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Threshold S -factor and Leptonic Decay Widths for a Composite System of Two Arbitrary-Mass Relativistic Spinor Quarks

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Plan of Talk

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- 2. The solutions of RQP equation in WKB approximation
- 3. The relativistic leptonic decay widths in WKB approximation
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1. Introduction

The present study is a continuation of studies performed earlier in [Yu. D. Chernichenko, V. V. Kondratjuk, *Phys. At. Nucl.* **81**, 51 (2018); Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 205 (2022); Yu. D. Chernichenko, L. P. Kaptari, in *Proceedings of the XXIX Anniversary Seminar “Nonlinear Phenomena in Complex Systems”, NPC’S’2022, June 21–24, 2022, Minsk, Belarus, Nonlinear Dynamics and Applications, Minsk, 2022. – Vol. 28. – P. 92*], where were obtained a relativistic expressions for the leptonic decay widths of vector mesons in the s -wave state ($\ell = 0$) to a lepton-antilepton pair on the basis of relativistic analog of semiclassical (WKB) method [N. B. Skachkov, I. L. Solovtsov, *Sov. J. Nucl. Phys.* **31**, 686 (1980); A. V. Sidorov, N. B. Skachkov, *Theor. Math. Phys.* **46**, 141 (1981); *Preprint JINR P2-80-45*, Dubna, 1980; V. I. Savrin, A. V. Sidorov, and N. B. Skachkov, *Hadronic J.* **4**, 1642 (1981)].

In this studies was used the LogunovTavkhelidze relativistic quasipotential (RQP) approach [A. A. Logunov and A. N. Tavkhelidze, *Nuovo Cim.* **29**, 380 (1963)] to the problem of a composite system of two relativistic spin particles to quantum field theory in the form proposed by V.G. Kadyshevsky [V. G. Kadyshevsky, *Nucl. Phys. B* **6**, 125 (1968)], which is based on the completely covariant Hamiltonian formulation of quantum field theory via a transition to the three-dimensional relativistic configuration representation (**r** representation) [V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, *Nuovo Cim. A* **55**, 233 (1968)].

In RQP approach [V. G. Kadyshevsky, *Nucl. Phys. B* **6**, 125 (1968); V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, *Nuovo Cim. A* **55**, 233 (1968)], the relativistic WKB expression for the leptonic decay widths of **vector meson** V with energy $M_Q = M_n$ for given **level** n and with **relative orbital moment** $\ell \geq 0$ for the case of two relativistic spinless particles with arbitrary masses m_1, m_2 , that interacts by means of potential

$$V(r) = V_{\text{conf}}(r) - \frac{\alpha_s}{r}, \quad (1)$$

$V_{\text{conf}}(r)$ is the confining potential ($V_{\text{conf}}(0) = 0$), α_s is the strong coupling constant, has following the form [Yu. D. Chernichenko, V. V. Kondratjuk, *Phys. At. Nucl.* **81**, 51 (2018)],

$$\begin{aligned} \Gamma_{n,\ell}(V \rightarrow e^+e^-) &= \frac{2\alpha^2 Q_V^2 (\hbar c)^3 \Gamma^2(\ell + 1)}{\pi \lambda'^3 g' m' c^2 \Gamma^2(2\ell + 2) M_n^2} \times \\ &\times \left(\frac{u'_{\text{rel},n}}{g'} \right)^{2\ell+1} L_{\text{RQP}}(u'_{\text{rel},n}) \frac{dM_n}{dn}, \end{aligned} \quad (2)$$

α is the fine-structure constant, Q_V is multiplier conditioned by isotopic structure of vector meson V and charges of quarks expressed in unit of the electric charge e and for the spinless quarks $Q_V = e_q$, where e_q is the quark charge in the units the electric charge e with the number of its colours $N_c = 3$; $\lambda' = \hbar/m'c$ is the Compton wavelengths of the effective relativistic particle with mass $m' = \sqrt{m_1 m_2}$, a relative three-momentum \mathbf{q}' and the energy $E_{q'} = c\sqrt{m'^2 c^2 + \mathbf{q}'^2}$, emerging instead of the system of two particles and carrying the total c.i.s. energy $\sqrt{s} = M_Q = \sqrt{Q^2} = \sqrt{(q_1 + q_2)^2}$ of the interacting particles proportional to the energy $E_{q'}$: $\sqrt{s} = c\sqrt{m_1^2 c^2 + \mathbf{q}^2} + c\sqrt{m_2^2 c^2 + \mathbf{q}^2} = 2g' E_{q'}$ and the factor g' gives by formula

$$g' = \frac{m'}{2\mu} = \frac{m_1 + m_2}{2\sqrt{m_1 m_2}} \quad (3)$$

$\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass,

$$u'_{\text{rel}} = \frac{2u}{\sqrt{1 - u^2}} \quad (4)$$

is the relative velocity of the effective relativistic particle of mass m' , the velocity u determine by expression

$$u = \sqrt{1 - \frac{4m'^2 c^4}{M_Q^2 - (m_1 - m_2)^2 c^4}}, \quad (5)$$

the relativistic spinless resummations Coulomb L and S factors give by expressions [O. P. Solovtsova, Yu. D. Chernichenko, *Phys. Atom. Nucl.* **73**, 1612 (2010); *Theor. Math. Phys.* **166**, 225 (2011)]

$$L_{\text{RQP}}(\chi') = \prod_{n=1}^{\ell} \left[1 + \left(\frac{\tilde{\alpha}_s}{2n \sinh \chi'} \right)^2 \right] S_{\text{RQP}}(\chi'), \quad \tilde{\alpha}_s = \frac{\alpha_s}{g' \hbar c}, \quad (6)$$

$$S_{\text{RQP}}(\chi') = \frac{X_{\text{RQP}}(\chi')}{1 - \exp[-X_{\text{RQP}}(\chi')]}, \quad X_{\text{RQP}}(\chi') = \frac{\pi \tilde{\alpha}_s}{\sinh \chi'}, \quad (7)$$

the value $\sinh \chi'$ is expressed through relative velocity in (4) as

$$\sinh \chi' = u'_{\text{rel}}/2g', \quad (8)$$

χ' is the rapidity, which is connected to the 3-momentum \mathbf{q}' , the energy $E_{q'}$ of the effective relativistic particle of mass m' and the total c.i.s. energy $M_{\mathcal{Q}}$ of the interacting particles as

$$\mathbf{q}' = m'c \sinh \chi' \mathbf{n}_{q'}, |\mathbf{n}_{q'}| = 1, E_{q'} = m'c^2 \cosh \chi', M_{\mathcal{Q}} = 2g'm'c^2 \cosh \chi'.$$

The relativistic spinless resummations Coulomb L and S factors have the correct relativistic ($u \rightarrow 1$) and ultrarelativistic ($m' \rightarrow 0$) limits equal to unit unlike the relativistic S factors in Refs. [A. H. Hoang, *Phys. Rev. D* **56**, 7276 (1997); J.-H. Yoon and Ch.-Y. Wong, *Phys. Rev. C* **61**, 044905 (2000); *J. Phys. G: Nucl. Part. Phys.* **31**, 149 (2005); A. B. Arbuzov, *Nuovo Cimento A* **107**, 1263 (1994)].

