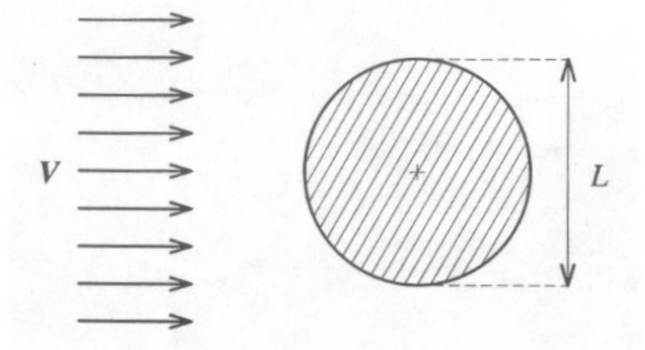
Fluid Dynamics

- dynamics: time-space behavior of velocity field $\mathbf{V}$
- turbulent flows in incompressible fluid ($\mathbf{div}\,\mathbf{V} = 0$, $\mathrm{Ma} \ll 1$)
- open system: injection and dissipation of kinetic energy
- Navier-Stokes equation:

$$\nabla_t \mathbf{V} = \nu_0 \Delta \mathbf{V} - \frac{\nabla p}{\rho} + \mathbf{F} \qquad \nabla_t \equiv \partial_t + (\mathbf{V}\nabla)$$



Uniform flow with velocity $\mathbf{V}$, incident on cylinder of diameter L

- Reynolds number: $\mathrm{Re} = \frac{\mathrm{VL}}{\nu_0}$
- turbulence: $\mathrm{Re}_{\mathrm{cr}} \gg 1$

Kármán vortex street $\mathrm{Re} \approx 500 \div 5000$

Teodor Kármán (1912)

Developed Turbulence

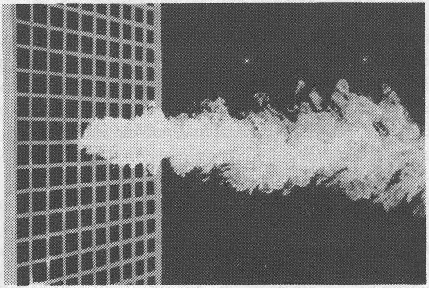
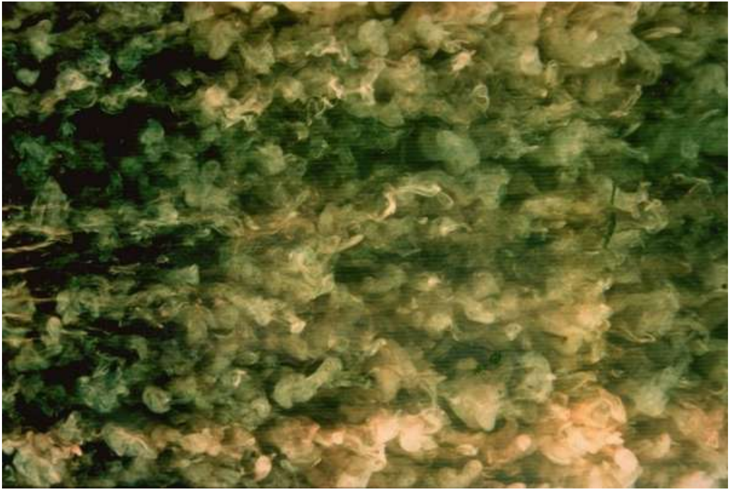


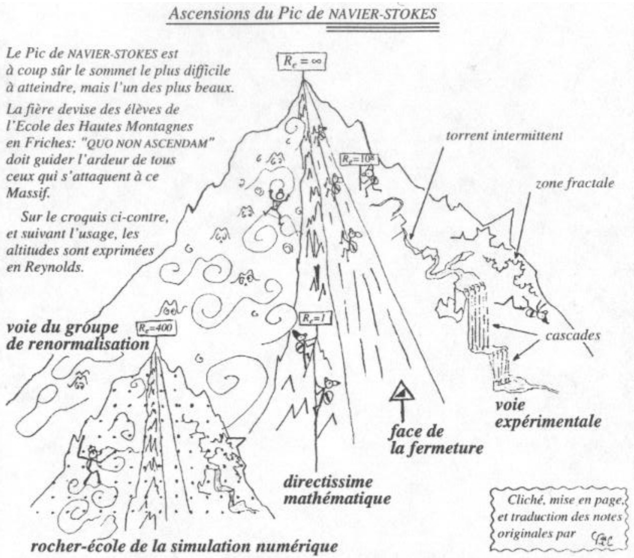
Figure I-3: turbulence created in a wind tunnel behind a grid. Here turbulence fills the whole apparatus, and a localized source of smoke has been placed on the grid to visualize the development of turbulence (picture by J.L. Balint, M. Ayrault and J.P. Schon, Ecole Centrale de Lyon; from Lesieur (1982), courtesy "La Recherche")

