Solution of the Schrödinger equation in the two Coulomb centers problem in the two-dimensions Lobachevsky space

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Quasi-angular Heun's equation

The quantum-mechanical problem of a particle motion in the field of two Coulomb centers with charges Z_1 and Z_2 was investigated in two-dimensional Euclidean space [1] and in two-dimensional Lobachevsky space [2]. The Schrödinger equation permits the separation of variables in both cases. In [2], two-dimensional Lobachevsky space was embedded in three-dimensional pseudo-Euclidean space. The separation of variables was realized by introducing quasi-angular x and quasi-radial z coordinates with the help of which a wave function can be written as $\Psi(x,z) = v(x)u(z)$. The quasi-angular function $v(x)$ was presented by the product

$$v(x) = (x-a)^{\alpha_+} f(x)$$

and the function $f(x)$ satisfies quasi-angular Heun's equation

$$\frac{d^2 f}{dx^2} + \left(\frac{1}{2x} + \frac{1}{2(x-1)} + \frac{2\alpha_+ + 1/2}{(x-a)}\right)\frac{df}{dx} + \frac{(Ax-p)f}{x(x-1)(x-a)} = 0. \quad (1)$$

Here E is energy, Λ is separation constant and

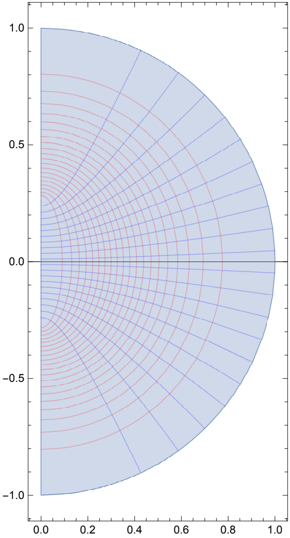
$$Z_- = Z_2 - Z_1, \quad \alpha_\pm = \frac{1}{4}\left(1 + \sqrt{-8E\rho^2 \pm 8Z_-\rho + 1}\right), \quad a = -\frac{(\gamma-1)^2}{4\gamma},$$

$$A = (\alpha_+ + \alpha_-)(\alpha_+ - \alpha_- + 1/2) = \alpha_+ + Z_-\rho, \quad p = \frac{A}{2} + \frac{\Lambda}{4\gamma}.$$

A parameter γ is defined via a curvature radius ρ and a distance R between charges by means of the formula

$$\gamma = \frac{e^{R/\rho}+1}{e^{R/\rho}-1}, \quad \gamma > 1. \qquad (2)$$

If $R/\rho \ll 1$ then $\gamma \to 2\rho/R$. In the used measurement system, Planck's constant, mass and charge of a particle moving in the field are equal to unit.



The presentation of solutions of the Heun's equation in the form of various series of hypergeometric functions is well known [3, 4]. In [2], the single expansion

$$f(x) = \sum_{j=-\infty}^{\infty} c_j F(-j, j; 1/2; x) \tag{3}$$

was considered. In the present paper, we examine two types of expansions in series of hypergeometric functions which in the limit $\rho \to \infty$ are converted to expansions for even and odd solutions investigated in [1].

Two expansions for solutions of quasi-angular equation

In accordance with [4], we obtain the expansion of the first type

$$f^{+}(x) = \sum_{j=0}^{\infty} c_j F(-j, j; 1/2; x). \tag{4}$$

The coefficients c_j obey the recurrence relations

$$\alpha_0 c_1 + \beta_0 c_0 = 0, \quad \alpha_1 c_2 + \beta_1 c_1 + 2\gamma_1 c_0 = 0, \tag{5}$$

$$\alpha_j c_{j+1} + \beta_j c_j + \gamma_j c_{j-1} = 0, \quad j > 1, \tag{6}$$

where

$$\begin{gathered} \alpha_j = 2\varepsilon(j+1-d_+)(j+1-d_-), \\ \beta_j = -j^2 + \lambda, \\ \gamma_j = 2\varepsilon(j-1+d_+)(j-1+d_-). \end{gathered} \tag{7}$$

Here we introduce notations

$$d_+ = \alpha_+ + \alpha_-, \quad d_- = \alpha_+ - \alpha_- + 1/2,$$

and

$$\lambda = \frac{\Lambda}{\gamma^2 + 1}, \quad \varepsilon = \frac{\gamma}{2(\gamma^2 + 1)} \tag{8}$$

Note that the relation (6) coincides with relation derived in [2].

