Functional solutions of stochastic problems

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Outline

Iterative solution of kinetic equation

Functional solution of kinetic equation

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Hydrodynamic kinetic equations

Evolution equations for macroscopic quantities: time-dependent Ginzburg-Landau (TDGL) equation for the order parameter φ

$$\frac{\partial\varphi}{\partial t} = -\gamma\,\left[-2g\nabla^2\varphi + 2a(T-T_c)\varphi + 4B\varphi^3 - h\right]\,.$$

Diffusion-limited rate equation of the reaction $A + A \to A$ for the concentration c (k is *rate constant*).

$$\frac{\partial\varphi}{\partial t} = D\nabla^2\varphi - k\varphi^2\,.$$

Passive scalar problem (θ concentration, temperature etc.)

$$\frac{\partial\theta}{\partial t} + \nabla\mathbf{v}\theta = D\nabla^2\theta\,.$$

All subject to fluctuations: stochastic problems follow.

Iterative solution of a nonlinear parabolic equation

Starting point of iteration is the linear TDGL equation

$$\frac{\partial\varphi^{(0)}}{\partial t} = -\gamma\,\left[-2g\nabla^2\varphi^{(0)} + 2a(T-T_c)\varphi^{(0)} - h\right]\,.$$

Solution is given by the *propagator* Δ

$$\varphi^{(0)}(t,\mathbf{x}) = \int d\mathbf{x}'\int dt'\,\Delta(t-t',\mathbf{x}-\mathbf{x}')\gamma h(t',\mathbf{x}')\,,$$

which is the Green function of the differential operator of TDGL

$$\left[\frac{\partial}{\partial t} - 2\gamma g\nabla^2 + 2a(T-T_c)\right]\Delta(t-t',\mathbf{x}-\mathbf{x}') = \delta_+(t-t')\delta(\mathbf{x}-\mathbf{x}')\,.$$

Iterative solution

Cast the TDGL equation into an integral equation

$$\varphi(t,\mathbf{x}) = \int d\mathbf{x}' \int dt' \Delta(t-t', \mathbf{x}-\mathbf{x}') \left[\gamma h(t',\mathbf{x}') - 4\gamma B \varphi^3(t',\mathbf{x}')\right] .$$

First order: put the zeroth-order (linear) solution in the right side to obtain

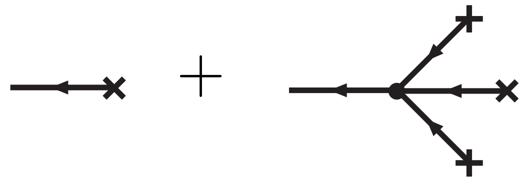
$$\varphi^{(1)}(t,\mathbf{x}) = -4\gamma B \int d\mathbf{x}' \int dt' \Delta(t-t', \mathbf{x}-\mathbf{x}') \left[\varphi^{(0)}(t',\mathbf{x}')\right]^3 ,$$

which then is substituted in the right side to obtain the second-order contribution etc.

Tree-graph solution

Solution in a graphical form: two leading terms

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \ldots =$$



$$+ \ldots$$

Directed line – Δ, the cross – γh, full dot – vertex factor.

The graphical solution consists of connected *tree graphs* only.

The propagator Δ is the free-field *response function*:

$$\chi(t - t', \mathbf{x} - \mathbf{x}') = \left. \frac{\delta \varphi(t, \mathbf{x})}{\delta h(t', \mathbf{x}')} \right|_{h=0} = \gamma \Delta(t - t', \mathbf{x} - \mathbf{x}') \, .$$

Functional solution of parabolic nonlinear equation

Let $\varphi[\tilde{A}]$ be solution of the generic kinetic equation

$$\frac{\partial\varphi}{\partial t} = -K\varphi + U(\varphi) + B(\varphi)\tilde{A}\,.$$

Generating function of solutions (only time integral explicit)

$$G(A) = e^{\int dt\, A\varphi[\tilde{A}]}\,,$$

Use functional δ function to introduce integral representation

$$G(A) = e^{\int dt\, A\varphi[\tilde{A}]} = \int \mathcal{D}\varphi\,\delta\left(\varphi - \varphi[\tilde{A}]\right)\, e^{\int dt\, A\varphi}$$

$$= \int \mathcal{D}\varphi\,\delta\left[-\partial_t\varphi - K\varphi + U(\varphi) + B(\varphi)\tilde{A}\right] |\,\det M|\, e^{\int dt\, A\varphi}\,.$$