Developed Turbulence



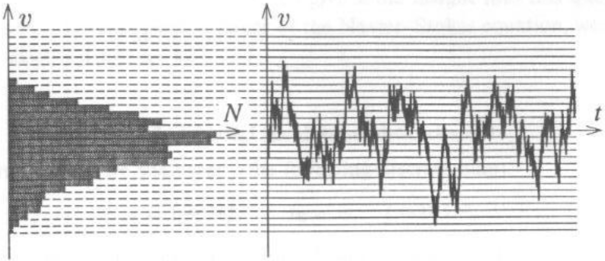
Past grid developed turbulence $\mathrm{Re} \gg \mathrm{Re}_{cr}$

Methods



Uriel Frish Turbulence: The Legacy of A. N. Kolmogorov (1996)

Developed Turbulent Flows - Signals



Construction of the histogram of a signal by binning.

- unpredictability of the details of motion
- random process
- starting point - stochastic differential equations

Stochastic Navier-Stokes equation

- Statistical discription: $\mathbf{V} =< \mathbf{V} > + \mathbf{v} \qquad < \mathbf{v} >= \mathbf{0}$
- Stochastic Navier-Stokes equation for fluctuating part $\mathbf{v}$:

$$\frac{\partial \, \mathrm{v}_i}{\partial t} = \nu_0 \Delta \mathrm{v}_i - (\mathrm{v}_s \nabla_s) \mathrm{v}_i - \nabla_i \mathrm{p} + \mathrm{f}_i$$

$$\nabla \mathbf{v} = 0, \;\; \mathrm{v}_i \equiv \mathrm{v}_i(x), \quad \mathrm{f}_i \equiv \mathrm{f}_i(x), \quad x \equiv (\mathbf{x}, t), \;\; \rho = 1, \;\; \mathrm{i} = 1, 2 ... \mathrm{d}$$

- External random forcing with Gaussian distribution

$$\mathcal{P}(\mathbf{f}) = e^{-\frac{1}{2} \mathbf{f} \, D^{-1} \, \mathbf{f}}, \quad < \mathrm{f}_i(x) \, \mathrm{f}_j(x') > \equiv D_{ij}(x, x')$$

Examples

- Langevin equation

$$\partial_t \varphi = \lambda \Delta \varphi + m\varphi - g\varphi^3 + f, \qquad \langle f(x) f(x') \rangle \equiv D(x, x') = 2\lambda\delta(x - x')$$

- Brownian motion

$$\frac{dv}{dt} = -\gamma v + f, \qquad \langle f(t) f(t') \rangle \equiv D(t, t') = 2\gamma\delta(t - t')$$

- Stochastic differential equations

$$\partial_t \tilde{\mathbf{v}}(x) = V(x,\phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x')$$

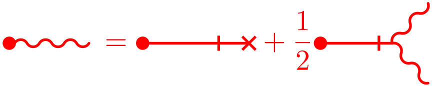
- $\phi(x) \equiv \phi(t,\mathbf{x})$ random fields
- equation valid for all space coordinates and for time instants $(-\infty\,,+\infty)$
- fields asymptotically vanish: $\phi \to 0$ as $t \to -\infty$ and as $|\mathbf{x}| \to \infty$ for arbitrary time instants t
- retardation condition
- t-local functional $V(\phi) = L\phi + N(\phi) + F$
- statistical averages $\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle$, $\quad \left\langle \frac{\delta^m [\phi(x_1)\ldots\phi(x_n)]}{\delta f(x_1')\ldots\delta f(x_m')} \right\rangle$

Solution of stochastic equation

- integral form

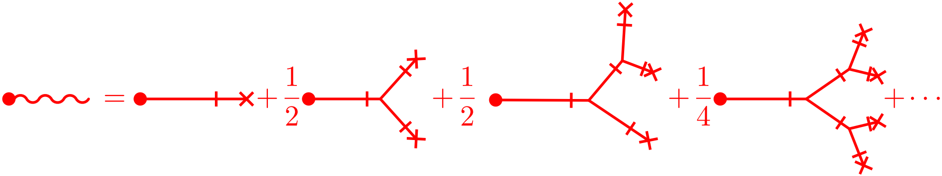
$$\phi = \Delta_{12}\left[F_r + f + N(\phi)\right], \quad \Delta_{12} \equiv \Delta_{12}(x,x') \equiv (\partial_t - L)^{-1}$$

