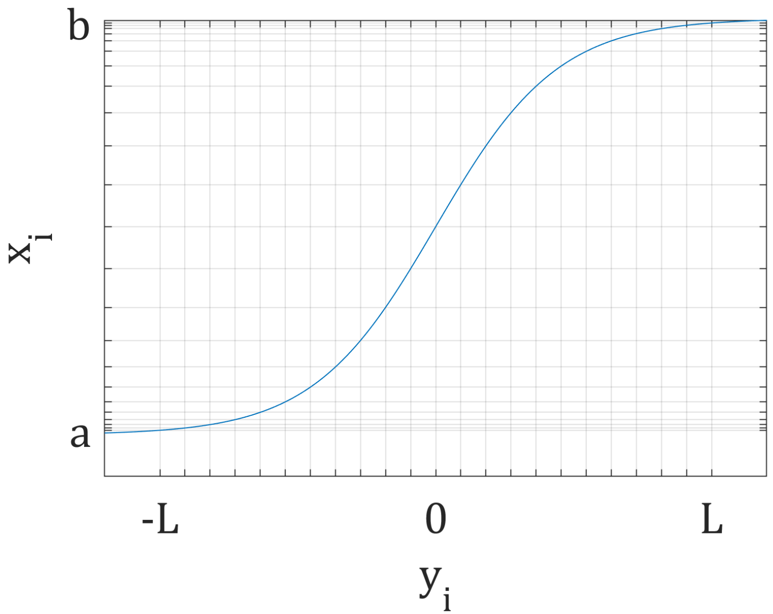
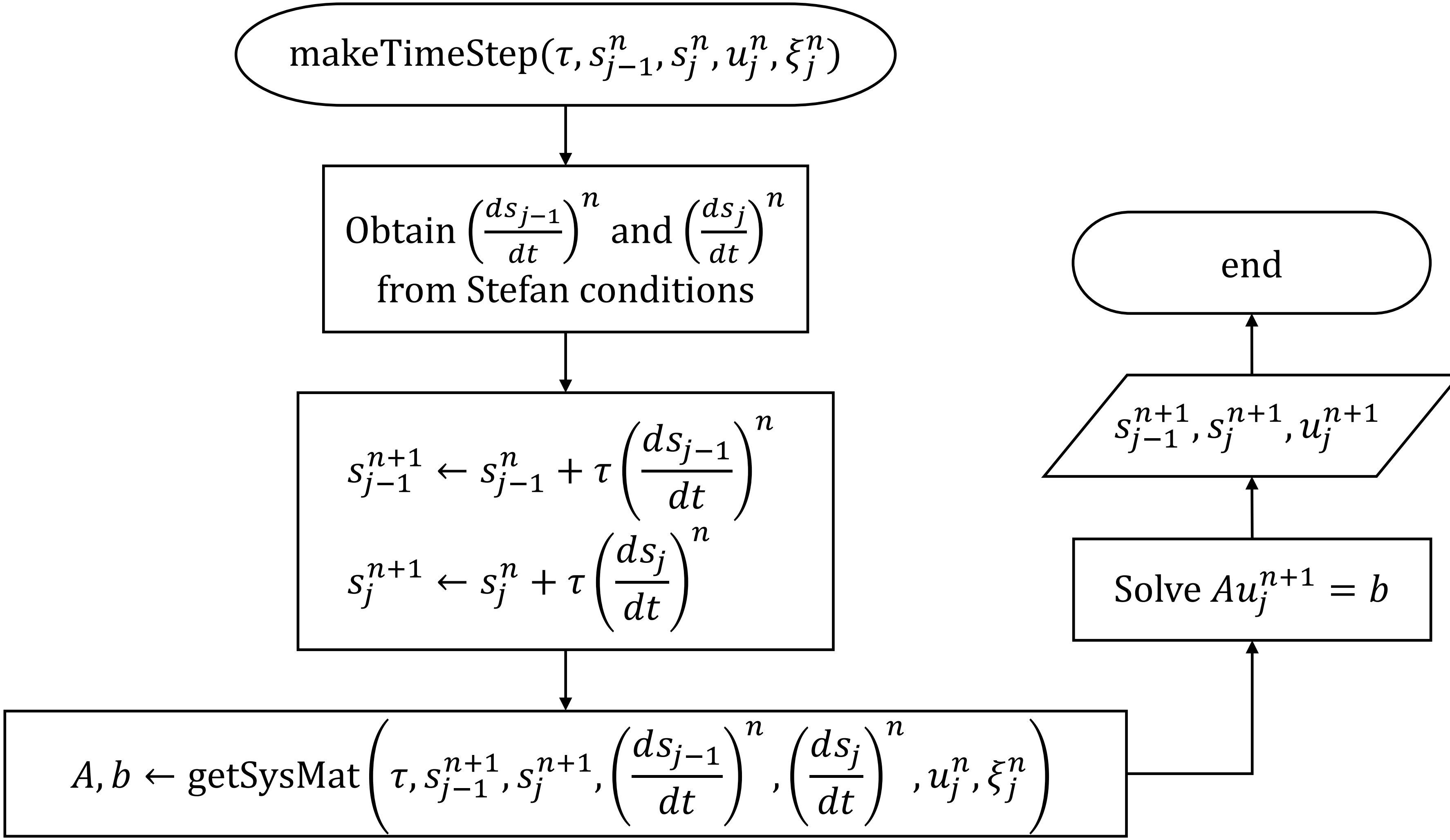