By applying substitution

$$f(x) = x^{1/2}(1-x)^{1/2}\tilde{f}(x)$$

we get the following Heun's equation

$$\frac{d^2\tilde{f}}{dx^2} + \left(\frac{3}{2x} + \frac{3}{2(x-1)} + \frac{2\alpha_+ + 1/2}{(x-a)}\right)\frac{d\tilde{f}}{dx} + \frac{(\tilde{A}x - \tilde{p})\tilde{f}}{x(x-1)(x-a)} = 0, \tag{9}$$

where

$$\tilde{A} = A + 2\alpha_+ + 3/2, \quad \tilde{p} = a + \alpha_+ + 1/4 + \frac{A}{2} + \frac{\Lambda}{4\gamma}.$$

Now according to [4], we represent the function $\tilde{f}(x)$ in the form of expansion in hypergeometric functions

$$\tilde{f}(x) = \sum_{j=1}^{\infty} \tilde{c}_j F(-j+1, j+1; 3/2; x).$$

The coefficients $\tilde{c}_j$ satisfy the recurrence relations

$$\tilde{\alpha}_1 \tilde{c}_2 + \tilde{\beta}_1 \tilde{c}_1 = 0, \tag{10}$$

$$\tilde{\alpha}_j \tilde{c}_{j+1} + \tilde{\beta}_j \tilde{c}_j + \tilde{\gamma}_j \tilde{c}_{j-1} = 0, \quad j > 1, \tag{11}$$

where

$$\begin{aligned} \tilde{\alpha}_j &= 2\varepsilon(j+1-d_+)(j+1-d_-)\frac{j}{(j+1)}, \\ \tilde{\beta}_j &= -j^2 + \lambda, \\ \tilde{\gamma}_j &= 2\varepsilon(j-1+d_+)(j-1+d_-)\frac{j}{(j-1)}. \end{aligned} \tag{12}$$

Hence, the second solution of Eq. (1) is given in the form of series of the second type

$$f^-(x) = \sum_{j=1}^{\infty} \tilde{c}_j x^{1/2}(1-x)^{1/2} F(-j+1, j+1; 3/2; x). \tag{13}$$

Emphasize that functions

$$y_j^+(x) = F(-j, j; 1/2; x), \quad y_j^-(x) = x^{1/2}(1-x)^{1/2}F(-j+1, j+1; 3/2; x) \tag{14}$$

are two linearly independent solutions of the hypergeometric equation

$$\frac{d^2y}{dx^2} + \left(\frac{1}{2x} + \frac{1}{2(x-1)}\right)\frac{dy}{dx} - \frac{j^2 y}{x(x-1)} = 0. \tag{15}$$

Eigenvalues of the separation constant

The recurrence relations (5), (6) and (10), (11) allow us to obtain the eigenvalues of λ. In the case of small intercenter distance, when $\varepsilon^2 \to R^2/16\rho^2$, we find the expansions of λ_m^+ and λ_m^- for the first and second solutions of Eq. (1) in powers of ε^2:

$$\lambda_0^+ = -\left(4E + 8Z_-^2\right)\rho^2\varepsilon^2$$

$$-\left[\left(18E + 36Z_-^2\right)\rho^2 + \left(2E^2 - 24EZ_-^2 - 56Z_-^4\right)\rho^4\right]\varepsilon^4 + O\left(\varepsilon^6\right),$$

$$\lambda_1^+ = 1 - \left[6 + \left(2E - \frac{20Z_-^2}{3}\right)\rho^2\right]\varepsilon^2$$

$$-\left[\frac{51}{2} + \left(10E - 28Z_-^2\right)\rho^2 + \left(\frac{E^2}{2} + \frac{242EZ_-^2}{9} + \frac{1526Z_-^4}{27}\right)\rho^4\right]\varepsilon^4 + O\left(\varepsilon^6\right),$$

$$\lambda_1^- = 1 - \left[6 + \left(6E + \frac{4Z_-^2}{3}\right)\rho^2\right]\varepsilon^2$$

$$-\left[\frac{51}{2} + \left(26E + 4Z_-^2\right)\rho^2 + \left(\frac{E^2}{2} - \frac{14EZ_-^2}{9} - \frac{10Z_-^4}{27}\right)\rho^4\right]\varepsilon^4 + O\left(\varepsilon^6\right),$$