- response function $\Delta_{12}(x,x') = 0$ at $t < t'$
- perturbation solution $N(\phi) = g\phi^2(x)/2$
- graphic representation



Solution of stochastic equation

- Perturbative solution with g^3 precision



- correlation functions - mutual multiplying of graphs for corresponding numbers of field ϕ and averaging over all realizations of random force f

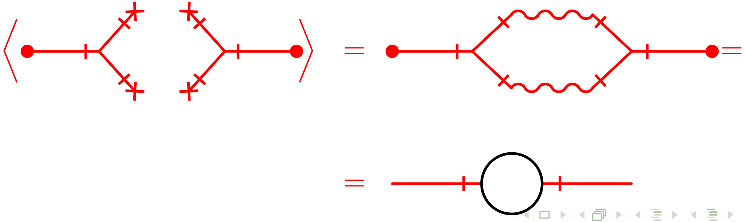
Solution of stochastic equation

- contraction of pairs creating noise D
- new graphical element

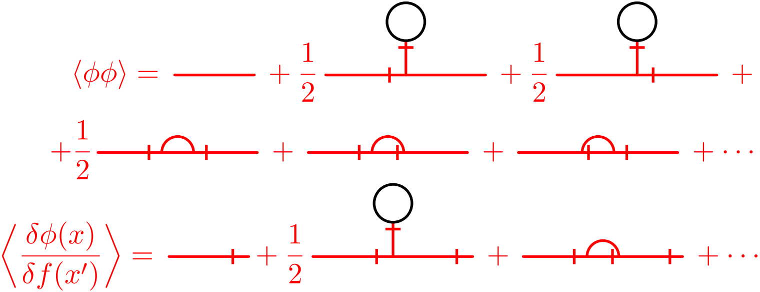
$$\Delta_{11} \equiv \langle \phi\phi \rangle_0 = \Delta_{12} D \Delta_{21} =$$

- wavy line bounded by vertical dash - noise D

$$\Delta_{21}(x, x') \equiv \Delta_{12}^{T}(x, x') = \Delta_{12}(x', x)$$



- illustration of perturbative scheme: a few first graphs for correlation pair function $\langle\phi\phi\rangle$ and response function $\langle\delta\phi(x)/\delta f(x')\rangle$:



- graphs do not contained closed loops of response function

Transition to quantum field model

- solution of SDE: $\tilde{\phi} = \tilde{\phi}(x; f)$
- generating functional $G(A^\phi)$

$$G(A^\phi) = \int \mathrm{D}f \exp\left[-\frac{fD^{-1}f}{2} + A^\phi\tilde{\phi}\right]$$

$$A^\phi\tilde{\phi} = \int \mathrm{d}x A^\phi(x)\tilde{\phi}(x)$$

$$fD^{-1}f = \iint \mathrm{d}x\,\mathrm{d}x' f(x)D^{-1}(x,x')f(x')$$

Transition to quantum field model

- Useful identity

$$\exp(A^\phi \tilde{\phi}) = \int D\phi \delta(\phi - \tilde{\phi}) \exp(A^\phi \phi)$$

- Functional δ-function

$$\delta(\phi - \tilde{\phi}) \equiv \prod_x \delta\left[\phi(x) - \tilde{\phi}(x)\right]$$

$$\phi = \tilde{\phi} \Leftrightarrow Q(\phi, f) \equiv -\partial_t \phi + V(\phi) + f = 0$$

$$\delta(\phi - \tilde{\phi}) = \delta\left[Q(\phi, f)\right] \det M \,, \quad M = \frac{\delta Q}{\delta \phi}$$

$$M(x, x') = \delta Q(x) / \delta \phi(x')$$

$$\delta\left[Q(\phi, f)\right] = \int D\phi' \, e^{[\phi' \, Q(\phi, f)]}$$

- ϕ' - auxiliary field

Transition to quantum field model

- Generating functional

$$G(A^\phi) = \int\int D\phi D\phi' \det M \exp\left[\phi' D\phi'/2 + \phi'\left(-\partial_t\phi + V(\phi)\right) + A^\phi\phi\right]$$

- contribution of the determinant

$$M = -\partial_t + L + \frac{\delta N(\phi)}{\delta\phi} = -\Delta_{12}^{-1}\left[1 - \Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right],$$