Fig. 1. Non-uniform grid generation with sigmoid function

Time-stepping algorithm



Appearance and the disappearance of phases of matter

- Phase boundaries does not “know” about each other
- Ice is always present
- **Phase disappearance**. If $s_1 < s_0$ (or $s_3 < s_2$) then lower (upper) phase is considered fully frozen.
- **Phase appearance.** If $u > U_F$ then phase is created . Phase size is determined using law of conservation of energy.

Solution of the three-phase Stefan problem

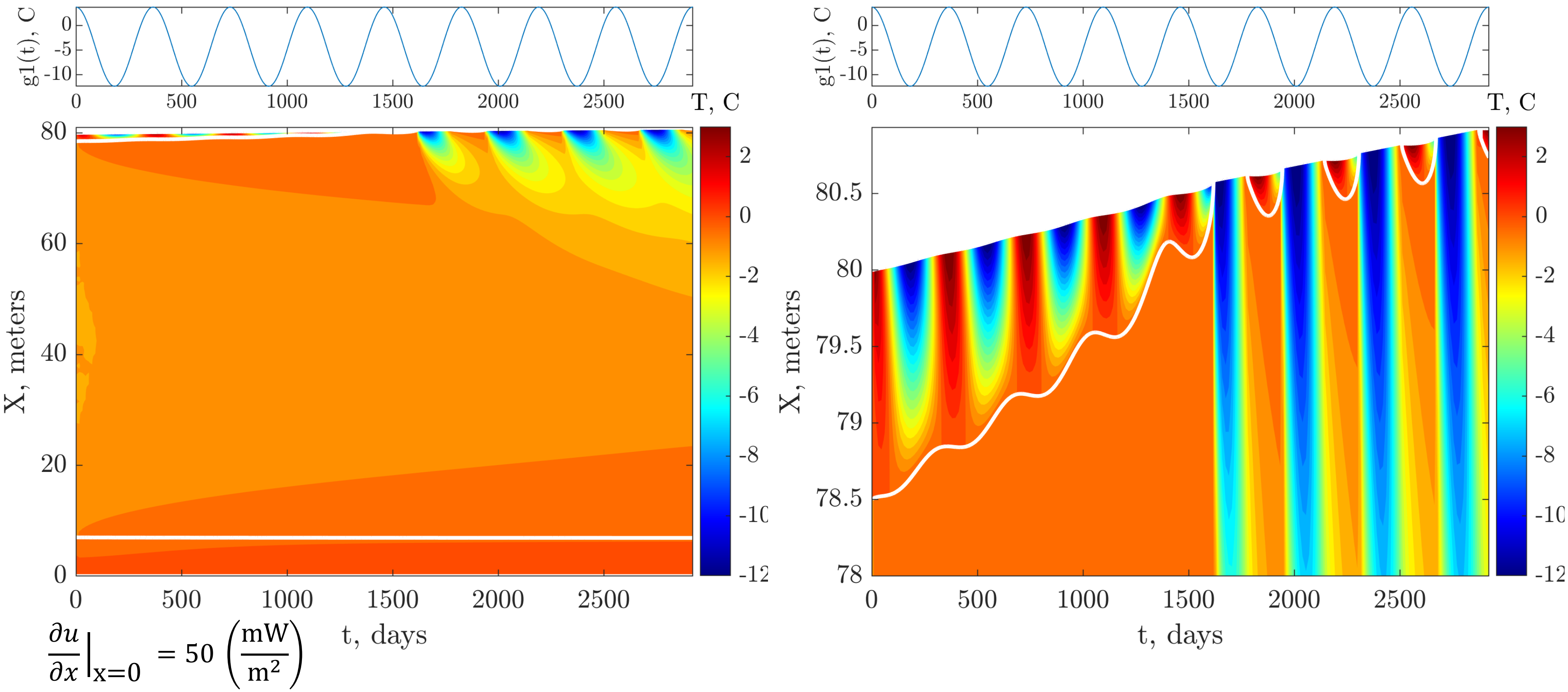


Fig. 2. Numerical solution of the three-phase Stefan problem. Above is a graph of surface temperature. The right figure is an zoomed in version of the left figure

Parallel implementation

- The numerical solution of the Stefan problem was implemented in MATLAB
