Introduction to the Functional Renormalization Group

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Advanced Methods of Modern Theoretical Physics:
Integrable and Stochastic Systems

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β-functions of O(N) model

LPA equation

$$\partial_t U_k = \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k \left[\frac{\delta^2 U_k}{\delta\phi_i\delta\phi_i} + R_k\right]^{-1}\right]$$

$O(N)$ model in the vicinity of transition

$$\phi = \left(\phi, \quad 0, \quad \dots\;, \quad 0\right)$$

$$\frac{\delta^2 U_k}{\delta\phi_i\delta\phi_i} = \frac{\delta U_k}{\delta\rho}\delta_{ij} + \frac{\delta^2 U_k}{\delta\rho^2}\phi_i\phi_j$$

Ansatz for potential

$$U_k(\phi) = m_k^2\phi^2 + \frac{\lambda_k}{4!}(\phi^2)^2$$

$$\partial_t U_k = \frac{1}{2}\int d^d q\,\partial_t R_k \left[\frac{1}{q^2 + R_k(q) + U_k' + 2\rho U_k''} + \frac{N-1}{q^2 + R_k(q) + U_k'}\right]$$

Expanding RHS in the powers of field and equating coefficients we obtain β functions

$$\partial_t \tilde{m}_t = -(d-2) + l^m_{d,\,N}(\tilde{m}_t, \tilde{\lambda}_t)$$

$$\partial_t \tilde{\lambda}_t = -(d-4) + l^\lambda_{d,\,N}(\tilde{m}_t, \tilde{\lambda}_t)$$

Optimisation and PMS

$$R_k(q) = \alpha\, Z_k\, k^2 e^{-q^2/k^2}$$

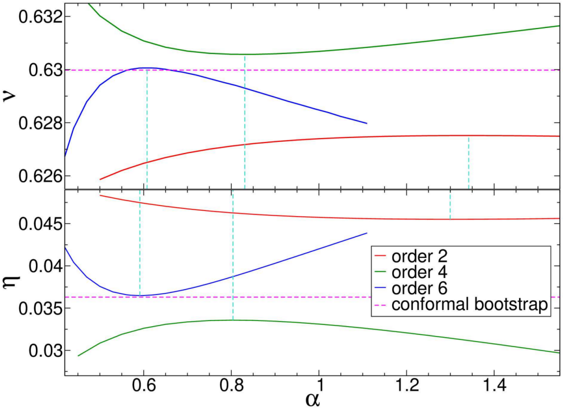

Figure 1: Picture from Balog et. al arXiv:1907.01829

The gradient expansion

Advantages

- ✓ Simplicity
- ✓ Captures all relevant IR physics even in the singular critical regime

Problems

- ✓ Gives no access to the momentum sector above $\sqrt{m_k^2 + k^2}$
- ✓ At criticality captures *only* the most IR physics

The vertex functions

The Wetterich equation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \operatorname{Tr}\left[\partial_t R_k \left[\Gamma^{(2)} + R_k\right]^{-1}\right]$$

The vertex functions

$$\Gamma_k[\phi] = \sum_n \frac{1}{n!} \int d^d x_1 \ldots d^d x_n \; \Gamma_k^{(n)}(x_1, \ldots, x_n) \phi(x_1) \ldots \phi(x_n)$$

$$\Gamma_k^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n \Gamma_k[\phi]}{\delta\phi(x_1) \ldots \delta\phi(x_n)}\bigg|_{\phi=0}$$

To obtain the flow of the vertex functions we shall variate the Wetterich equation with respect to ϕ

The flow of vertex functions

Short hand notation

$$G_k(p) = \left[\Gamma^{(2)} + R_k\right]^{-1}(p)$$

The flow of the first vertex

$$\partial_t \Gamma_k^{(1)}(p) = -\frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q) G_k(q)\;\Gamma^{(3)}(p,q,-q)\;G_k(q)\right]$$