- $det\, M = exp[tr\ \ ln M]$

Transition to quantum field model

- generating functional

$$G(A^{\phi}) = \iint \mathrm{D}\phi \, \mathrm{D}\phi' \, \mathrm{e}^{S(\phi,\phi')+A^{\phi}\phi},$$

- action

$$S(\phi,\phi') = \operatorname{tr}\ln\left(1 - \Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right) + \frac{\phi' D \phi'}{2} + \phi'\left(-\partial_t\phi + L\phi + N(\phi)\right)$$

$$\operatorname{tr}\ln\left(1 - \Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right) = -\operatorname{tr}\left[\Delta_{12}\frac{\delta N(\phi)}{\delta\phi} + \frac{1}{2}\Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\Delta_{12}\frac{\delta N(\phi)}{\delta\phi} + \cdots\right]$$

Transition to quantum field model

- excercise: to find the solution and form of equal-time response function

$$(\partial_t - L)\Delta_{12}(x, x') = \delta(x - x')$$

$$S(\phi, \phi') = \frac{\phi' D \phi'}{2} + \phi' \left(-\partial_t \phi + L\phi + N(\phi)\right)$$

Transition to quantum field model

- Final action

$$S(\phi,\phi') = \iint \mathrm{d}x\,\mathrm{d}x'\,\frac{\phi'(x)D(x,x')\phi'(x')}{2} + \int \mathrm{d}x\,\phi'(x)\left[-\partial_t\phi(x) + V(\phi(x))\right]$$

$$S(\Phi) = S_0(\Phi) + S_I(\Phi), \qquad S_I(\Phi) = \int \mathrm{d}x\,\phi'(x)N(\phi(x)), \qquad \Phi \equiv \phi,\phi'$$

$$S_0(\Phi) = -\frac{1}{2}\Phi K\Phi \equiv -\frac{1}{2}\begin{pmatrix}\phi \\ \phi'\end{pmatrix}\begin{pmatrix}0 & (\partial_t - L)^T \\ \partial_t - L & -D\end{pmatrix}\begin{pmatrix}\phi \\ \phi'\end{pmatrix}$$

$$\Phi K\Phi \equiv \iint \mathrm{d}x\,\mathrm{d}x'\,\Phi(x)K(x,x')\Phi(x'), \quad K^T(x,x') \equiv K(x',x) = K(x,x')$$

- Final generating functional

$$G(A) = \int \mathrm{D}\Phi \; \exp\left[S(\Phi) + A\Phi\right], \; A\Phi \equiv \int \mathrm{d}x \left[A^{\phi}(x)\phi(x) + A^{\phi'}(x)\phi'(x)\right]$$

- Wick theorem and Feynman graphs

$$G(A) = \exp\left(\frac{1}{2}\frac{\delta}{\delta\Phi}\Delta\frac{\delta}{\delta\Phi}\right)\; \exp[S_I(\Phi) + A\Phi]|_{\Phi=0}], \quad \Delta = K^{-1}$$

$$\frac{\delta}{\delta\Phi}\Delta\frac{\delta}{\delta\Phi} \equiv \begin{pmatrix} \frac{\delta}{\delta\phi} \\ \frac{\delta}{\delta\phi'} \end{pmatrix} \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta}{\delta\phi} \\ \frac{\delta}{\delta\phi'} \end{pmatrix}$$

$$\Delta_{12} = \Delta_{21}^{T} = (\partial_t - L)^{-1}, \;\; \Delta_{11} = \Delta_{12} D \Delta_{21}, \;\; \Delta_{22} = 0$$

Transition to quantum field model

- Green functions

$$G(A) = \sum_{n=0}^{\infty} \frac{1}{n!} G_n A^n = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdot \int dx_1 ... dx_n G_n(x_1, ... x_n) A(x_1) ... A(x_n)$$

$$G_n(x_1, ... x_n) = \frac{\delta^n G(A)}{\delta A(x_1) ... A(x_n)} |_{A=0} = \int \mathrm{D}\Phi \, \Phi(x_1) ... \Phi(x_n) e^{S(\Phi)}$$

- Response function

$$\langle \phi(x) \phi'(x') \rangle = \frac{\delta^2 G(A)}{\delta A^{\phi}(x) \, \delta A^{\phi'}(x')} |_{A=0} = \int \mathrm{D}\Phi \, \phi(x) \phi'(x') e^{S(\Phi)}$$