2. The solutions of RQP equation in WKB approximation

The basis of our consideration is completely covariant RQP equation in the \mathbf{r} representation [Yu. D. Chernichenko, *Phys. At. Nucl.* **84**, 339 (2021)] for the radial RQP wave function $\varphi_\ell(r, \chi')$ that describes the composite system of two relativistic spin particles of arbitrary masses m_1, m_2 with the relative orbital moment $\ell \geq 0$

$$\left(\hat{H}_{0,\ell}^{\text{rad}} - \cosh \chi'\right) \varphi_\ell(r, \chi') = -V(r) \hat{A} \left(\hat{H}_{0,\ell}^{\text{rad}}\right) \varphi_\ell(r, \chi'), \quad (9)$$

$$\hat{H}_{0,\ell}^{\text{rad}} = \cosh \left(i\lambda' \frac{d}{dr}\right) + \frac{\lambda'^2 \ell(\ell+1)}{2r(r+i\lambda')} \exp \left(i\lambda' \frac{d}{dr}\right)$$

is the radial part of operator free Hamiltonian

$$\begin{aligned} \hat{H}_0 = 2m'c^2 \left[\cosh \left(i\lambda' \frac{\partial}{\partial r}\right) + \frac{i\lambda'}{r} \sinh \left(i\lambda' \frac{\partial}{\partial r}\right) - \right. \\ \left. - \frac{\lambda'^2}{2r^2} \Delta_{\theta,\varphi} \exp \left(i\lambda' \frac{\partial}{\partial r}\right) \right], \end{aligned}$$

while $\Delta_{\theta,\varphi}$ is its the angular part, potential $V(r)$ is local in the sense of Lobachevsky geometry and for simplicity depends not from energy $M_{\mathcal{Q}}$, the operator \hat{A} is

$$\hat{A} \left(\hat{H}_{0,\ell}^{\text{rad}} \right) = \frac{1}{4} \left[a' \left(\hat{H}_{0,\ell}^{\text{rad}} \right)^2 + b' \right],$$

$$a' = \frac{1}{2}g'^2, \quad b' = \frac{3}{4} - \frac{1}{2}g'^2 \text{ for } \hat{O} = \gamma_{\mu} \text{ (vector)}, \quad (10)$$

χ' is the rapidity

$$\begin{aligned} \Delta_{q',m'\lambda_{\mathcal{Q}}} &= m'c \sinh \chi' \mathbf{n}_{\Delta_{q',m'\lambda_{\mathcal{Q}}}}, \quad |\mathbf{n}_{\Delta_{q',m'\lambda_{\mathcal{Q}}}}| = 1, \\ M_{\mathcal{Q}} &= 2g' \Delta_{q',m'\lambda_{\mathcal{Q}}}^0, \quad \Delta_{q',m'\lambda_{\mathcal{Q}}}^0 = m'c^2 \cosh \chi'. \end{aligned} \quad (11)$$

We shall remind that here all the 4-momentums belong to the upper sheet of the mass hyperboloid $\Delta_{q',m'\lambda_Q}^2 = \Delta_{q',m'\lambda_Q}^{02} - c^2 \Delta_{q',m'\lambda_Q}^2 = m'^2 c^4$, where $\lambda_Q = (\lambda_Q^0; \boldsymbol{\lambda}_Q) = Q/\sqrt{Q^2}$ is the 4-velocity of a composite particle with the 4-momentum $Q = q_1 + q_2$, and $\Delta_{q',m'\lambda_Q}^0, \Delta_{q',m'\lambda_Q}$ are, respectively, the time and spatial components of the 4-vector $\Lambda_{\lambda_Q}^{-1} q' = \Delta_{q',m'\lambda_Q}$ from the Lobachevsky space [Yu. D. Chernichenko, *Phys. At. Nucl.* **84**, 339 (2021)].

We note that the values of the parameters a', b' in the Eq. (10) at $m_1 = m_2 = m$ coincides with the values of their the analogs a, b that were obtained for the case of spin and equal masses [Yu. D. Chernichenko, *Phys. At. Nucl.* **80**, 707 (2017)].

We will seek the WKB solution of RQP Eq. (9) in the usual form [Yu. D. Chernichenko, V. V. Kondratjuk, *Phys. At. Nucl.* **81**, 51 (2018); Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 205 (2022); Yu. D. Chernichenko, L. P. Kaptari, in *Proceedings of the XXIX Anniversary Seminar “Nonlinear Phenomena in Complex Systems”, NPCS’2022, June 21–24, 2022, Minsk, Belarus*, Nonlinear Dynamics and Applications, Minsk, 2022. – Vol. 28. – P. 92; N. B. Skachkov, I. L. Solovtsov, *Sov. J. Nucl. Phys.* **31**, 686 (1980); A. V. Sidorov, N. B. Skachkov, *Theor. Math. Phys.* **46**, 141 (1981); *Preprint JINR P2-80-45*, Dubna, 1980; V. I. Savrin, A. V. Sidorov, and N. B. Skachkov, *Hadronic J.* **4**, 1642 (1981)].

$$\varphi_\ell(r, \chi') = \exp \left[\frac{i}{\hbar} g(r) \right], \quad (12)$$

$$g(r) = g_0(r) + \frac{\hbar}{i} g_1(r) + \left(\frac{\hbar}{i} \right)^2 g_2(r) + \dots$$