The flow of the second vertex

$$\partial_t \Gamma_k^{(2)}(p,-p) = \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q) G_k(q)\,\Gamma^{(3)}(p,q,-q-p)\;G_k(q+p)\,\Gamma^{(3)}(-p,q+p,-q)\;G_k(q)\right] -$$

$$-\frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q) G_k(q)\;\Gamma^{(4)}(p,q,-q,-p)\;G_k(q)\right]$$

In general the flow of the $\Gamma_k^{(n)}$ depends on the first $(n+2)$ vertex functions. So we have an infinite tower of equations

Loop momentum suppression

The factor $\partial_k R_k(q)$ suppresses contributions from the momentum $q^2 > k^2$. Hence the reasonable approximation is to neglect dependence on the loop momentum for any vertex except propagator $G_k(q)$. This approximation seems especially reasonable for large external momenta.

The approximated flow of the second vertex

$$\partial_t \Gamma_k^{(2)}(p,-p) = \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q) G_k(q)\ \Gamma^{(3)}(p,0,-p)\ G_k(q+p)\ \Gamma^{(3)}(-p,p,0)\ G_k(q)\right] -$$

$$-\frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q) G_k(q)\ \Gamma^{(4)}(p,0,0,-p)\ G_k(q)\right]$$

Background field formalism

By the definition

$$\Gamma_k^{(n+1)}(p_1, \ldots, p_n, q, \phi) = \frac{\delta \Gamma_k^{(n)}(p_1, \ldots, p_n, \phi)}{\delta \phi(q)}$$

It is intuitively clear that

$$\Gamma_k^{(n+1)}(p_1, \ldots, p_n, 0; \phi) = \frac{\delta \Gamma_k^{(n)}(p_1, \ldots, p_n; \phi)}{\delta \phi(0)} = \left. \frac{\partial \Gamma_k^{(n)}(p_1, \ldots, p_n; \phi)}{\partial \phi} \right|_{\phi = const}$$

Expand around arbitrary constant background ϕ_0

$$\Gamma_k[\phi] = \sum_n \frac{1}{n!} \int d^d x_1 \ldots d^d x_n \; \Gamma_k^{(n)}(x_1, \ldots, x_n; \phi_0) \left[\phi(x_1) - \phi_0\right] \ldots \left[\phi(x_n) - \phi_0\right]$$

Approximation for higher vertexes

$\Gamma_k[\phi]$ does not depend on ϕ_0, so that

$$\frac{\partial \Gamma_k[\phi]}{\partial \phi_0} = 0$$

Substituting here the expansion around ϕ_0 one gets

$$\int dy \, \Gamma_k^{(n+1)}(x_1, \ldots, x_n, y; \phi_0) = \frac{\partial \Gamma_k^{(n)}(x_1, \ldots, x_n; \phi_0)}{\partial \phi_0}$$

Performing Fourier transform at vanishing momentum we prove our statement, which allows us to enclose our infinite tower of flow equations[1]

[1] Blaizot, Méndez-Galain, Wschebor, Phys Let B,Vol. 632(4), 2006, pp. 571-578

BMW approximation for 2-point correlation function

$$\partial_t \Gamma_k^{(2)}(p,-p;\phi) = \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q)\Big(G_k(q)\Big)^2\ \Big(\partial_\phi \Gamma^{(2)}(p,-p;\phi)\Big)^2\ G_k(q+p)\right] -$$

$$-\frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k(q)\Big(G_k(q)\Big)^2\ \Big(\partial_\phi^2 \Gamma^{(2)}(p,-p;\phi)\Big)\right]$$

This equation allows us to track momentum dependence of inverse propagator at any scale. On the other hand, note that

$$\Gamma_k^{(2)}(0,0;\phi) = U_k'(\rho) + 2\rho U_k''(\rho)$$

while

$$\partial_{p^2}\Gamma_k^{(2)}(p,-p;\phi)\Big|_{p=0;\phi=0} = Z_k$$

So that LPA' type results are easily recovered

Criticality within BMW framework

$$\Gamma^{(2)}_\Lambda(p,-p;\phi) = p^2 + m^2_\Lambda + \frac{\lambda_\Lambda}{2}\phi^2$$