Functional Jacobi determinant

Loop expansion of $\det M = e^{\mathrm{Tr}\ln M}$ yields the representation

$$\det M = \det\left(\partial_t + K - U' - B'\tilde{A}\right) = \det\left(\partial_t + K\right) e^{-\int dt\,\Delta(0)\left(U' + B'\tilde{A}\right)}\,.$$

Here, the shorthand notation stands for

$$\int dt\,\Delta(0)\left(U' + B'\tilde{A}\right) = \int dt \int d\mathbf{x} \int d\mathbf{x}'\Delta(t,\mathbf{x};t,\mathbf{x}') \times \int du \left[\frac{\delta U(\mathbf{x}',\varphi(t))}{\delta\varphi(u,\mathbf{x})} + \tilde{A}(t,\mathbf{x}')\frac{\delta B(\mathbf{x}',\varphi(t))}{\delta\varphi(u,\mathbf{x})}\right]\,.$$

Diagonal value of the propagator (response function of φ)
$\Delta(0) := \Delta(t,\mathbf{x};t,\mathbf{x}')$ remains an explicit free parameter.

Note that all this does not require random $\tilde{A}$. Contrary to widespread view, this ambiguity has nothing to do with the white-noise problem.

Ambiguous functional integral

The resulting (formal) functional integral

$$G(A)=\int\mathcal{D}\varphi\int\mathcal{D}\tilde{\varphi}\left|\det\left(\partial_t+K\right)e^{-\int dt\,\Delta(0)\left[U'+B'\tilde{A}\right]}\right|e^{\int dt\,\tilde{\varphi}\left[-\partial_t\varphi+V+B\tilde{A}\right]}\,e^{\int dt\,A\varphi}\,.$$

defines the *dynamic action*, ambiguous in a deterministic problem already

$$S=\int dt\,\left\{\tilde{\varphi}\left[-\partial_t\varphi-K\varphi+U(\varphi)+B(\varphi)\tilde{A}\right]-\Delta(0)\left[U'(\varphi)+B'(\varphi)\tilde{A}\right]\right\}\,.$$

The corresponding *S-matrix functional* contains well-defined quantities:

$$G(A)=\exp\left(\int dt\,\int dt'\,\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\Big[\int dt\,\tilde{\varphi}\left(U+B\tilde{A}\right)$$
$$-\int dt\,\Delta(0)\left(U'+B'\tilde{A}\right)+\int dt\,\tilde{\varphi}\tilde{A}+\int dt\,A\varphi\Big]\Big|_{\tilde{\varphi}=\varphi=0}\,.$$

Perturbation expansion for S-matrix functional

Cast the functional integral for S-matrix functional $H(\phi)$ into Gaussian integral with the aid of the shift operator

$$\left(\det\frac{K}{2\pi}\right)^{\frac{-1}{2}} H(\phi) = \int \mathcal{D}\varphi \,\exp\int dt \,\Big[-\frac{1}{2}\varphi K\varphi + (\varphi+\phi)\,A$$
$$+\sum_{n=3}^{\infty}\frac{1}{n!}\,v_n\,(\varphi+\phi)^n\Big] = \int \mathcal{D}\varphi \,\exp\int dt\,\left(-\frac{1}{2}\varphi K\varphi + \varphi\frac{\delta}{\delta\phi}\right)\,\exp\int dt\,V(\phi)\,.$$

Calculation of the Gaussian integral yields the relation $(K\Delta = 1)$

$$H(\varphi) = \exp\left(\frac{1}{2}\int dt\,\int dt'\,\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}\right)\exp\int dt\,V(\varphi)$$

defining the functional integral as perturbation expansion with propagator Δ.

Perturbation expansion is unambiguous

Independence of $\Delta(0)$ is revealed by the loop theorem

$$\exp\left(\iint dt dt' \, \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\tilde\varphi}\right)\prod_{n=1}^{N} F_n(\varphi,\tilde\varphi) = \exp\left(\iint dt dt' \, \frac{\delta}{\delta\varphi}\Delta'\frac{\delta}{\delta\tilde\varphi}\right)\prod_{n=1}^{N} F'_n(\varphi,\tilde\varphi)\,.$$

Reduction operator $\exp\left(\int dt \; \int dt' \, \frac{\delta}{\delta\varphi}\Delta'\frac{\delta}{\delta\tilde\varphi}\right)$ spans lines only between different functionals in the *normal form* $F'(\varphi,\tilde\varphi)$ (diagonal terms are summed up)

$$F'(\varphi,\tilde\varphi) \equiv \exp\left(\iint dt dt' \, \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\tilde\varphi}\right) F(\varphi,\tilde\varphi)\,.$$