With two first terms of the expansion in (12) the WKB solutions with the left, r_L , and the right, r_R , of the classical turning points in the inner region $r_L \leq r \leq r_R$ has then the form

$$\varphi_\ell^{L,R}(r, \chi') = \frac{C_{L,R}(\chi')}{2 \sqrt[4]{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} \times \times \left\{ \exp \left[i\alpha_+^{L,R}(r) \mp \frac{i\pi}{4} \right] + \exp \left[i\alpha_-^{L,R}(r) \pm \frac{i\pi}{4} \right] \right\}, \quad (13)$$

$$\alpha_\pm^{L,R}(r) = \frac{1}{\lambda'} \int_{r_{L,R}}^r dr' \chi_\pm(r'), \quad (14)$$

$$\chi_\pm(r) = \ln \left[\mathcal{X}(r) \pm \sqrt{\mathcal{X}^2(r) - R^2(r)} \right], \quad R(r) = \sqrt{1 + \frac{\lambda'^2 \Lambda^2}{r^2}},$$

$$\Lambda = \ell + 1/2, \quad \mathcal{X}(r) = \frac{2X(r)}{1 + \sqrt{1 + a'V(r)X(r)}}, \quad X(r) = \cosh \chi' - \frac{b'}{4}V(r),$$

$C_{L,R}$ are the normalization constants, the turning points, $r_{L,R}$, are branch points of root in the WKB solutions (13), (14) that lead to

$$\mathcal{X}(r_{L,R}) = R(r_{L,R}). \quad (15)$$

Condition of applicability of the relativistic WKB solutions (13)

$$\lambda' \left| \frac{\cosh \chi_{\text{eff}}(r)}{\chi_+(r) \sinh \chi_{\text{eff}}(r)} \frac{d\chi_+(r)}{dr} \right| \ll 1, \quad (16)$$

$$\chi_{\text{eff}}(r) = \text{arcosh} \mathcal{X}_{\text{eff}}(r) = \ln \left(\mathcal{X}_{\text{eff}}(r) + \sqrt{\mathcal{X}_{\text{eff}}^2(r) - 1} \right),$$

$$\mathcal{X}_{\text{eff}}(r) = \cosh \chi_{\text{eff}}(r) = \frac{\mathcal{X}(r)}{R(r)}.$$

In the case of $\ell = 0$, the condition (16) is converted in inequality

$$\lambda' \left| \frac{\cosh \chi_S(r)}{\chi_S(r) \sinh \chi_S(r)} \frac{d\chi_S(r)}{dr} \right| \ll 1, \quad (17)$$

$$\chi_S(r) = \text{arcosh} \mathcal{X}(r) = \ln \left[\mathcal{X}(r) + \sqrt{\mathcal{X}^2(r) - 1} \right]. \quad (18)$$

WKB quantization condition of levels energy [Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 488 (2022)]

$$\int_{r_L}^{r_R} dr [\chi_+(r) - \ln R(r)] = \pi \lambda' \left(n + \frac{1}{2} \right), \quad n = 0, 1, \dots, \ell \geq 0. \quad (19)$$

At $\alpha' = 0, b' = 2/g'm'c^2$ the WKB quantization condition of levels energy in Eq. (19) coincides with the analogous expression obtained in the case of spinless particles of arbitrary masses [Yu. D. Chernichenko, V. V. Kondratjuk, *Phys. At. Nucl.* **81**, 51 (2018)].

In the case of equal masses $m_1 = m_2 = m$ ($g' = 1$) the expression (19) moves over to the WKB quantization condition of levels energy that was obtained in the case of spin particles with the equal masses in [Yu. D. Chernichenko, *Phys. At. Nucl.* **83**, 488 (2020)].

3.The relativistic leptonic decay widths in WKB approximation

Within the RQP approach, the relativistic modification of nonrelativistic expression for the leptonic decay widths of vector mesons as a bound system of two spin quarks of equal mass m with energy M_n and relative orbital moment $\ell = 0$, proposed by Matveev, Struminskii, and Tavkhelidze [V. A. Matveev, B. V. Struminsky, A. N. Tavkhelidze, *Preprint JINR P-2524*, Dubna, 1965] or by Van Royen and Weisskopf [W. F. Weisskopf, R. Van Royen, *Nuovo Cim.* **50**, 617 (1967); **51**, 583 (1967)], also Refs. [B. Durand and L. Durand, *Phys. Rev. D* **30**, 1904 (1984); E. Etim and L. Schülke, *Nuovo Cim. A* **77**, 347 (1983); R. Barbieri, R. Gatto, R. Kögerler and Z. Kunszt, *Phys. Lett. B* **57**, 455 (1975)], can be represented through solution (13) in the form

[Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 205 (2022); Yu. D. Chernichenko, L. P. Kaptari, in *Proceedings of the XXIX Anniversary Seminar "Nonlinear Phenomena in Complex Systems", NPC'S'2022, June 21–24, 2022, Minsk, Belarus*, Nonlinear Dynamics and Applications, Minsk, 2022. – Vol. 28. – P. 92]

$$\begin{aligned} \Gamma_{n,\ell=0}(V \rightarrow e^+e^-) &= \frac{16\pi\alpha^2 Q_V^2 f_1^2(t)}{M_n^2} \times \\ &\times \lim_{r \rightarrow i\lambda'} \left| e^{-\pi\tilde{\rho}'/2} \Gamma(1 + i\tilde{\rho}') \frac{\varphi_0^L(r, \chi_n)}{r} \right|^2, \\ \tilde{\rho}' &= \frac{\tilde{\alpha}'_s a' \cosh \chi'}{4}, \quad \tilde{\alpha}'_s = \frac{\alpha_s}{\lambda'}, \end{aligned} \quad (20)$$

$f_1(t)$ is the form factor of quark in expression of the current operator of a quark, as function of the square of the 4-momentum transfer

$$t = (p' - p)^2 = 2M_Q^2 c^4 (1 - \cosh \chi_\Delta), \quad (21)$$

p, p' are the initial and final momentums, χ_Δ is the rapidity, $f_1(t)$ depends on the anomalous magnetic moment of quark μ_{anom} , into the nonrelativistic case [W. F. Weisskopf, R. Van Royen, *Nuovo Cim.* **50**, 617 (1967); **51**, 583 (1967)] $\mu_{\text{anom}} = 0 \Rightarrow f_1(t) = 1$. In the case of relativistic quarks $f_1(t)$ we choose as

$$f_1(t) = \frac{\sinh \chi_\Delta}{\chi_\Delta}, \quad (22)$$

$\chi_\Delta / \sinh \chi_\Delta$ is the relativistic geometric factor in the RQP approach [V. G. Kadyshevsky, *Nucl. Phys. B* **6**, 125 (1968)], which for the first time validly came up in Ref. [N. B. Skachkov, *Theor. Math. Phys.* **25**, 1154 (1975)] for invariant description of the spatial structure of the particles in the three-dimensional relativistic \mathbf{r} representation [V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, *Nuovo Cim. A* **55**, 233 (1968)] and it serves the contribution measure of relativistic effects of quarks.