- Connecting and IP-irreducible Green functions

$$W(A) = \ln G(A), \quad \Gamma(\alpha) = W(A) - \alpha A, \quad \alpha(x) = \frac{\delta W(A)}{\delta A(x)}$$

$$\frac{\partial\, \mathrm{v}_i}{\partial t} = \nu_0 \Delta \mathrm{v_i} - (\mathrm{v_s} \nabla_\mathrm{s})\mathrm{v_i} - \nabla_\mathrm{i}\mathrm{p} + \mathrm{f_i}, \qquad \nabla \mathbf{v} = 0$$

- Action

$$\begin{aligned} S(\mathbf{v},\mathbf{v}') &= \frac{1}{2}\int dx dx' \mathrm{v}'_i(x) D_{ij}(x,x')\mathrm{v}_j(x') + \\ & \int dx \mathrm{v}'_i(x)[-\partial_t \mathrm{v}_i(x) + \nu_0 \triangle \mathrm{v}_i(x) - \mathrm{v}_s(x)\nabla_s \mathrm{v}_i(x)] \end{aligned}$$

$$S(\mathbf{v},\mathbf{v}') = \frac{1}{2}\mathbf{v}D\mathbf{v} + \mathbf{v}'[-\partial_t \mathbf{v} + \nu_0 \triangle \mathbf{v} - (\mathbf{v}\nabla)\mathbf{v}]$$

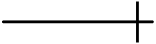
- Generating functional

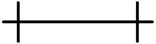
$$G(\mathbf{A},\mathbf{A}') = \int D\mathbf{v} D\mathbf{v}' e^{S(\mathbf{v},\mathbf{v}') + \mathbf{A}\mathbf{v} + \mathbf{A}'\mathbf{v}'}$$

$$\mathbf{A}\mathbf{v} + \mathbf{A}'\mathbf{v}' \equiv \int d\mathrm{x}[\mathrm{A}_i(\mathrm{x})\mathrm{v}_i(\mathrm{x}) + \mathrm{A}'_i(\mathrm{x})\mathrm{v}'_i(\mathrm{x})]$$

Turbulence: Elements of Feynman graphs

v_i $v_j \; = \; \langle v_i v_j \rangle_0 \equiv \Delta^{vv}_{ij}(\omega_k, \boldsymbol{k})$

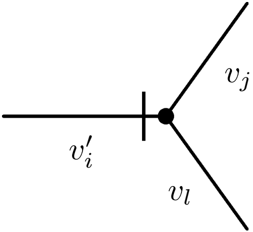
v_i  $v'_j \; = \; \langle v_i v'_j \rangle_0 \equiv \Delta^{vv'}_{ij}(\omega_k, \boldsymbol{k})$

v'_i  $v'_j \; = \; \langle v'_i v'_j \rangle_0 \equiv \Delta^{v'v'}(\omega_k, \boldsymbol{k})$

$$\Delta^{\mathrm{vv}}_{ij}(\mathbf{k}, \omega_k) = \frac{P_{ij}(\mathbf{k}) D(k)}{(i\omega_k + \nu k^2)(-i\omega_k + \nu k^2)}, \quad \Delta^{\mathrm{v'v'}}_{ij}(\mathbf{k}, \omega_k) = 0$$

$$\Delta^{\mathrm{vv'}}_{ij}(\mathbf{k}, \omega_k) = \frac{P_{ij}(\mathbf{k})}{-i\omega_k + \nu k^2}, \quad \Delta^{\mathrm{v'v}}_{ij}(\mathbf{k}, \omega_k) = \frac{P_{ij}(\mathbf{k})}{i\omega_k + \nu k^2}$$

Turbulence: Feynman graphs



$$\equiv V_{ijl} = i(k_j \delta_{il} + k_l \delta_{ij})$$

- Vertex responsible for nonlinear interactions among velocity fluctuations

Ward identities

- Galilean tansformations $\phi \equiv \varphi', \varphi;\qquad \phi \to \phi_v$

$$\varphi_v(x) = \varphi(x_v) - v(t), \qquad \varphi'_v(x) = \varphi'(x_v)$$

$$x \equiv (t, \boldsymbol{x}), \quad x_v \equiv (t, \boldsymbol{x}+u(t)), \quad u(t) = \int_{-\infty}^{t} dt' v(t') = \int_{-\infty}^{\infty} dt' \theta(t-t') v(t')$$