Generating function in the normal form does not contain $\Delta(0)$:

$$G(A) = \exp\left(\iint dt dt' \, \frac{\delta}{\delta\varphi}\Delta'\frac{\delta}{\delta\tilde\varphi}\right)\exp\left\{\tilde\varphi\left[U(\varphi)+B(\varphi)\tilde A\right]+\tilde\varphi\tilde A + A\varphi\right\}\Big|_{\tilde\varphi=\varphi=0}\,.$$

Solution of discrete nonlinear Volterra equation

Discrete Volterra equation for $\partial_t\varphi = V(\varphi)$ $(0 \leq \vartheta \leq 1)$:

$$\varphi_N = \varphi_0 + \sum_{i=1}^{N} V\left(\vartheta\varphi_i + (1-\vartheta)\varphi_{i-1}\right)(t_i - t_{i-1}) \ , \quad \varphi_i \equiv \varphi(t_i) \, .$$

Form of integral sum in the form due Stratonovich (from stochastic integral). Valid $\forall N$, replace by the equivalent system

$$\varphi_i = \varphi_{i-1} + V\left(\vartheta\varphi_i + (1-\vartheta)\varphi_{i-1}\right)(t_i - t_{i-1}) \ , \quad i = 1, \ldots, N \, .$$

Repeat continuum case: δ functions for these equations bring about the Jacobi matrix ($\overline{\varphi}_i = \vartheta\varphi_i + (1-\vartheta)\varphi_{i-1}$)

$$M_{ij} = \delta_{i\,j} - \delta_{i\,j-1} - (\vartheta\delta_{i\,j} + (1-\vartheta)\delta_{i\,j-1})V'(\overline{\varphi}_i)\,(t_i - t_{i-1}) \ .$$

Ambiguity comes from the definition of the integral!

Jacobi determinant of the Volterra equation

Determinant of triangular Jacobi matrix is product of diagonal terms and in continuum limit yields the closed-loop contribution of the functional Jacobi determinant

$$\det M = \prod_{i=1}^{N} [1 - \vartheta V'(\overline{\varphi}_i)\,(t_i - t_{i-1})] \xrightarrow{N\to\infty} \exp\left[-\int dt\,\vartheta V'(\varphi)\right] ,$$

since in the limit $N \to \infty$, $t_i - t_{i-1} \sim 1/N \to 0\ \forall i$. Ambiguity comes not from the propagator, but the integral sum! Fixing rules of integration is called for.

Ambiguity has nothing to do with randomness, this is a deterministic problem!

Putting $\Delta(0) = 0$ is misleading (yields degenerate propagator). But nothing prohibits the choice $\vartheta = 0$ in the integral.

Intermediate results

Generating function of the deterministic evolution equation

$$\frac{\partial \varphi}{\partial t} = V(\varphi) = -K\varphi + U(\varphi) + B(\varphi)\tilde{A}$$

is expressed as the functional integral

$$G(A) = \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi} \, \exp \int dt \left\{ \tilde{\varphi} \left[-\partial_t \varphi - K\varphi + U(\varphi) \right] - \vartheta \left[-K + U'(\varphi) + B'(\varphi)\tilde{A} \right] + A\varphi \right\},$$

with the integral sum of the form ($\overline{\varphi}_i = \vartheta\varphi_i + (1-\vartheta)\varphi_{i-1}$, $0 \le \vartheta \le 1$)

$$\sum_{i=1}^{N} \tilde{\varphi}_i \left\{ -\varphi_i + \varphi_{i-1} + \left[V\left(\overline{\varphi}_i\right) + \vartheta V'\left(\overline{\varphi}_i\right) \right] (t_i - t_{i-1}) \right\} .$$

Random processes

Random process (field) described by joint PDFs $p\left(\varphi_1, t_1; \varphi_2, t_2; \ldots; \varphi_n, t_n\right)$.

Markov process: no memory. All PDFs expressed in terms of $p(t_1, \varphi_1)$ and $p\left(\varphi_1, t_1 | \varphi_2, t_2\right)$ (conditional PDF):

$$
\begin{aligned}
p\left(\varphi_1, t_1; \varphi_2, t_2; \ldots; \varphi_n, t_n\right) &= p\left(\varphi_n, t_n | \varphi_{n-1}, t_{n-1}\right) \\
&\times p\left(\varphi_{n-1}, t_{n-1} | \varphi_{n-2}, t_{n-2}\right) \cdots p\left(\varphi_2, t_2 | \varphi_1, t_1\right) p\left(\varphi_1, t_1\right)
\end{aligned}
$$

provided $t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_{n-1} \leq t_n$.

