Introduction to the Functional Renormalization Group

N. M. Lebedev

Advanced Methods of Modern Theoretical Physics:
Integrable and Stochastic Systems

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Introductory literature

- ✓ J. Berges, N. Tetradis, C. Wetterich, “Non-Perturbative Renormalization Flow in Quantum Field Theory and Statistical Physics”

  arXiv:hep-ph/0005122

- ✓ B. Delamotte, “An Introduction to the Nonperturbative Renormalization Group”

  arXiv:cond-mat/0702365

Setting up the language

The generating functional of Green's functions (Partition function)

$$Z[J] = \int \mathcal{D}\varphi \exp\{-S[\varphi] + J\varphi\}$$

Average of some operator

$$\langle \varphi(x)\varphi(y) \rangle = Z^{-1} \int \mathcal{D}\phi \; \varphi(x)\varphi(y) \; \exp\{-S[\varphi]\}$$

Which is equivalent to

$$\langle \varphi(x)\varphi(y) \rangle = Z^{(2)}[J=0] = \left. \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} \right|_{J=0}$$

Connected and 1PI functionals

- ✓ The generating functional of connected Green's functions (Helmholtz free energy)

$$W[J] = \ln Z[J].$$

By definition

$$\frac{\delta W[J]}{\delta J} = \langle \varphi \rangle = \phi$$

- ✓ The Legendre transformation – 1PI Green functions (Gibbs free energy)

$$\Gamma[\phi] = J\phi - W[J],$$

Where J meets the equation

$$\left.\frac{\delta W[J]}{\delta J}\right|_{J=J(\phi)} = \phi.$$

Useful identities

Exponentiating Legendre transform

$$\exp\{-\Gamma[\phi]\} = \int \mathcal{D}\varphi \exp\{-S[\varphi] + J(\varphi - \phi)\} =$$
$$= \int \mathcal{D}\varphi' \exp\left\{-S[\varphi' + \phi] + \frac{\delta\Gamma}{\delta\phi}\varphi'\right\}$$

Expanding $S[\varphi' + \phi]$ up to quadratic term in φ' and calculating Gaussian integral

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\,\mathrm{Tr}\ln S^{(2)}[\phi] + \dots$$

Another usefull identity

$$\delta(x-y) = \frac{\delta^2 W[J]}{\delta J(x)\delta\phi(y)} = \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta J(z)}{\delta\phi(y)} = \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(y)}$$

Shorthand notations

$$W^{(2)} = [\Gamma^{(2)}]^{-1}$$

Mode decoupling

The idea is to separate stochastic microscopic variables φ into rapid $\varphi(p > k)$ and slow $\varphi(p < k)$ modes in such a way that fluctuations of slow modes sufficiently suppressed while rapid modes are unaffected and can be integrated out.

The generating functional of connected Green's functions

$$W_k[J] = \ln \int \mathcal{D}\varphi \exp\left\{-S[\varphi] - \Delta S_k[\varphi] + J\varphi\right\},$$

with the quadratic additive

$$\Delta S_k[\phi] = \frac{1}{2}\phi(\mathbf{p})\, R_k(\mathbf{p})\, \phi(-\mathbf{p}).$$

We want $R_k(p)$ behave as momentum dependent mass term.

The properties of cut-off kernel $R_k(q)$

- ✓ $R_k(\mathbf{p}) \to \infty$ (or Λ) as $k \to \infty$ (or Λ): all fluctuations are frozen
- ✓ $R_k(\mathbf{p}) \to 0$ as $k \to 0$: all fluctuations are integrated out
- ✓ $R_k(\mathbf{p}) \to 0$ as $p >> k$: rapid modes are unaffected
- ✓ $R_k(\mathbf{p}) \simeq k^2$ as $p << k$: slow modes acquire large mass

Widely used kernels:

- the exponential shape

$$R_k(\mathbf{p}) = \frac{p^2}{e^{p^2/k^2} - 1}$$

- the theta-regulator

$$R_k(\mathbf{p}) = (k^2 - p^2)\Theta(k^2 - p^2)$$

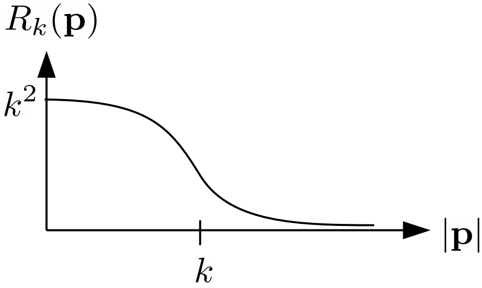

Figure 1: A typical shape of the cut-off function.