The additional factor $\exp(-\pi\tilde{\rho}'/2)\Gamma(1+i\tilde{\rho}')$ ensures a correct relativistic limit equal to unit for the threshold resummation S factor for the composite system of two relativistic arbitrary-masses of spin quarks for $\chi' \rightarrow +\infty$ ($u \rightarrow 1$) and the transition to the spinless case at $a' = 0$ and $b' = 2/g'm'c^2$ [Yu. D. Chernichenko, *Phys. At. Nucl.* **84**, 339 (2021)]. In the spinor case, the function

$$\psi_0(r, \chi) = e^{-\pi\tilde{\rho}/2}\Gamma(1+i\tilde{\rho})\varphi_0^L(r, \chi)$$

is the physical wave function of s -wave state of the composite system that consists of two relativistic spinor particles of arbitrary-masses, interacting by means of potential (1).

For potential (1) $\varphi_\ell^L(r, \chi_n)$ in region $r \in (r_L; r_R)$ in accordance with Eqs. (13) and (14) can be presented as

$$\varphi_\ell^L(r, \chi_n) = \frac{C_\ell(\chi_n)}{\sqrt[4]{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} \times \times \sin \left\{ \frac{1}{\lambda'} \int_{r_L}^r dr' [\chi_+(r') - \ln R(r')] \right\}, \quad (23)$$

the normalization constant $C_\ell(\chi_n)$ is found from the condition

$$4\pi \int_0^\infty dr |\varphi_\ell^L(r, \chi_n)|^2 = 1, \ell \geq 0. \quad (24)$$

In the region of applicability of the relativistic WKB method the argument of sine in (23) is the quickly oscillating function. Therefore, the square of sine in (24) can be replaced, either as in the nonrelativistic case, on its the average importance equal $1/2$.

Instead of Eq. (24) we then obtain

$$2\pi|C_\ell(\chi_n)|^2 \int_{r_L}^{r_R} \frac{dr}{\sqrt{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} = 1. \quad (25)$$

Differentiation of the WKB quantization condition of energy levels (19) on $M_n = 2g'm'c^2 \cosh \chi_n$ under $\ell \geq 0$, where potential $V_{\text{conf}}(r)$ does not depend on M_n , and taking into consideration determinations (14) and condition (15) for the turning points $r_{L,R}$, we come to the condition

$$\int_{r_L}^{r_R} \frac{dr}{\sqrt{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} = 2\pi\lambda'g'm'c^2 \frac{dn}{dM_n}. \quad (26)$$

From Eqs. (25) and (26) we find

$$|C_\ell(\chi_n)|^2 = \frac{1}{4\pi^2\lambda'g'm'c^2} \frac{dM_n}{dn}. \quad (27)$$

Then, either as in the Refs. [Yu. D. Chernichenko, V. V. Kondratjuk, *Phys. At. Nucl.* **81**, 51 (2018); Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 205 (2022); Yu. D. Chernichenko, L. P. Kaptari, in *Proceedings of the XXIX Anniversary Seminar “Nonlinear Phenomena in Complex Systems”, NPCS’2022, June 21–24, 2022, Minsk, Belarus, Nonlinear Dynamics and Applications, Minsk, 2022. – Vol. 28 – P. 92*; J. S. Bell and P. Pasupathy, *Z. Phys. C* **2**, 183 (1979)], the WKB radial wave function (23) of the potential (1) at $\ell = 0$ for enough of the large value of $\rho = r/\lambda', r \in (r_L; r_R)$, but such, where the Coulomb interaction $V_{\text{Coul}} = -\alpha_s/r$ will dominate in the potential (1), can be approximated by the Coulomb radial *s-wave* function for which its the exact form is the known [Yu. D. Chernichenko, *Phys. At. Nucl.* **84**, 339 (2021)]

$$\begin{aligned} \varphi_0^{\text{Coul}}(\rho, \chi') = 2\pi C_0^{\text{Coul}}(\chi') e^{iB'\chi' - \chi' + i(\rho - \tilde{\rho}')\chi'} (\rho - \tilde{\rho}') \times \\ \times F(1 - iB', 1 - i(\rho - \tilde{\rho}'); 2; 1 - e^{-2\chi'}), \end{aligned} \quad (28)$$

$F(a, b; c; z)$ is the hypergeometric function, $C_0^{\text{Coul}}(\chi')$ is the normalization constant, the parameter B' is defined as

$$B' = \frac{\tilde{\alpha}'_s(a' \cosh^2 \chi' + b')}{4 \sinh \chi'}, \quad (29)$$

at $\chi' = i\kappa_n$ the parameter B' is connected with the quantization condition of the energy levels for the Coulomb potential by expression [Yu. D. Chernichenko, *Phys. At. Nucl.* **84**, 339 (2021)]

$$\frac{\tilde{\alpha}'_s(a' \cos^2 \kappa_n + b')}{4 \sin \kappa_n} = n, \quad \ell = 0, \quad n = 1, 2, \dots, \quad 0 < \kappa_n < \pi/2.$$

Comparing of asymptotic expression for Coulomb function in (28),

$$\varphi_0^{\text{Coul}}(\rho, \chi')|_{\rho \gg 1} \sim \frac{2\pi C_0^{\text{Coul}}(\chi') e^{-\pi B'/2}}{\sinh \chi' |\Gamma(1 - iB')|} \sin \left[\rho \chi' + \delta_0^{\text{Coul}, S}(\chi') \right],$$

with the asymptotic form of WKB solution in (23), taken at $\ell = 0$,

$$\varphi_0^L(\rho, \chi')|_{\rho \gg 1} \sim \frac{C_0(\chi')}{\sqrt{\sinh \chi'}} \sin \left[\rho \chi' + \delta_0^{\text{Coul}, \text{WKB}, S}(\chi') \right],$$

gives the relationship between the normalization constants

$$|2\pi C_0^{\text{Coul}}(\chi')|^2 = \sinh \chi' e^{\pi B'} |\Gamma(1 - iB')|^2 |C_0(\chi')|^2; \quad (30)$$

$$\delta_0^{\text{Coul}, S}(\chi') = B' \ln (2\rho \sinh \chi') - \tilde{\rho}' \chi' + \arg \Gamma(1 - iB') \quad (31)$$

$$\delta_0^{\text{Coul}, \text{WKB}, S}(\chi') = B' \ln \left(\frac{2\rho \sinh \chi'}{B'} \right) - \tilde{\rho}' \chi' \quad (32)$$

are the phases of the Coulomb wave functions in (28) and in the WKB approximation [Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 488 (2022)].