- Galilean invariance $\delta_v G(A) = 0$

$$\int D\phi \delta_v e^{S(\phi_v)+A\phi_v} = 0, \qquad D\phi = D\phi_v$$

Ward identities

$$\int D\phi \left[\delta_v S(\phi_v) + A\delta_v \phi\right] e^{S(\phi)+A\phi} = 0$$

$$\delta_v S(\phi_v) = \varphi' \partial_t v$$

$$\delta_v \varphi(x) = u\partial\varphi(x) - v, \qquad \delta_v \varphi'(x) = u\partial\varphi'(x)$$

$$\delta_v \partial_t \varphi(x) = u\partial\partial_t\varphi(x) + v\partial\varphi(x) - \partial_t v$$

$$\langle\langle \varphi' \partial_t v + A\delta_v \phi \rangle\rangle = 0$$

$$\int dt \int d\boldsymbol{x} v(t) \left\langle\left\langle -\partial_t \varphi'(x) + \int_{-\infty}^{t} A(\boldsymbol{x}, t') \frac{\partial \phi(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - A_\varphi(x) \right\rangle\right\rangle = 0$$

Ward identity

$$\int d\boldsymbol{x} \left\langle\!\!\left\langle -\partial_t \phi'(x) + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x},t') \frac{\partial \phi(\boldsymbol{x},t')}{\partial \boldsymbol{x}} - A_\varphi(x) \right\rangle\!\!\right\rangle = 0$$

$$\phi \qquad \text{in} \qquad \langle\langle \quad \rangle\rangle \Leftrightarrow \frac{\delta}{\delta A}$$

$$\int d\boldsymbol{x} \left\langle\!\!\left\langle -\partial_t \frac{\delta}{\delta A_{\varphi'}(x)} + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x},t') \frac{\partial}{\partial \boldsymbol{x}} \frac{\delta}{\delta A(\boldsymbol{x},t')} - A_\varphi(x) \right\rangle\!\!\right\rangle = 0$$

- Functional of connected Green functions $G = e^W$

$$\int d\boldsymbol{x} \left[-\partial_t \frac{\delta W}{\delta A_{\varphi'}(x)} + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x},t') \frac{\partial}{\partial \boldsymbol{x}} \frac{\delta W}{\delta A(\boldsymbol{x},t')} - A_\varphi(x) \right] = 0$$

Ward identity

- Functional of I-particle irreducible Green functions

$$\Gamma(\alpha) = W(A) - \alpha A, \quad \alpha(x) = \frac{\delta W(A)}{\delta A(x)}, \quad A(x) = -\frac{\delta \Gamma(\alpha)}{\delta \alpha(x)}$$

$$\int d\boldsymbol{x} \left[-\partial_t \alpha_{\varphi'}(x) + \int_{-\infty}^{\infty} \theta(t-t') \frac{\delta \Gamma(\alpha)}{\delta \alpha(\boldsymbol{x}, t')} \frac{\partial \alpha(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - \frac{\delta \Gamma(\alpha)}{\delta \alpha_{\varphi}(x)} \right] = 0$$

$$\Gamma(\alpha) = \alpha_{\varphi'} \Gamma_{\varphi' \varphi} \alpha_{\varphi} + \frac{1}{2} \alpha_{\varphi'} \Gamma_{\varphi' \varphi \varphi} \alpha_{\varphi}^2 + \dots$$

$$\alpha_{\varphi'} \Gamma_{\varphi' \varphi \varphi} \alpha_{\varphi}^2 \equiv \int \int \int dx_1 dx_2 dx_3 \alpha_{\varphi_i'}(x_1) \Gamma_{\varphi_i' \varphi_s \varphi_l}(x_1, x_2, x_3) \alpha_{\varphi_s}(x_2) \alpha_{\varphi_l}(x_3)$$

$$\Gamma_{\varphi_i' \varphi_s}(x_1, x_2) \equiv \Gamma_{is}(x_1, x_2), \; \Gamma_{\varphi_i' \varphi_s \varphi_l}(x_1, x_2, x_3) \equiv \Gamma_{isl}(x_1, x_2, x_3)$$

Ward identity

$$\int d\boldsymbol{x}\, \Gamma_{isl}(x_1, x_2, x) + \left[\theta(t - t_1)\frac{\partial}{\partial x_{1l}} + \theta(t - t_2)\frac{\partial}{\partial x_{2s}}\right]\Gamma_{is}(x_1, x_2) = 0$$

$$\int dx\, \Gamma_{isl}(x_1, x_2, x) + (t_2 - t_1)\frac{\partial \Gamma_{is}(x_1, x_2)}{\partial x_{1l}} = 0$$