Effective average action

As we have seen, at zero order Γ_k defined through Legendre transform coincides with the (modified) action functional

$$\Gamma_k[\phi] \simeq S[\phi] + \Delta S_k[\phi]$$

Modified Legendre transform

$$\Gamma_k[\phi] = J_k(\phi)\phi - W_k[J_k(\phi)] - \Delta S_k[\phi],$$

where $J(\phi)$ meets the equation

$$\left.\frac{\delta W_k[J]}{\delta J}\right|_{J=J_k(\phi)} = \phi.$$

EAA as an interpolation functional

As $k \to 0$ term $\Delta S_k[\phi]$ vanishes, the model appears to be unmodified and hence

$$\Gamma_{k=0}[\phi] = \Gamma[\phi]$$

To find opposite limit lets employ modified version of our useful identity

$$\exp\left\{-\Gamma_k[\phi]\right\} = \int \mathcal{D}\varphi' \exp\left\{-S[\varphi' + \phi] + \frac{\delta \Gamma_k}{\delta \phi}\varphi' - \frac{1}{2}\varphi' R_k \varphi'\right\}$$

As $k \to \infty$

$$\exp\left\{-\Gamma_{k\to\infty}[\phi]\right\} = \int \mathcal{D}\varphi' \; \delta(\varphi') \exp\left\{-S[\varphi' + \phi] + \frac{\delta \Gamma_{k\to\infty}}{\delta \phi}\varphi'\right\} = \exp\left\{-S[\phi]\right\}$$

So we have natural initial condition

$$\Gamma_{k=\Lambda}[\phi] = S[\phi]$$

As scale k varies the functional $\Gamma_k[\phi]$ interpolates between mean field approximation of the Gibbs free energy and its full functional

The Wilson-Polchinski Equation

By definition

$$\exp\{W_k\} = \int \mathcal{D}\varphi \exp\{-S[\varphi] - \Delta S_k[\varphi] + J\varphi\}$$

Hence

$$\begin{aligned}\partial_k \exp\{W_k[J]\} &= -\frac{1}{2}\int \mathcal{D}\varphi\,(\varphi\,\partial_k R_k\,\varphi)\,\exp\{-S[\varphi] - \Delta S_k[\varphi] + J\varphi\} = \\ &= \left(-\frac{1}{2}\int dq\,\partial_k R_k(\mathbf{q})\frac{\delta}{\delta J(\mathbf{q})}\frac{\delta}{\delta J(-\mathbf{q})}\right)\exp\{W_k[J]\}\end{aligned}$$

Taking variations we arrive to the flow equation

$$\partial W_k[J] = -\frac{1}{2}\,\mathrm{Tr}\left[\partial_k R_k\left(\frac{\delta^2 W_k}{\delta J \delta J} + \frac{\delta W_k}{\delta J}\frac{\delta W_k}{\delta J}\right)\right]$$

The flow of EAA

Modified Legendre transform

$$\Gamma_k[\phi] = J_k\phi - W_k[J_k] - \Delta S_k[\phi],$$

$$\partial_k\Gamma_k[\phi] = \partial_k J_k\phi - \partial_k W_k[J_k] - \frac{\delta W_k}{\delta J_k}\partial_k J_k - \partial_k \Delta S_k[\phi]$$

By the definition of $J_k(\phi)$: $\frac{\delta W_k}{\delta J_k} = \phi$

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\partial_k R_k\left(\frac{\delta^2 W_k}{\delta J_k \delta J_k} + \frac{\delta W_k}{\delta J_k}\frac{\delta W_k}{\delta J_k}\right)\right] - \partial_k \Delta S_k[\phi]$$

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\partial_k R_k\left(\frac{\delta^2 W_k}{\delta J_k \delta J_k} + \phi\phi\right)\right] - \frac{1}{2}\phi\,\partial_k R_k\,\phi$$

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\partial_k R_k\left[\Gamma^{(2)} + R_k\right]^{-1}\right]$$

The Wetterich equation

For a scaling form of evolution we can take logarithmic derivative with respect to scale

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[k\partial_k R_k\left[\Gamma^{(2)} + R_k\right]^{-1}\right]$$

It is convenient to introduce RG time

$$t = -\ln\frac{k}{\Lambda}$$

So that the Wetterich equation will take the form

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k\left[\Gamma^{(2)} + R_k\right]^{-1}\right]$$

The gradient expansion

The scale k plays the role of IR regulator in a sence

$$\Gamma^{(2)} \sim q^2(q^2 + k^2)^{-\eta/2}$$

As long as k stays finite we can safely expand Γ_k in powers of gradients around

$$q^2 = 0$$

The derivative expansion is expected to converge at least up to the scale

$$q^2 \sim k^2$$

Singularities start to build up as k is lowered. So one can hope that lower orders of gradient expansion will already capture large scale physics even near the criticality.