Finally, taking into consideration Eqs. (20), (27), (28), and (30), we get expression for the relativistic leptonic decays widths of the vector mesons in *s-wave state* ($\ell = 0$) and with energy M_n as the composite system of two relativistic spinor quarks of arbitrary masses m_1, m_2 , interacting by means of potential (1):

$$\Gamma_{n,\ell=0}(V \rightarrow e^+e^-) = \frac{4\alpha^2 Q_V^2 f_1^2(t) (\hbar c)^3 \sinh \chi_n}{\pi \lambda'^3 g' m' c^2 M_n^2} S_{\text{RQP},S}(\chi_n) \frac{dM_n}{dn}, \quad (33)$$

$$S_{\text{RQP},S}(\chi') = \frac{X_{\text{RQP},S}(\chi')}{1 - \exp[-X_{\text{RQP},S}(\chi')]} e^{-\pi \tilde{\rho}'} \times \quad (34)$$

$$\times |\Gamma(2 + i\tilde{\rho}') F(1 + iB', -i\tilde{\rho}'; 2; 1 - e^{-2\chi'})|^2$$

is the resummation Coulomb *S* factor in considered RQP approach for the composite system of two relativistic spin particles of arbitrary masses, that interacts by means of the Coulomb potential.

$$X_{\text{RQP},S}(\chi') = 2\pi B' = \frac{\pi \tilde{\alpha}'_s (a' \cosh^2 \chi' + b')}{2 \sinh \chi'}, \quad (35)$$

which can be presented by means of (8) in term of the velocities (4) and (5)

$$\begin{aligned} X_{\text{RQP},S}(u) &= \frac{\pi \tilde{\alpha}'_s \sqrt{1-u^2}}{2g'u} \left[g'^2(a' + b') + \frac{a'u^2}{1-u^2} \right] = \\ &= \frac{\pi \tilde{\alpha}'_s}{g'u'_{\text{rel}}} \left[g'^2(a' + b') + \frac{a'}{4} u'^2_{\text{rel}} \right]. \end{aligned} \quad (36)$$

We note that at $a' = 0, b' = 2/g'm'c^2$ Eq. (33) moves over to Eq. (2) for the case spinless, taken at $\ell = 0, Q_V = e_q, f_1(t) = 1$.

4. The study of the influence spin parameters of vector mesons on behavior of their the leptonic decay widths

We shall conduct the study of the influence spin parameters a' and b' in (10) on importances of the relativistic leptonic decay widths for s -state of vector mesons. On Fig. 1, we present the behavior of $R = R(u)$. $R = R(u)$ is defined as the ratio of the relativistic leptonic decay widths for s -state of vector mesons, Eqs. (33), (34) and (36), to its the relativistic spinless of analogue, Eqs. (2) and (7), at $\ell = 0$, $f_1(t) = 1$, $Q_V = e_q$, $\hbar = c = 1$, $\tilde{\alpha}_s = \tilde{\alpha}'_s = \alpha_s = 0.2$. The curves on Fig. 1 correspond importances of g' : the solid curve corresponds to $g' = 1$, the dashed and the dot-dashed curves correspond to $g' = 1.5$ and $g' = 2.5$; the dotted line answers to importance $\alpha_s = 0$.