Wiener process is the basis of the interpolation construction of functional integral.Its conditional probability density is

$$
p\left(W_2, t_2 | W_1, t_1\right) = \frac{1}{[4\pi(t_2 - t_1)]^{1/2}} \, e^{-(W_2 - W_1)^2/2(t_2 - t_1)} .
$$

Kinetic equation with white noise (Wiener process)

In SDE the external field is a representative of a Wiener process

$$\varphi(t) = \varphi(0) + \int_0^t \left[-K\varphi + U(\varphi)\right] du + \int_0^t B(\varphi) dW \,,$$

defining the measure in the stochastic integral (last term of right side). With the reference point at the right of the elementary interval ($\vartheta = 0$) this is the famous stochastic integral of Itô

$$\sum_{i=1}^{N} B\left(\varphi_{i-1}\right)\left(W_i - W_{i-1}\right) \,.$$

With the midpoint reference ($\vartheta = 1/2$) we arrive at the Stratonovich integral. Both are widely used.

Jacobi determinant due to Wiener process

Jacobi determinant now contains representatives of the Wiener process

$$\det M(W) = \prod_{i=1}^{N} \left\{1 + \vartheta \left[K - U'(\overline{\varphi}_i)\right](t_i - t_{i-1}) - \vartheta B'(\overline{\varphi}_i)\,(W_i - W_{i-1})\right\} \ ,$$

present in the exponential of the generating function as well:

$$\exp \sum_{i=1}^{N} \tilde{\varphi}_i B\left(\overline{\varphi}_i\right)\left(W_i - W_{i-1}\right) \ .$$

Expectation value wrt Wiener process readily calculated with the use the characteristic function (Pourier transform of the PDF) satisfying

$$\langle \exp\left[ip(W_2 - W_1)\right]\rangle = \exp\left[-\frac{1}{2}(t_2 - t_1)p^2\right] \ , \qquad t_2 > t_1 \, .$$

Characteristic function of the Langevin equation

Expectation value of the factor containing the Wiener process in the integrand of the generating function ($\overline{\varphi}_i = \vartheta\varphi_i + (1-\vartheta)\varphi_{i-1}$, $0 \le \vartheta \le 1$)

$$\left\langle \det M(W) \exp\left[\sum_{i=1}^{N} \tilde{\varphi}_i B\left(\overline{\varphi}_i\right)\left(W_i - W_{i-1}\right)\right]\right\rangle$$

$$= \prod_{i=1}^{N} \left\{1 + \vartheta\left[K - U'(\overline{\varphi}_i)\right](t_i - t_{i-1}) - \vartheta\tilde{\varphi}B(\overline{\varphi}_i)B'(\overline{\varphi}_i)(t_i - t_{i-1})\right\}$$

$$\times \exp\frac{1}{2}\sum_{i=1}^{N}\left[\tilde{\varphi}_i B\left(\overline{\varphi}_i\right)\right]^2 (t_i - t_{i-1})$$

$$\approx \exp\sum_{i=1}^{N}\left\{\frac{1}{2}\left[\tilde{\varphi}_i B\left(\overline{\varphi}_i\right)\right]^2 + \vartheta\left[K - U'(\overline{\varphi}_i) - \tilde{\varphi}B(\overline{\varphi}_i)B'(\overline{\varphi}_i)\right]\right\}(t_i - t_{i-1})\ .$$

De Dominicis-Janssen action of Langevin equation

Generating function of the Langevin equation (Wiener process)

$$d\varphi = [-K\varphi + U(\varphi)]\, dt + B(\varphi)dW$$

is expressed as the functional integral

$$G(A) = \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi}\, \exp \int dt \left\{ \frac{1}{2} \left[\tilde{\varphi} B\left(\varphi\right)\right]^2 + \tilde{\varphi}\left[-\partial_t \varphi - K\varphi + U\left(\varphi\right)\right] \right.$$
$$\left. - \vartheta\left[-K + U'(\varphi) + \tilde{\varphi} B(\varphi) B'(\varphi)\right] + A\varphi \right\}.$$

Ambiguity remains even with normal form: Itô interpretation of SDE $\vartheta \to 0$; Stratonovich $\vartheta \to \frac{1}{2}$ (upon that, no explicit closed loops in PT)

$$S' = \int dt \left\{ \frac{1}{2} \left[\tilde{\varphi} B\left(\varphi\right)\right]^2 + \tilde{\varphi}\left[-\partial_t \varphi - K\varphi + U\left(\varphi\right)\right] + \vartheta\, \tilde{\varphi} B(\varphi) B'(\varphi) \right\}$$

Langevin equation with coloured noise

Does appear. White-noise limit interesting as such.