Ansatz

✓ Local Potential Approximation (LPA)

$$\Gamma_k = \int d^d x \left(\frac{1}{2} (\partial \phi)^2 + U_k[\phi] \right)$$

✓ LPA'

$$\Gamma_k = \int d^d x \left(\frac{1}{2} Z_k (\partial \phi)^2 + U_k[\phi] \right)$$

✓ $\mathcal{O}(\partial^2)$

$$\Gamma_k = \int d^d x \left(\frac{1}{2} Z_k[\phi] (\partial \phi)^2 + U_k[\phi] \right)$$

The flow of the local potential

Evaluating EAA at uniform field ϕ we get the flow equation for local potential

$$\partial_t U_k(\phi) = \frac{1}{2} \int d^d q \, \frac{\partial_t R_k(q)}{q^2 + R_k(q) + \frac{\partial^2 U_k(\phi)}{\partial \phi^2}}$$

It is conveniet to introduce $\rho = \frac{1}{2}\,\phi^2$

$$\partial_t U_k(\rho) = \frac{1}{2} \int d^d q \, \frac{\partial_t R_k(q)}{q^2 + R_k(q) + U_k'(\rho) + 2\rho U_k''(\rho)}$$

Taking regulator in the form

$$R_k(q) = (k^2 - q^2)\theta(k^2 - q^2)$$

We can perform momentum integration analytically

$$\partial_t U_k(\rho) = \frac{v_d}{d} \, \frac{k^{d+2}}{k^2 + U_k'(\rho) + 2\rho U_k''(\rho)}$$

Scaling variables

To study critical behaviour it is necessary to pass to dimensionless variables

$$
\begin{aligned}
&y = \frac{q^2}{k^2} \\
&\tilde{x} = kx \\
&R_k(q) = q^2\, r(y) = k^2\, y\, r(y) \\
&\tilde{\phi}(\tilde{x}) = k^{\frac{2-d}{2}} \phi(x) \\
&\tilde{\rho}(\tilde{x}) = k^{2-d} \rho(x) \\
&\tilde{U}_t(\tilde{\rho}(\tilde{x})) = k^{-d}\, U_k(\rho(x))
\end{aligned}
$$

The flow of dimensionless potential

$$
\partial_t \tilde{U}_t = -d\tilde{U}_t + (d-2)\tilde{\rho}\tilde{U}_t' + \frac{4v_d}{d}\, \frac{1}{1 + \tilde{U}_t' + 2\tilde{\rho}\tilde{U}_t''}
$$

Possible physical regimes

- ✓ The system is in the broken phase ($T < T_c$)

$$\phi_{spontanious} = \sqrt{2\rho_0(k=0)}; \quad \rho_0(k) = k^{d-2}\tilde{\rho}_0(t)$$

Minimum of $\tilde{U}_t(\tilde{\rho}(\tilde{x}))$ flows to infinity

- ✓ The system is in the critical regime ($T = T_c$)

$$\rho_0(k \to 0) = k^{d-2}\tilde{\rho}_0(t \to \infty) \to 0; \quad \tilde{\rho}_0(t \to \infty) - \text{finite}$$

- ✓ The system is in the symmetric phase ($T > T_c$)

$$\tilde{\rho}_0(t < t_\xi) > 0; \quad \tilde{\rho}_0(t > t_\xi) = 0$$

Potential approaching convexity in broken phase

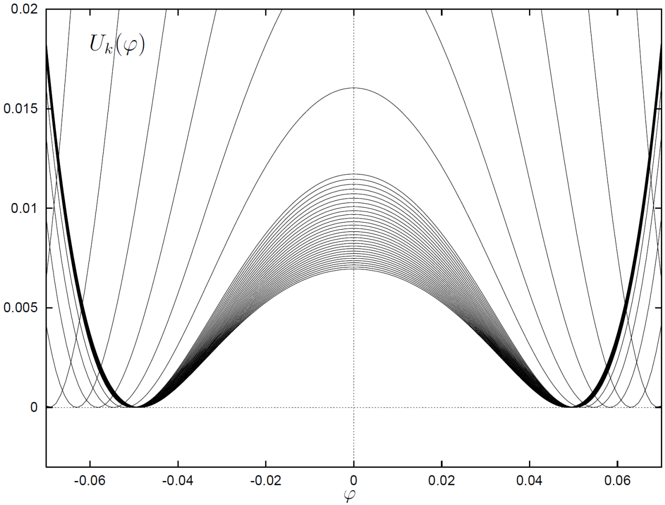

Figure 2: Flow of dimensionless potential in the broken phase

Search for the fixed potential

Generic initial potential

$$\tilde{U}_{t=0}(\tilde{\rho}) = \frac{\lambda_{t=0}}{2}(\tilde{\rho} - \tilde{\rho}_0(t=0))^2 + \frac{u_{t=0}}{6}(\tilde{\rho} - \tilde{\rho}_0(t=0))^3 + \dots$$