$$\lambda_2^+ = 4 - \left[24 + \left(4E - \frac{8Z_-^2}{15}\right)\rho^2\right]\varepsilon^2$$
$$- \left[102 + \left(17E - \frac{22Z_-^2}{5}\right)\rho^2 - \left(\frac{5E^2}{3} + \frac{124EZ_-^2}{45} + \frac{1732Z_-^4}{3375}\right)\rho^4\right]\varepsilon^4 + O\left(\varepsilon^6\right)$$

$$\lambda_2^- = 4 - \left[24 + \left(4E - \frac{8Z_-^2}{15}\right)\rho^2\right]\varepsilon^2$$
$$- \left[102 + \left(19E - \frac{2Z_-^2}{5}\right)\rho^2 + \left(\frac{E^2}{3} + \frac{76EZ_-^2}{45} + \frac{1268Z_-^4}{3375}\right)\rho^4\right]\varepsilon^4 + O\left(\varepsilon^6\right),$$

at $m > 2$

$$\lambda_m^{\pm} = m^2 - \left[6m^2 + \left(4E - \frac{8Z_-^2}{4m^2-1}\right)\rho^2\right]\varepsilon^2$$
$$- \left[\frac{51m^2}{2} + \left(18E - \frac{36Z_-^2}{4m^2-1}\right)\rho^2\right.$$
$$\left. - \left(2E^2 + \frac{24EZ_-^2}{4m^2-1} + \frac{8(20m^2+7)Z_-^4}{(4m^2-1)^3}\right)\frac{\rho^4}{m^2-1}\right]\varepsilon^4 + O\left(\varepsilon^6\right).$$

Indexing of the eigenvalues is realized by means of number m. The obtained eigenvalues of separation constant are converted to eigenvalues derived in [1] in the case of the Euclidean space if we perform the limiting procedure ($\rho \to \infty$) in our formulas. On the other hand, if we reject terms proportional to ε^4 in obtained eigenvalues λ_m^+ then we get eigenvalues derived in [2].

Comparison with the Euclidean case

Let us remark that the functions y_j^+ and y_j^- can be expressed via the Chebyshev polynomials of the first and second kind [5]:

$$y_j^+(x) = T_j(1-2x), \quad y_j^-(x) = x^{1/2}(1-x)^{1/2}\frac{U_{j-1}(1-2x)}{j}.$$

This circumstance is the sufficient reason to introduce new angular variable θ with the help of the formula

$$\cos(\theta) = 1 - 2x. \tag{16}$$

We find that expansions of two solutions of Eq. (1) are presented as series of the trigonometrical functions

$$y_j^+(\theta) = \cos(j\theta), \quad y_j^-(\theta) = \frac{\sin(j\theta)}{2j}. \tag{17}$$

From here it is seen that expansions (4) and (13) in the case of the Lobachevsky space are analogues of expansions used in the case of the Euclidean space [1].

It should be stressed that the hypergeometric equation (15) acquires the simplest form

$$-\frac{d^2y}{d\theta^2} = j^2 y \tag{18}$$

if we use variable θ. Eq. (1) is transformed in the following way

$$(-1+4\varepsilon\cos(\theta))\frac{d^2f}{d\theta^2}-4\varepsilon(2\alpha_+ +1/2)\sin(\theta)\frac{d\,f}{d\theta}-4\varepsilon A\cos(\theta)f = \lambda f. \tag{19}$$

In the limit $\rho \to \infty$, Eq. (19) is converted to equation

$$-\frac{d^2f}{d\theta^2} - R(-2E)^{1/2}\sin(\theta)\frac{d\,f}{d\theta} - R((-2E)^{1/2}/2 + Z_-)\cos(\theta)f = \lambda f, \tag{20}$$

coinciding with the equation considered in [1] in the case of the Euclidean space.

Conclusion

Two solutions of the quasi-angular equation in the two Coulomb centers problem in the two-dimensional Lobachevsky space have been obtained in the form of two series of hypergeometric functions. These results are the essential addition to the results presented in [2]. It is shown that the solutions in the case of the Euclidean space [1] can be obtained from our solutions in the limit $\rho \to \infty$. At last, we intend to use derived eigenvalues of the separation constant for solution of the quasi-radial equation and for calculation of the energy spectrum.

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