- Ward identity in wave number - frequency representation $p \equiv \boldsymbol{k}, \omega$:

$$\Gamma_{is}(x_1, x_2) = \frac{1}{(2\pi)^{2+d}}\int dp\, \Gamma_{is}(p) e^{ip(x_1 - x_2)}$$

$$\Gamma_{isl}(x_1, x_2, x_3) =$$

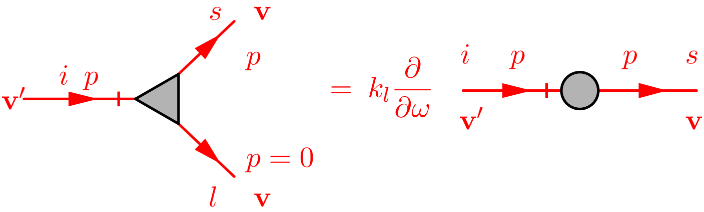
$$\frac{1}{(2\pi)^{3(2+d)}}\int dp_1 dp_2 dp_3 \delta(p_1 + p_2 + p_3)\Gamma_{isl}(p_1, p_2, p_3) e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3)}$$

$$\int dx\, \Gamma_{isl}(x_1, x_2, x) = \frac{1}{(2\pi)^{(2+d)}}\int dp\, \Gamma_{isl}(p, p, 0) e^{ip(x_1 - x_2)}$$

Ward identity

- Galilean symmetry: Ward identity

$$\Gamma_{is\,l}(p,p,0) = k_l \frac{\partial}{\partial\omega}\Gamma_{is}(p)$$



Energy-balance equation

- Invariance of functional measure, Swinger equations

$$\int D\Phi F(\Phi) = \int D\Phi F(\Phi + \Phi_0) \quad \Rightarrow \quad \int D\Phi \frac{\delta F(\Phi)}{\delta \Phi} = 0$$

- Swinger equation in field-theoretic model of developed turbulence

$$\int D\mathbf{v} D\mathbf{v}' \frac{\delta}{\delta \mathrm{v}'_i(x)} \mathrm{v}_i(\mathrm{x}) e^{S(\mathbf{v},\mathbf{v}')+\mathbf{A}\mathbf{v}+\mathbf{A}'\mathbf{v}'} = 0$$

$$-\mathrm{v}_i \partial_t \mathrm{v}_i + \mathrm{v}_i \nu_0 \Delta \mathrm{v}_i - \mathrm{v}_i(\mathrm{v}_s \nabla_s)\mathrm{v}_i - (\mathrm{v}_s \nabla_s)p + \mathrm{v}_i D_{is} \mathrm{v}'_s + \mathrm{v}_i \mathrm{A}'_i = 0$$

Energy-balance equation

$$\partial_t E + \partial_i S_i = -\mathcal{E} + \mathrm{v}_i D_{is}\mathrm{v}'_s + \mathrm{v}_i \mathrm{A}'_i$$

- energy density, vector of energy flow density, energy dissipation rate

$$E = \frac{1}{2}\mathrm{v}^2,\; S_i = p\mathrm{v}_i - \nu_0 \mathrm{v}_k(\partial_i \mathrm{v}_k + \partial_k \mathrm{v}_i) + \frac{1}{2}\mathrm{v}_i \mathrm{v}^2,\; \mathcal{E} = \frac{1}{2}\nu_0(\partial_i \mathrm{v}_k + \partial_k \mathrm{v}_i)^2$$

- Averaging equation

$$\partial_t \langle E \rangle + \partial_i \langle S_i \rangle = -\langle \mathcal{E} \rangle + \langle \mathrm{v}_i D_{is} \mathrm{v}' \rangle$$

$$-\overline{\mathcal{E}} + \iint d\mathbf{x}' dt' \left\langle \mathrm{v}_i(x)\mathrm{v}'_s(x')\right\rangle D_{is}(x, x') = 0, \qquad \overline{\mathcal{E}} \equiv \langle \mathcal{E} \rangle$$

Energy-balance equation

- Fourier representation of forcing noise

$$D_{ij}(x,x') = \frac{\delta(t-t')}{(2\pi)^d}\int d\mathbf{k}\, D(k) P_{ij}(\mathbf{k}) e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \equiv \delta(t-t') d_{is}(\mathbf{x},\mathbf{x}')$$

$$\overline{\mathcal{E}} = \int d\mathbf{x}' \,\langle \mathrm{v}_i(x)\mathrm{v}'_s(x')\rangle\,|_{t=t'}\, d_{is}(\mathbf{x},\mathbf{x}'), \quad \langle \mathrm{v}_i(x)\mathrm{v}'_s(x')\rangle\,|_{t=t'} = \frac{1}{2}\delta(\mathbf{x}-\mathbf{x}')P_{is}$$

$$\overline{\mathcal{E}} = \frac{d-1}{2(2\pi)^d}\int d\mathbf{k}\, d(k), \quad P_{ii}(\mathbf{k}) = d-1, \quad k \equiv |\mathbf{k}|$$

$$d(k) = D_0 \mathrm{k}^{4-d-2\epsilon} h(m/k), \quad h(0) = 1, \qquad m \equiv \mathrm{L}^{-1}$$