Typically proximity to the critical point is equivalent

$$\tilde{\rho}_0(t=0) - \tilde{\rho}_{0_c} \sim T_c - T$$

One can either

- ✓ Integrate flow equation directly fine tuning the parameter $\tilde{\rho}_0(t=0)$
- ✓ Directly search for the fixed potential by the shooting method

Shooting method

Fixed point condition

$$\partial_t \tilde{U}^* = 0$$

Is equivalent to the equation

$$0 = -d\tilde{U}^* + (d-2)\tilde{\rho}\tilde{U}^{*\prime} + \frac{4v_d}{d}\frac{1}{1+\tilde{U}^{*\prime}+2\tilde{\rho}\tilde{U}^{*\prime\prime}}$$

Initial condition

$$\tilde{U}^{*\prime}(\tilde{\phi}=0) = 0$$

Follows from $\mathbb{Z}_2$ symmetry, while second initial condition $\tilde{U}^{*\prime\prime}(\tilde{\phi}=0)$ should be tuned by shooting so that $\tilde{U}^{*\prime}(\tilde{\phi})$ stays finite for all values of ϕ

Fixed point potential

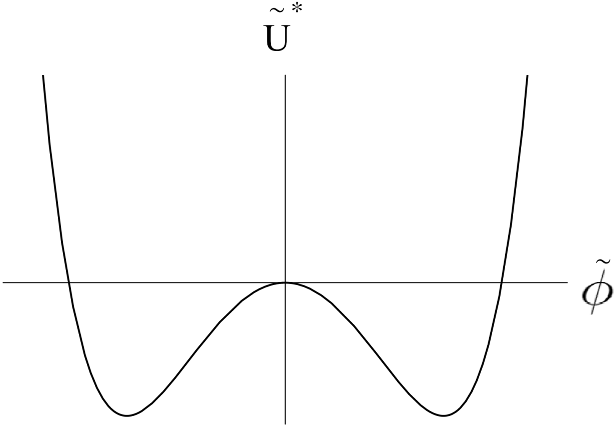

Figure 3: Fixed point potential

Critical behavior

RG transformation and fixed point

$$\partial_t \tilde{U} = T[\tilde{U}]; \qquad \partial_t \tilde{U}^* = 0 = T[\tilde{U}^*]$$

Lets consider small vicinity of fixed point $\tilde{U}^* + \delta\tilde{U}$

$$\partial_t \left(\tilde{U}^* + \delta\tilde{U} \right) = T[\tilde{U}^* + \delta\tilde{U}]$$

$$\partial_t \delta\tilde{U} = T'[\tilde{U}^*]\delta\tilde{U}$$

Lets suppose that there is complete set of eigenvectors of the operation

$$T'[\tilde{U}^*]V_i = \lambda_i V_i$$

And expand our perturbation

$$\delta\tilde{U} = \sum_i \delta g_i V_i$$

Critical behavior

The the flow equation in the vicinity of the fixed point will take the form

$$\partial_t \Big[\sum_i \delta g_i V_i \Big] = \sum_i \lambda_i \delta g_i V_i$$

Which is equivalent to the set of the equations

$$\partial_t \delta g_i = \lambda_i \delta g_i \quad \Rightarrow \quad \delta g_i = e^{\lambda_i t}$$

Latter means that we can from the very beginning look for eigenvalues in the form

$$\delta \tilde{U} = \sum_i e^{\lambda_i t} V_i$$

Or even

$$\delta \tilde{U} = e^{\lambda_i t} V_i$$

Then we end up with the eigenproblem

$$\lambda_i V_i = T'[\tilde{U}^*] V_i$$

Critical exponents

In the LPA approximation there is no wave function (field) renormalization, which automatically means

$$\eta_{LPA} = 0$$

As for the index $1/\nu$ we can introduce small perturbation to the fixed potential in a form

$$\tilde{U}(\tilde{\rho}) = \tilde{U}^*(\tilde{\rho}) + \epsilon\, e^{t/\nu}\, V(\tilde{\rho})$$

Substituting this Ansatz into the flow equation and keeping track only of linear in ϵ terms we end up with the eigenproblem

$$0 = -\Big(d + \frac{1}{\nu}\Big)V(\tilde{\rho}) + (d-2)\tilde{\rho}V'(\tilde{\rho}) - \frac{4v_d}{d}\frac{V'(\tilde{\rho}) + 2\tilde{\rho}V''(\tilde{\rho})}{1 + \tilde{U}^{*\prime} + 2\tilde{\rho}\tilde{U}^{*\prime\prime}}$$

Solving this equation one obtains

$$\nu_{LPA} = 0.65$$

To be compared with $\nu = 0.6297$ obtained by Monte Carlo simulations

Thank you for attention!

And I hope to see you tomorrow(=