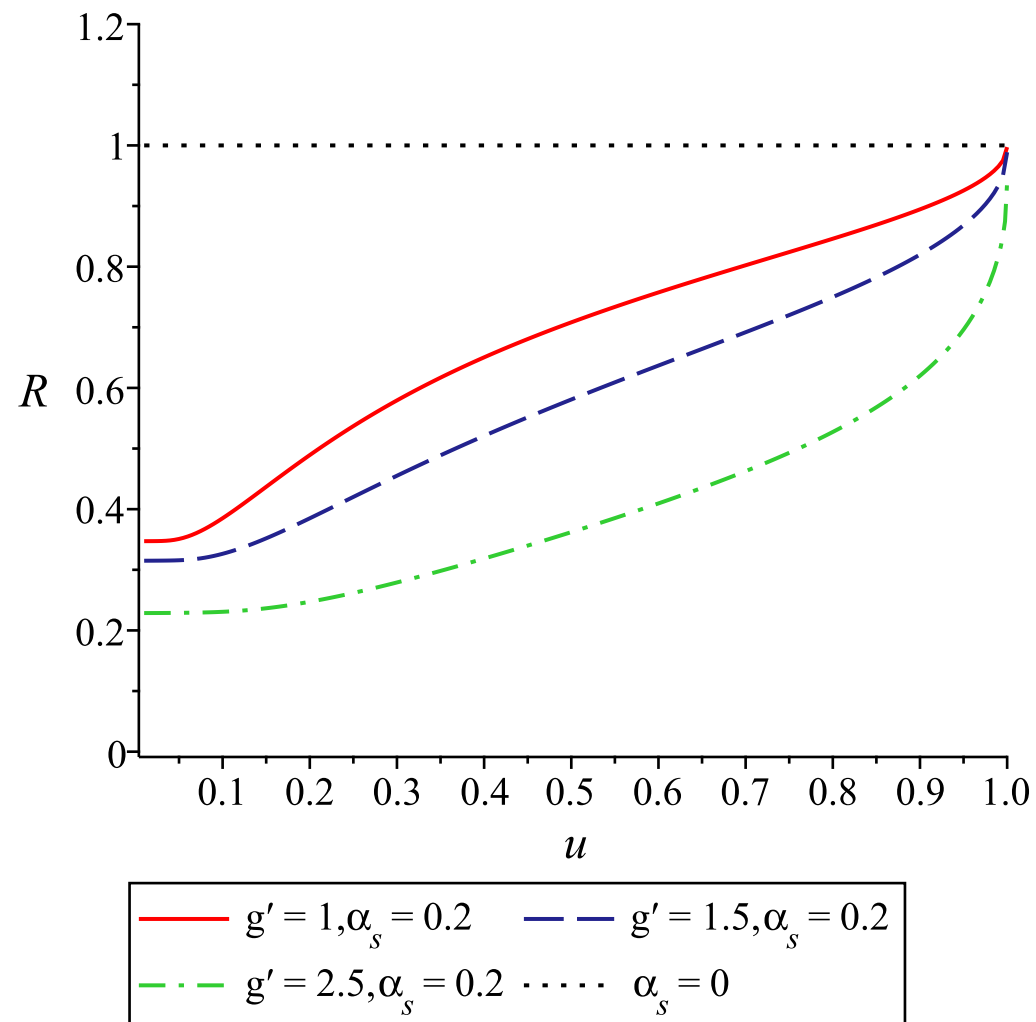


Fig. 1

From Fig. 1 we see that at $u \ll 1$ the spin and masses of quarks (the factor g'), which form the spinor parameters a' and b' of vector mesons, affects substantially the behavior of $R = R(u)$ and, hence, on the behavior of the leptonic decay widths of vector mesons. When the velocity u grows, the influence of the spinor parameters a' and b' of vector mesons on the behavior of $R = R(u)$ becomes weaker, and $R \rightarrow 1, u \rightarrow 1$, that is, this influence disappears. Also, Fig. 1 shows that the behavior of $R = R(u)$ depends on g' in a broad region of importance of u , and this dependence becomes weaker in the relativistic region of importance of the velocity $u : R \rightarrow 1, u \rightarrow 1$.

Such the behavior of function R is connected with difference in the behavior of spinless S -factor (7) and its the spin of analogue (34), which come up for Eqs. (2) and (33) for the relativistic leptonic decay widths of the s -state of vector mesons. In considered RQP-approach the S -factor (34) reproduces as the well-known nonrelativistic result in the nonrelativistic limit ($v \rightarrow 0$) in the spinless case, so and expected the relativistic ($v \rightarrow 1$) and ultrarelativistic ($m \rightarrow 0$) limits, when $m_1 = m_2 = m$, $a' = a = 0$ and $b' = b = 2/mc^2$. In the relativistic limit of $\chi' \rightarrow +\infty$ ($u \rightarrow 1$) the asymptotic behavior of S -factor (34) has the form

$$\begin{aligned}
S_{\text{RQP},S}(\chi')|_{\chi' \gg 1} &\approx \frac{2\pi(B' - \tilde{\rho}')}{1 - \exp[-2\pi(B' - \tilde{\rho}')] }|_{\chi' \gg 1} \approx \\
&\approx 1 + \frac{\pi\tilde{\alpha}'_s}{4}(a' + 2b')e^{-\chi'} \xrightarrow{\chi' \rightarrow +\infty} \\
&\xrightarrow{\chi' \rightarrow +\infty} \begin{cases} 1 + 0 & \text{for } \hat{O} = \gamma_5 \text{ (pseudoscalar) and } 1 \leq g' \leq \sqrt{2}, \\ 1 - 0 & \text{for } \hat{O} = \gamma_5 \text{ (pseudoscalar) and } g' > \sqrt{2}, \\ 1 + 0 & \text{for } \hat{O} = \gamma_\mu \text{ (vector) and } 1 \leq g' \leq \sqrt{3}, \\ 1 - 0 & \text{for } \hat{O} = \gamma_\mu \text{ (vector) and } g' > \sqrt{3}; \\ 1 + 0 & \text{for } \hat{O} = \gamma_5 \gamma_\mu \text{ (pseudovector) and } g' \geq 1, \end{cases}
\end{aligned}$$

where for the vector mesons the spin parameters a' and b' are given in (10), which are the functions of parameter g' and, hence, of the quarks masses (the details in Refs. [Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 488 (2022); Yu. D. Chernichenko, L. P. Kaptari, O. P. Solovtsova, *Eur. Phys. J. Plus.* **136**, Art. 132 (2021)[ArXiv : 2012. 13128v1 (hep-ph)]]).

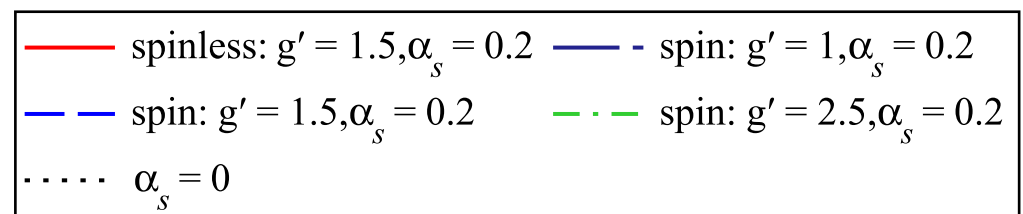
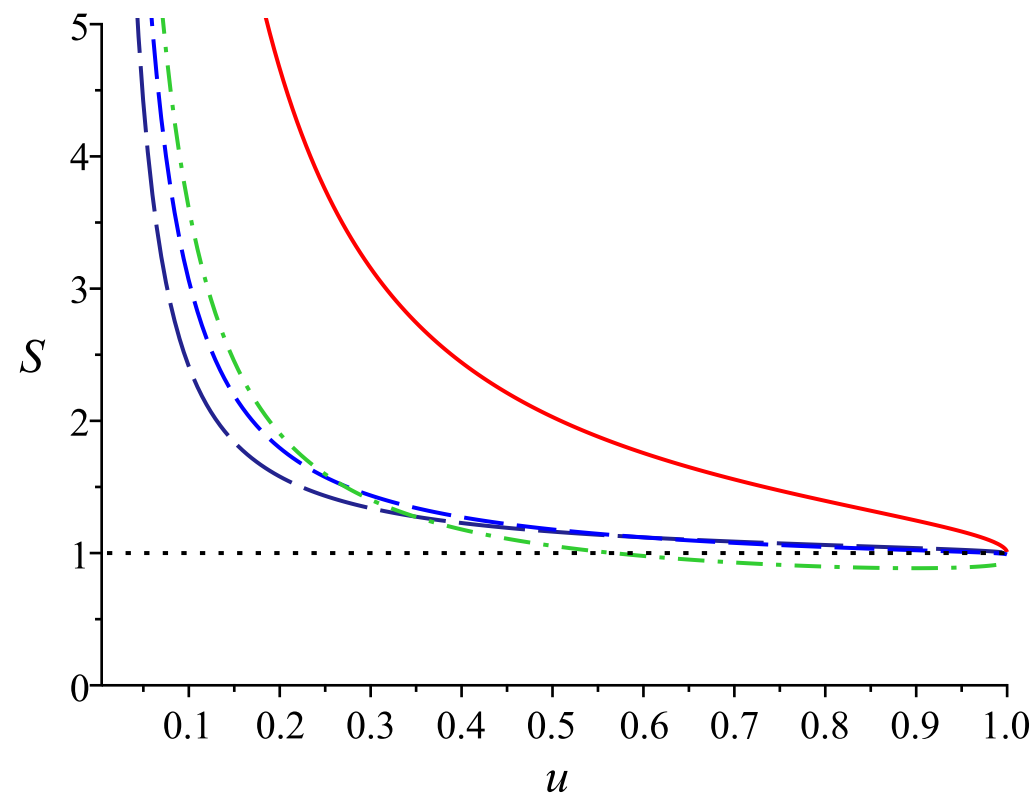