Now a smooth random field f in the Jacobi determinant

$$\det M(W) = \prod_{i=1}^{N} \left\{1 + \vartheta \left[K - U'(\overline{\varphi}_i) - B'(\overline{\varphi}_i) f_i\right](t_i - t_{i-1})\right\} \ ,$$

and in exponential of the generating function $\exp \sum_{i=1}^{N} \tilde{\varphi}_i B\left(\overline{\varphi}_i\right) f_i \left(t_i - t_{i-1}\right)$ .Characteristic function of coloured noise (C_{mn} – correlation matrix)

$$\left\langle \exp\left[i \sum_m p_m f_m\right]\right\rangle = \exp\left[-\frac{1}{2}\sum_{m,n} p_m C_{mn} p_n\right] \ .$$

Coloured noise nonlocal

Coloured noise: two-fold integrals. Average Jacobian to second order in time increments to collect relevant terms ($U'(\overline{\varphi}_i) \to U'_i$, $t_i - t_{i-1} \to \Delta t_i$)

$$\prod_{i=1}^{N} \{1 + \vartheta\,[K - U'_i - B'_i f_i]\,\Delta t_i\} = 1 + \sum_{i=1}^{N} \vartheta\,[K - U'_i - B'_i f_i]\,\Delta t_i$$

$$+ \frac{1}{2} \sum_{i \neq j = 1}^{N} \vartheta^2\,[K - U'_i - B'_i f_i]\left[K - U'_j - B'_j) f_j\right] \Delta t_i \Delta t_j + O(\Delta t^3)\,.$$

$$\left\langle \det M(W) \exp \sum_{i=1}^{N} \tilde{\varphi}_i B_i f_i \Delta t_i \right\rangle \approx \exp \left\{ \sum_{i=1}^{N} \vartheta\,[K - U'_i]\,\Delta t_i \right.$$

$$\left. + \sum_{i,j=1}^{N} \left[\frac{1}{2} \tilde{\varphi}_i B_i C_{ij} \tilde{\varphi}_j B_j - \vartheta \tilde{\varphi}_i B_i C_{ij} B'_j + \frac{1}{2} \vartheta^2 B'_i C_{ij} B'_j \right] \Delta t_i \Delta t_j \right\}.$$

Coloured noise: De Dominicis-Janssen action

Generating function of the Langevin equation (coloured noise)

$$d\varphi = [-K\varphi + U(\varphi)]\, dt + B(\varphi) f\, dt$$

is expressed as the functional integral ($B(\varphi(t)) \to B_t$ etc.)

$$G(A) = \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi}\, \exp \int dt \left\{ \tilde{\varphi}\, [-\partial_t \varphi - K\varphi + U] - \vartheta\, [-K + U'] + A\varphi \right\}$$
$$+ \int dt \int dt' \left\{ \frac{1}{2} \tilde{\varphi}_t B_t C(t,t') \tilde{\varphi}_t B_{t'} - \vartheta \tilde{\varphi}_t B_t C(t,t') B'_{t'} + \frac{1}{2} \vartheta^2 B'_t C(t,t') B'_{t'} \right\}.$$

Coloured noise: perturbation theory unambiguous

Normal-form action independent of $\vartheta = \Delta_{12}(0)$ (but contains explicit propagator)

$$S' = \int dt \left\{ \tilde{\varphi} \left[-\partial_t \varphi - K\varphi + U \right] + A\varphi \right\}$$
$$+ \int dt \int dt' \left\{ \frac{1}{2} \tilde{\varphi}_t B_t C(t,t') \tilde{\varphi}_t B_{t'} + \tilde{\varphi}_t B'_t \Delta_{12}(t,t') C(t,t') B_{t'} \right\} .$$

White-noise limit unambiguous and yields Stratonovich form

$$C(t,t') \to \delta(t-t') \,, \quad \Delta_{12}(t,t') C(t,t') \to \frac{1}{2} \delta(t-t') \,,$$

Perturbation theory relies on normal form, interpolation formulas for functional integral are trickier (apart from the Itô case).