- MATLAB supports multiprocessing by means of MATLAB workers
- MATLAB workers is a MATLAB session that runs without GUI
- Master process distributes computational work to workers and retrieves the results

- Our model allows full data parallelism
- One worker per physical core
- Each worker solves a Stefan problem. Master process retrieves the results and writes it to disk

Writing results to disk

- txt – slow and files take up a lot of space.
- mat (HDF5 variant) – poor implementation in MATLAB, too slow and files take up a lot of space. Time complexity $O(N)$
- bin – our choice. Read $\sim$0.1 us. Write $\sim$1.5 us. Read/write time complexity $O(1)$

Scalability Test. Methodology

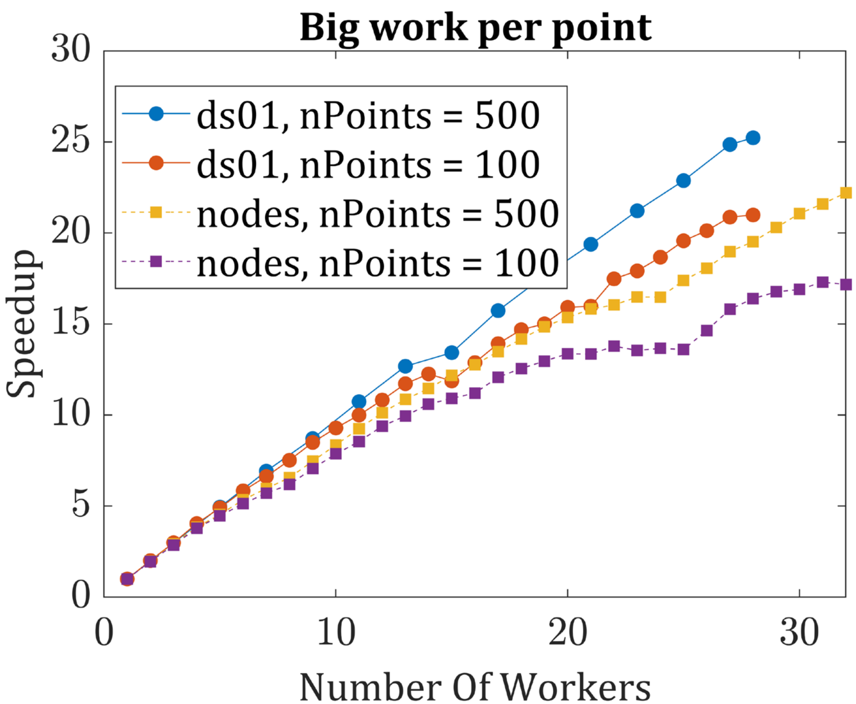
We tested parallel implementation using following computing nodes (SPBU Computing Centre):