Fig. 2

As an example of the application of Eq. (33), on Fig. 3 we presented the behavior of the relativistic leptonic decay widths for the ρ_0 -, ω -, and ϕ -mesons in the s -wave state and level $n = 1$, as a functions of the rapidity $\chi' = \chi$. The quarks of vector mesons interact by means of the linear potential with Coulomb-like (chromodynamical) potential,

$$V(r) = \sigma r - \frac{\alpha_s}{r}.$$

The curves on Fig. 3 correspond the different importances of the parameters for ρ_0 -, ω -, and ϕ -mesons from the Tabl. 1.

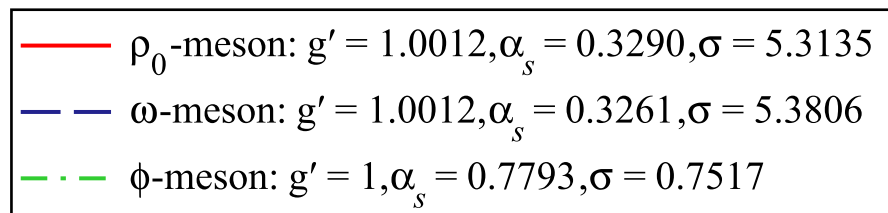
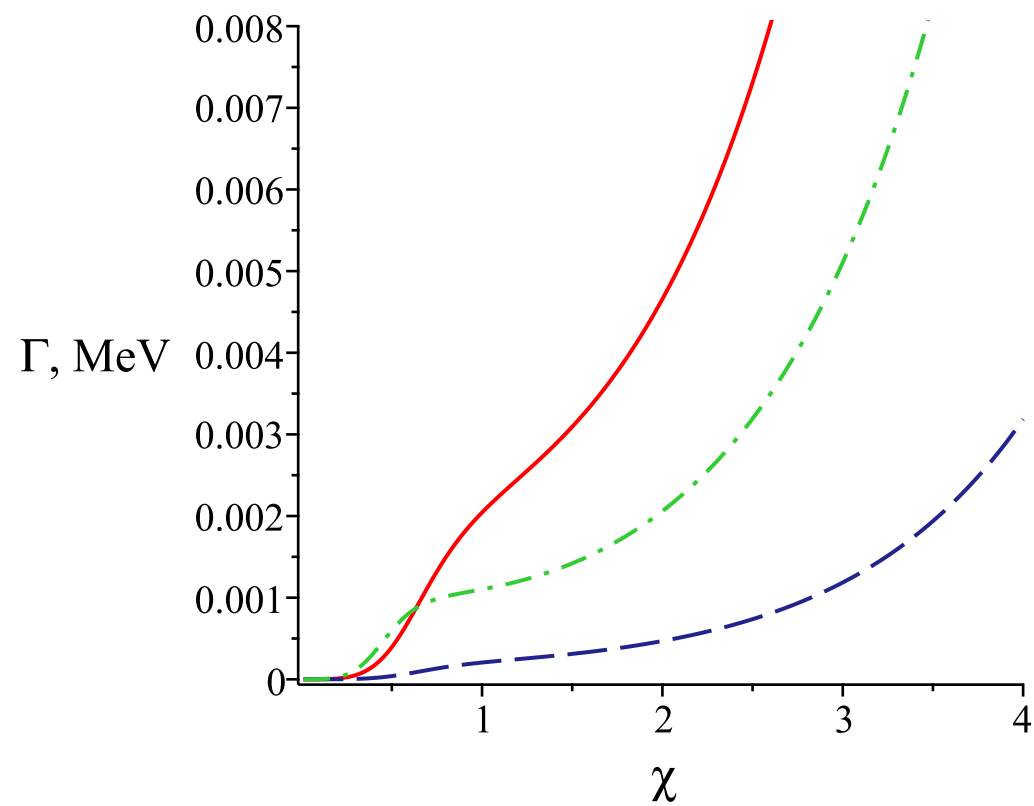


Fig. 3

Masses for the ρ_0 -, ω - and ϕ -mesons were chosen equals [K. A. Olive *et al.* (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014)]: $M_{\rho_0} = 775.26 \text{ MeV}$, $M_{\omega} = 782.65 \text{ MeV}$ and $M_{\phi} = 1019.461 \text{ MeV}$.

Importances of g' , which is defined by formula (3) through relations of m_u, m_d, m_s for the u -, d -, s -quarks, that form the ρ_0 -, ω - and ϕ -mesons, were chosen equals: $g'_{\rho_0} = g'_{\omega} = 1.0012$ either as in Ref. [Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 488 (2022)] for π^{\pm} -meson, $g'_{\phi} = 1$ because of equal masses.

Importances of m_u, m_d, m_s for the u -, d -, s -quarks were found in Ref. [Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 488 (2022)] at study of the spectrum of masses pseudoscalars of π^{\pm} -, K^{\pm} - and K_0 -mesons:
 $m_u = 62.57 \text{ MeV}$, $m_d = 69.00 \text{ MeV}$, $m_s = 262.29 \text{ MeV}$.