Tuning the value of m_Λ by shooting we can approach the critical regime

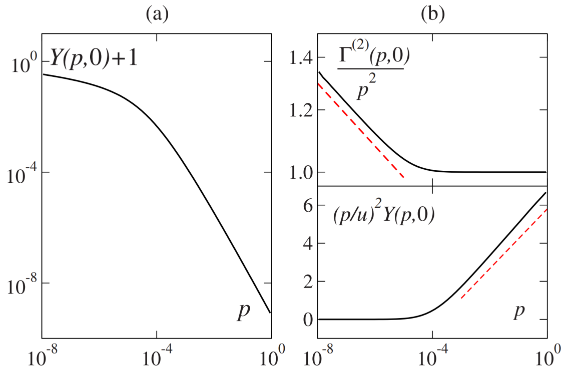

Figure 2: Picture from Phys. Rev. E 80, 030103R (2009)

Critical exponents

Exponent	BMW	Monte Carlo
η	0.039	0.0368(2)
ν	0.632	0.6302(1)
ω	0.78	0.821(5)

At the end of the day not only we calculated non-universal function but also improved estimates for universal exponents!

Stochastic problems

$$\partial_t\varphi(x) = U(x,\varphi) + f(x); \quad \langle f(x)f(x')\rangle = D(x,x')$$

Correlation functions

$$\langle\varphi(x_1)\ldots\varphi(x_n)\rangle$$

Response functions

$$\left\langle \frac{\delta^m\big[\varphi(x_1)\ldots\varphi(x_n)\big]}{\delta f(y_1)\ldots\delta f(y_m)} \right\rangle$$

Equivalent field theoretic model

$$S(\varphi,\varphi') = \frac{1}{2}\varphi' D\varphi' + \varphi'\big(-\partial_t + U(x,\varphi)\big)$$

Correlation functions

$$\langle\varphi(x_1)\ldots\varphi(x_n)\rangle$$

Response functions

$$\langle\varphi(x_1)\ldots\varphi(x_n)\varphi'(y_1)\ldots\varphi'(y_m)\rangle$$

Regulator

$$\Delta S_k(\varphi,\varphi') = \begin{pmatrix}\varphi \\ \varphi'\end{pmatrix} \begin{pmatrix} 0 & R_k^{(1,1)}(\omega,q^2) \\ R_k^{(1,1)}(-\omega,q^2) & R_k^{(0,2)}(\omega,q^2)\end{pmatrix} \begin{pmatrix}\varphi \\ \varphi'\end{pmatrix}$$

- ✓ $R_k^{(1,1)}$ introduces running correlation length and time in the model
- ✓ $R_k^{(0,2)}$ introduces modifies noise correlator

Almost always $\hat{R}_k$ can be chosen time independent. There are two (rather exotic) exceptions:

- ✓ There are frequency divergences in the model
- ✓ We expect that the radius of convergence of the DE expansion is smaller than the range of frequencies contributing to the flow

In those exceptional cases due to causality $R_k^{(1,1)}(t>0)=0$, for example

$$R_k^{(1,1)}(t,\mathbf{x}) = k^2\theta(-t)e^{t/k^2}r(\mathbf{x})$$

Diagonal terms

Often $R_k^{(0,2)}$ can be set to zero, except for the cases when its presence required by the symmetry of the model.