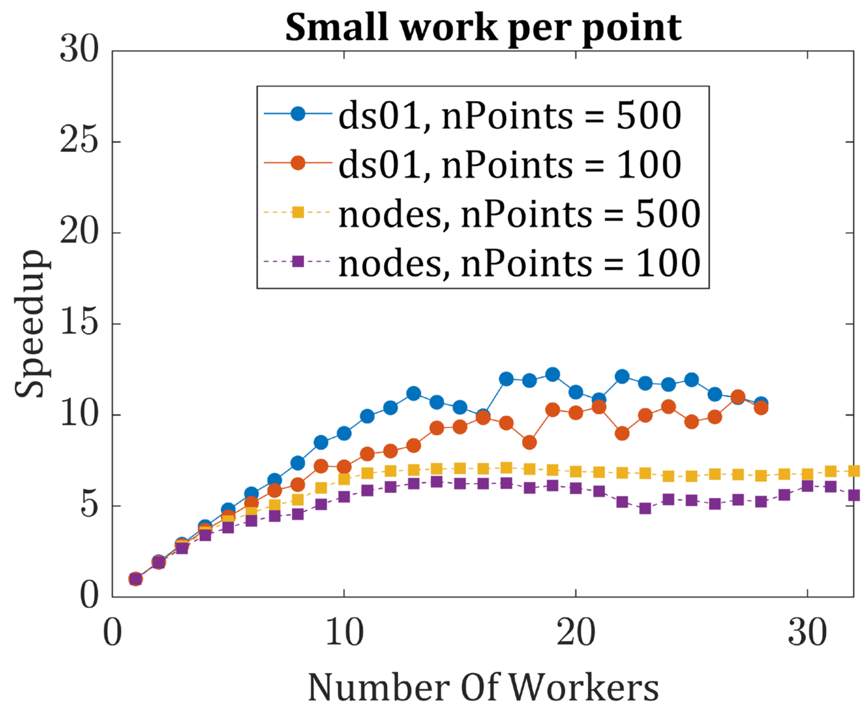
1. Node ds01: Intel(R) Xeon(R) CPU E5-2690 v4 @ 2.60 GHz, 28 cores, 250 GB RAM, 1 Gbit Ethernet, NFS
2. Nodes node-3, node-25, node-43, node-44: Intel(R) Xeon(R) CPU E5335 @ 2.00 GHz, 8 Cores, 16 GB RAM, 1 Gbit Ethernet, NFS

Methodology:

1. Computational work per geographical point
   - Big work per point: $\tau = 1$ month, $t_{max} = 1000$ years
   - Small work per point: $\tau = 4$ month, $t_{max} = 500$ years
2. We considered two cases: 500 Stefan problems and 100 Stefan problems

Scalability Test. Results





Simulation of basal melt of Antarctic ice sheet

- We used Bedmap2 [Fretwell et al., 2013] data to obtain initial sub- and supraglacial surfaces (~ 120 000 geographical points)
- Surface temperature changes according to sinusoidal law
- Geothermal heat flux is constant [Martos et al., 2017]
- Initial ice temperature is at pressure melting point