Importances of χ' , that corresponds importances of M, m_u, m_d, m_s and the factor g' for the ρ_0 -, ω - and ϕ -mesons, we find according of formulas (3) and (11) from expressions

$$M_{\rho_0} = 2m'_{\rho_0}g'_{\rho_0} \cosh \chi_{\rho_0}, M_{\omega} = 2m'_{\omega}g'_{\omega} \cosh \chi_{\omega}, M_{\phi} = 2m'_{\phi}g'_{\phi} \cosh \chi_{\phi}.$$

Importances of the multiplier Q_V , conditioned by isotopic structure of vector mesons V and charges of quarks expressed in unit of the electric charge e , for the ρ_0 -, ω - and ϕ -mesons were taken from Refs. [V. A. Matveev, B. V. Struminsky, A. N. Tavkhelidze, *Preprint JINR P-2524*, Dubna, 1965] or by Van Royen and Weisskopf [W. F. Weisskopf, R. Van Royen, *Nuovo Cim.* **50**, 617 (1967); **51**, 583 (1967)]: $Q_{\rho_0} = 1/\sqrt{2}, Q_{\omega} = 1/3\sqrt{2}, Q_{\phi} = -1/3$.

By using the quantization conditions of the energy levels for vector mesons [Yu. D. Chernichenko, *Phys. At. Nucl.* **85**, 488 (2022)]

$$\begin{aligned} & \frac{4}{a' \tilde{\sigma}'} \left[\frac{\cosh \chi'}{\sqrt{b'/a'} \sqrt{1 + b'/a'}} \operatorname{arctanh} \left(\frac{\tanh \chi' \sqrt{b'/a'}}{\sqrt{1 + b'/a'}} \right) - \right. \\ & \quad \left. - \frac{1}{\sqrt{1 + b'/a'}} \operatorname{arctan} \left(\frac{\sinh \chi'}{\sqrt{1 + b'/a'}} \right) \right] = \\ & = \pi \left(n + \frac{\ell}{2} + \frac{3}{4} \right) - \delta_{\ell}^{\text{Coul, WKB, } S}(\chi'), \quad n = 0, 1, \dots, \ell \geq 0, \end{aligned}$$

we calculated the importances of $\tilde{\sigma}' = \sigma$ and $\tilde{\alpha}'_s = \alpha_s$ of interaction constants and value of $dM_n/dn = 2m'g' \sinh \chi'_n d\chi'_n/dn$ for the ρ_0 -, ω - and ϕ -mesons in s -state and level $n = 1$ across importances of g' and χ' from [Tabl. 1](#), where the phase of the Coulomb radial RQP wave function in the WKB approximation, $\delta_{\ell}^{\text{Coul, WKB, } S}(\chi')$ at $\ell = 0$, gives by [Eq. \(32\)](#) taken in the turning point $\rho_+ = 4(\cosh \chi' - 1)/\tilde{\sigma}'(a' + b')$.

Importances of χ_Δ for form factor $f_1(t)$ in (22) for the ρ_0 -, ω - and ϕ -mesons were chosen equals: $\chi_{\Delta_{\rho_0}} = 3.2042$, $\chi_{\Delta_\omega} = 3.0081$ and $\chi_{\Delta_\phi} = 2.4054$. This importances of χ_Δ correspond to importances of the relativistic leptonic decay widths for the ρ_0 -, ω -, and ϕ -mesons in the s -wave state and level $n = 1$ from Tabl. 1 [K. A. Olive *et al.* (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014)].

Table 1: Importances of leptonic decay widths and parameters for the ρ_0 -, ω - and ϕ -mesons

Mesons	M , MeV	m_u , MeV	m_d , MeV	m_s , MeV	m' , MeV
ρ_0	775.26	62.57	69.00		65.71
ω	782.65	62.57	69.00		65.71
ϕ	1019.461			262.29	262.29

Mesons	g'	χ'	Q_V	σ	α_s
ρ_0	1.0012	2.4595	$1/\sqrt{2}$	5.3135	0.3290
ω	1.0012	2.4691	$1/3\sqrt{2}$	5.3806	0.3261
ϕ	1	1.2836	$-1/3$	0.7517	0.7793

Mesons	χ_Δ	$-t, \text{ GeV}^2$	u	$\Gamma_{\text{theor}}, \text{ keV}$	$\Gamma_{\text{exp}}, \text{ keV}$
ρ_0	3.2042	13.63	0.9855	7.04	7.04 ± 0.06
ω	3.0081	11.21	0.9858	0.60	0.60 ± 0.02
ϕ	2.4054	9.53	0.8574	1.251	1.251 ± 0.021

From [Tabl. 1](#) is seen that quarks, that form the ρ_0 -, ω - and ϕ -mesons in s -state and level $n = 1$, are the relativistic ($u > 0.85$).

4. Conclusions

In the present study, the new relativistic expression for the leptonic decay widths of vector mesons in *s*-wave state have been obtained on the basis of the RQP approach in the relativistic semiclassical approximation. The present analysis has been performed for the case where relativistic spin quarks of arbitrary-masses that form vector mesons interact via a funnel-like potential including a purely confining part, which is not singular, and a singular part in the form of a Coulomb-like chromodynamical potential.

For this aim the fully covariant finite-difference RQP equation in the three-dimensional relativistic \mathbf{r} representation for the case of interaction between two relativistic spin particles of arbitrary-masses has been solved by the relativistic WKB method. The condition of applicability of the WKB approximation has been established.

It has been shown that, at $a' = 0$ and $b' = 2/g'm'c^2$, the new expression for the leptonic decay widths of vector mesons reduces to its relativistic spinless analog.

The comparison of the new expression with its relativistic spinless analogue is executed.

The influence of the spin and masses of quarks on the behavior of leptonic decay widths of vector mesons in the s -wave state has been explored.

This the influence of the spin and masses of quarks is essential in the region of small values of the velocity u (it the so-called Sommerfeld effect); when the velocity u grows, its influence becomes weaker, and in the relativistic limit, $u \rightarrow 1$, its the influence disappears.

Importances of the leptonic decay widths for the ρ_0 -, ω -, and ϕ -mesons in the s -wave state and level $n = 1$ were culculated.

Since the new relativistic expression for the relativistic semiclassical leptonic decay widths of vector mesons has been obtained within a fully covariant method and has a correct connection with the Bethe-Salpeter function, one can expect that this expression takes into account more adequately both the relativistic character of interacting particles and their spin and masses.

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