Fluctuation-Dissipation Theorem

$$\langle\varphi(x)\varphi'(y)\rangle - \langle\varphi(y)\varphi'(x)\rangle = -\partial_t\langle\varphi(x)\varphi(y)\rangle$$

Requires

$$R_k^{(0,2)}(\omega, q^2) = \frac{R_k^{(1,1)}(-\omega, q^2) - R_k^{(1,1)}(\omega, q^2)}{2i\omega}$$

The absence of $R_k^{(2,0)}$ together with Ito prescription guaranties that

$$\Gamma_k[\phi, \phi' = 0] = 0$$

The KPZ equation

The stochastic equation

$$\partial_t \varphi(x) = \nu \partial^2 \varphi + \frac{\lambda}{2}(\partial \varphi)^2 + f(x); \quad \langle f(x) f(x') \rangle = D\delta(t - t')\delta(\mathbf{x} - \mathbf{x}')$$

Corresponding field theory

$$S(\varphi, \varphi') = \frac{1}{2}\varphi' D \varphi' + \varphi'\big(- \partial_t \varphi(x) + \nu \partial^2 \varphi + \frac{\lambda}{2}(\partial \varphi)^2 \big)$$

The symmetries

1. $\varphi(t, \mathbf{x}) \to \varphi(t, \mathbf{x}) + c(t)$
2. $\varphi(t, \mathbf{x}) \to \mathbf{x} \cdot \mathbf{v} + \varphi(t, \mathbf{x} + t\lambda \mathbf{v})$
   $\varphi'(t, \mathbf{x}) \to \varphi'(t, \mathbf{x} + t\lambda \mathbf{v})$
3. for $d = 1$
   $\varphi(t, \mathbf{x}) \to -\varphi(-t, \mathbf{x})$
   $\varphi'(t, \mathbf{x}) \to \varphi(-t, \mathbf{x}) + \frac{\nu}{2D}\partial^2 \varphi(-t, \mathbf{x})$

The Ward Identity

Invariance of the functional measure

$$\int \mathcal{D}\Phi \; \delta_c \; e^{S(\Phi)+j\varphi+j'\varphi'} = 0$$

Calculating variation

$$\int \mathcal{D}\Phi \Big(c(t) j - \varphi' \partial_t c(t) \Big) e^{S(\Phi)+j\varphi+j'\varphi'} = 0$$

$$\Big(c(t) j - \frac{\delta}{\delta j'} \partial_t c(t) \Big) e^{W[j,j']} = 0$$

$$\int dt \, d^d x \Big(\frac{\delta \Gamma[\phi, \phi']}{\delta \phi} + \partial_t \phi' \Big) c(t) = 0$$

Since the function $c(t)$ is arbitrary the functional $\Gamma[\phi, \phi'] - \phi \partial_t \phi'$ is invariant

Galilean invariance

Analogously from Galilean invariance follows the relation

$$i\partial_{\mathbf{p}}\Gamma^{(m+1,n)}(\omega=0,\mathbf{p}=0,p_q,\ldots,\mathbf{p}_{m+n-1})=$$

$$=\lambda(\mathbf{p}_1\partial_{\omega_1}+\cdots+\mathbf{p}_{m+n-1}\partial_{\omega_{m+n-1}})\Gamma^{(m,n)}(\mathbf{p}_1,\ldots,\mathbf{p}_{m+n-1})$$

Which in practice means that ∂_t always enters Γ only in the form of the Gallilean covariant derrivative

$$D_t=\partial_t-\lambda(\partial\phi)\,\partial$$

Structure function

$$C(L,t)=\langle\left[\varphi(L,t)-\varphi(0,0)\right]^2\rangle=L^{2\chi}f(t/L^z)$$

The exact relation holds

$$\chi+z=2$$

DE for KPZ

Ansatz

$$\Gamma_k(\phi,\phi') = \frac{1}{2}\phi' D_k \phi' + \phi'\big(-\partial_t\phi(x) + \nu_k \partial^2\phi + \frac{1}{2}(\partial\phi)^2\big)$$

Regulator

$$R_k(q^2) = r(\frac{q^2}{k^2}) \begin{pmatrix} 0 & \nu_k q^2 \\ \nu_k q^2 & D_k \end{pmatrix}$$