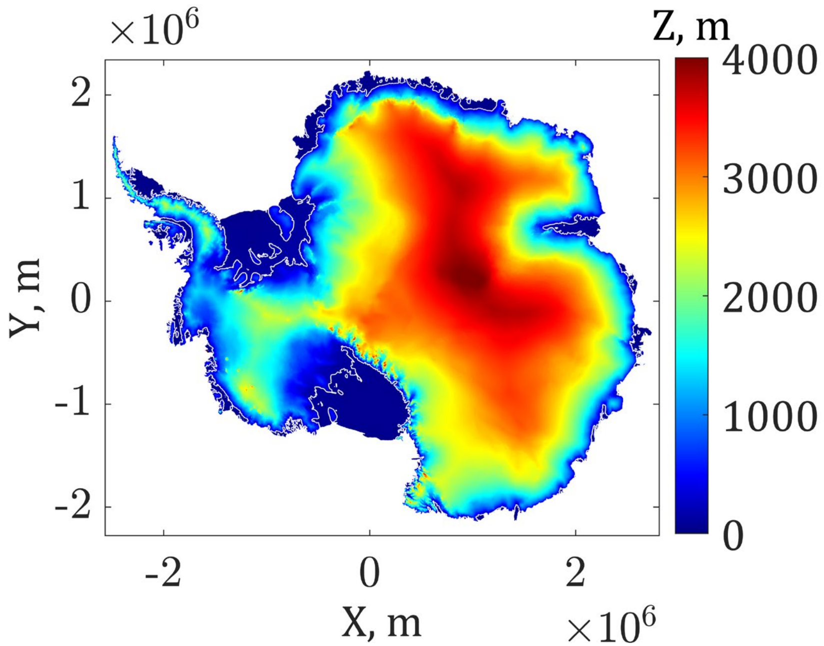
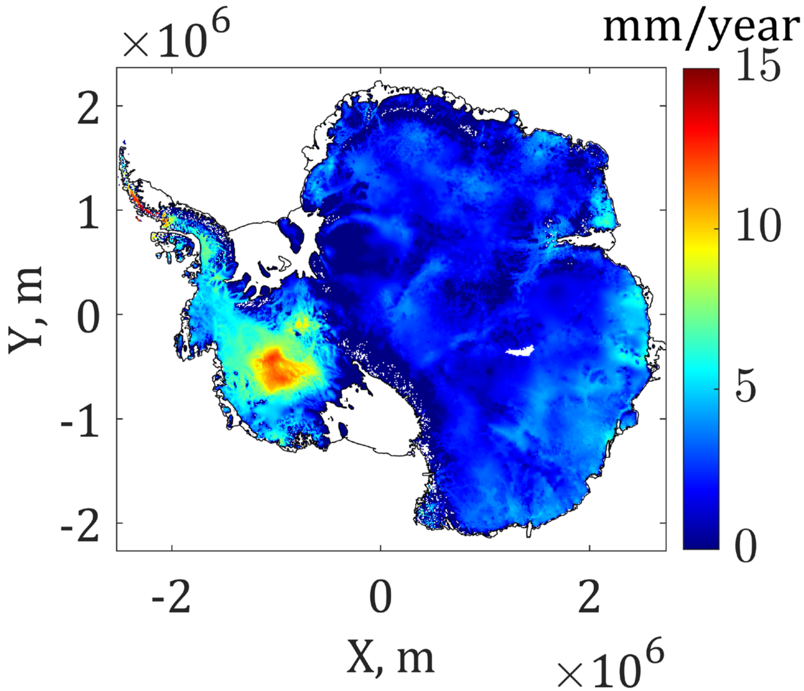