Running parameters

$$\nu_k = \partial_{q^2} \frac{\delta^2\Gamma_k}{\delta\phi\phi'} \qquad D_k = \frac{\delta^2\Gamma_k}{\delta\phi'^2}$$

Inserting such Ansatz into the Wetterich equation and passing to the scaling variables one can search for the fixed point solutions for the effective coupling

$$g_k = \frac{D_k}{\nu_k^3}$$

Strong Coupling fixed point

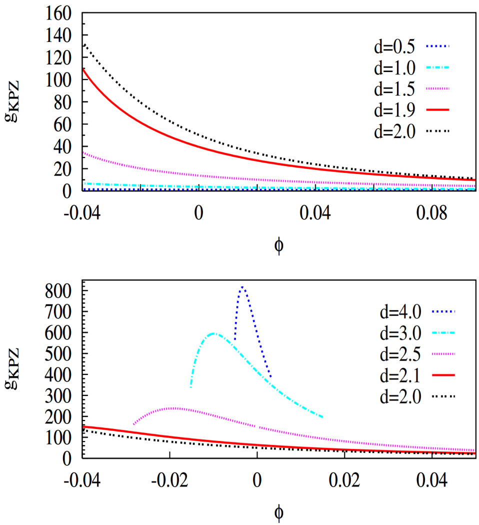

Figure 3: Picture from L.Canet arXiv:cond-mat/0509541

BMW-like scheme

Background field formalism

$$\Gamma_k^{(m,n)}(\mathbf{q}_1 = 0, \{\mathbf{q}_i\}; \phi, \phi') = \partial_\phi \Gamma_k^{(m-1,n)}(\{\mathbf{q}_i\}; \phi, \phi')$$

$$\Gamma_k^{(m,n)}(\{\mathbf{q}_i\}, \mathbf{q}_{m+n} = 0; \phi, \phi') = \partial_{\phi'} \Gamma_k^{(m,n-1)}(\{\mathbf{q}_i\}; \phi, \phi')$$

But for the function $\Gamma_k^{(2,1)}$ one should keep its signature momenta dependence

$$\Gamma_k^{(2,1)}(\mathbf{q}_1, \mathbf{q}_2) \sim \mathbf{q}_1 \cdot \mathbf{q}_2$$

External frequencies and vertex frequencies are set to zero.

Expansion around $\phi' = 0$ that keeps only the leading term

$$\Gamma_k^{(1,1)}(\mathbf{q}, \omega, \phi') = i\omega + q^2 \gamma_k^{(1,1)}(q^2)$$
$$\Gamma_k^{(0,2)}(\mathbf{q}, \omega, \phi') = \gamma_k^{(0,2)}(q^2)$$
$$\Gamma_k^{(1,1)}(\mathbf{q}, \omega, \phi') = -q^2 \phi'$$

Phase diagram of KPZ from modified BMW

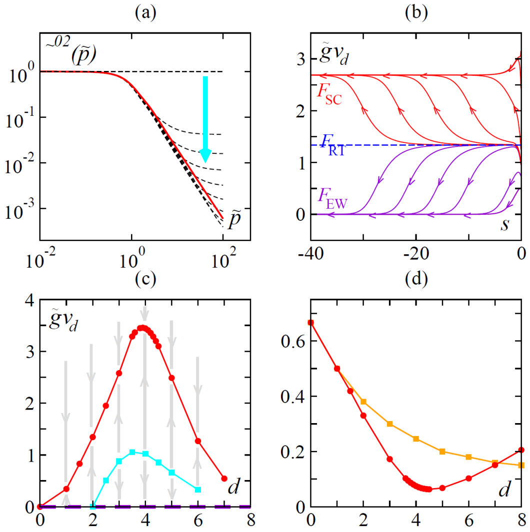

Figure 4: Picture from L.Canet et al arXiv:0905.1025

Thank you for attention!