Fig. 3. Antarctic ice sheet surface, Bedmap2 data

Simulation results



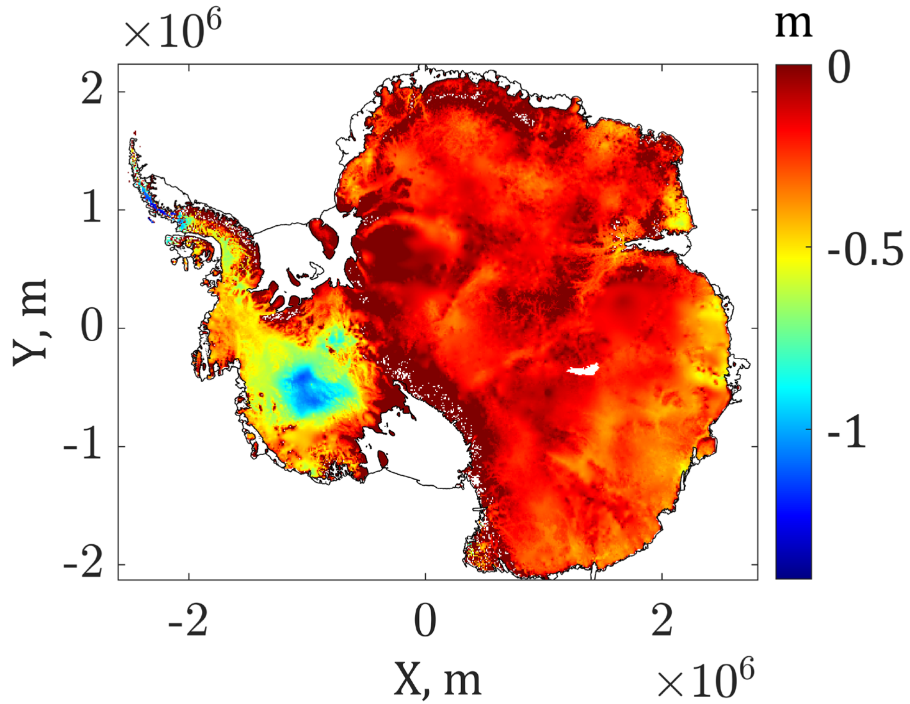
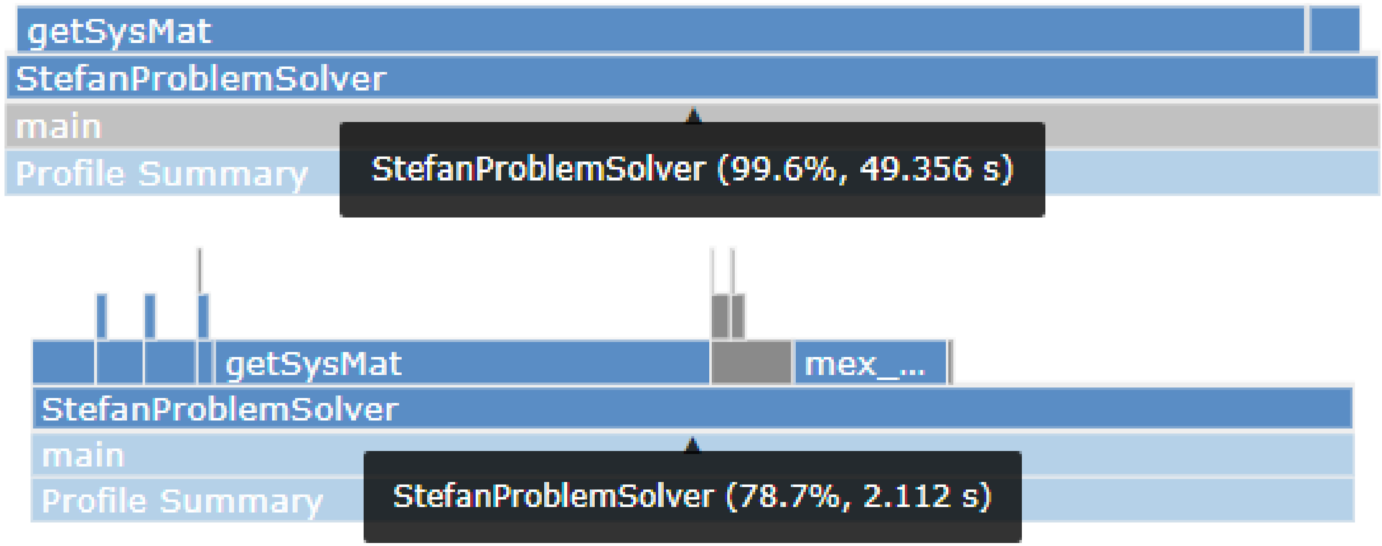


Fig. 4. Left: basal melt rate. Right: supraglacial surface change excluding precipitation accumulation.

Mean total basal melt rate is 29 Gt/year (2.5 mm/year)

Optimization



- **Initial calculation time:** 12 hours and 55 minutes
- We used built-in MATLAB profiler and found out that procedures of obtaining and solving systems of linear equations are hotspots
- **After TMA with C MEX API:** 2 hours and 32 minutes (five times faster)
- **Algebraic simplifications:** 1 hour and 42 minutes

Conclusions

1. We implemented simple parallel numerical model for basal melt of Antarctic ice sheet
2. We obtained mean basal melt rate of 29 Gt/year.
   Similar models provide basal melt rate up to 4000 Gt/year [Seroussi et al., 2020], but with ice shelves included.
3. The relatively low basal melt rate is due to absence of ice shelves in our model. Model [Pattyn, 2010] does not include ice shelves and provides basal melt rate of 65 Gt/year
4. MATLAB have proved to be simple and useful toolset for our problem

Thank you for your attention

References

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