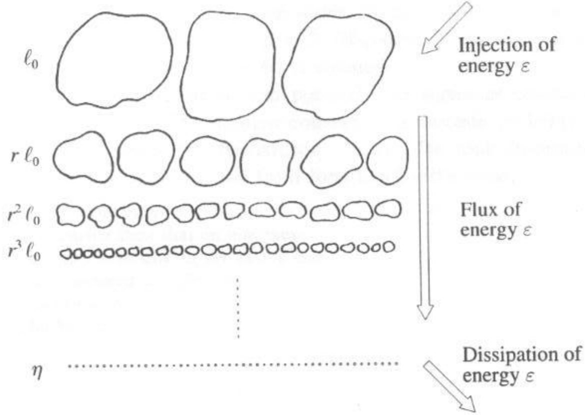
$$[\overline{\mathcal{E}}] = \mathrm{cm}^2/\mathrm{s}^3, \quad [\nu_0] = \mathrm{cm}^2/\mathrm{s}, \quad \mathrm{l} = \left(\frac{\nu_0^3}{\overline{\mathcal{E}}}\right)^{1/4}, \quad \mathrm{Re} = \frac{\mathrm{VL}}{\nu_0} = \left(\frac{\mathrm{L}}{\mathrm{l}}\right)^{4/3}$$

Kolmogorov-Obukhov theory

Three typical ranges:

- energy-containing range: characteristic outer (correlation) length L
- disipative range: characteristic inter (dissipation) length l
- inertial range: $l << r << L$, $r = |\mathbf{x} - \mathbf{x}'|$

The general physical picture of the developed turbulence has been proposed by L. F. Richardson and mathematically formulated by A. N. Kolmogorov and A. M. Obukhov. Very schematically in this picture eddies of various sizes are represented as blobs stacked in decreasing sizes. The upper most eddies have the outer turbulent scale L. The smallest eddies have scales proportional to the Kolmogorov dissipative scale l. Energy introduced to the turbulent system on the outer scale is “cascading” down this hierarchy of eddies and is removed by dissipation. The main advantage of this cascade picture is that it brings out basic assumptions of Kolmogorov-Obukhov phenomenological theory (K41) and their possible violation.



The cascade according to the Kolmogorov theory 1941

Kolmogorov scaling

- Structure functions S_p of velocity field v

$$S_p(\mathrm{r}) \equiv \langle [\mathrm{v}_r(\mathrm{x}) - \mathrm{v}_r(\mathrm{x}')]^p \rangle\,, \quad \mathrm{r} \equiv |\mathrm{x} - \mathrm{x}'|\,, \quad \mathrm{v}_r \equiv \mathbf{vr}/\mathrm{r}$$

- Second Kolmogorov hypothesis

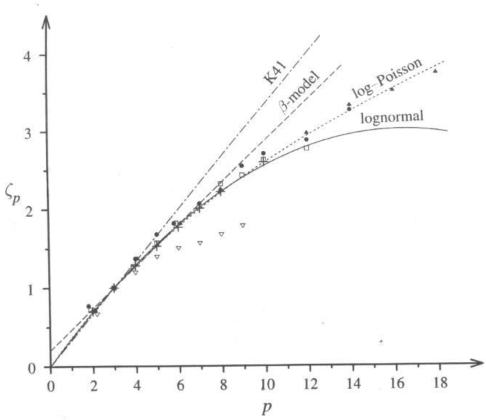
$$S_p(\mathrm{r}) = (\overline{\mathcal{E}}\mathrm{r})^{p/3} f_p(\mathrm{r}/\mathrm{L})\,, \qquad \mathrm{r} \gg \mathrm{l}$$

- Kolmogorov power laws (first Kolmogorov hypothesis)

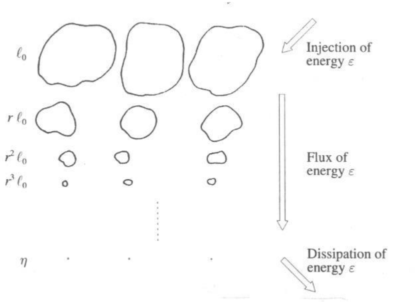
$$S_p(\mathrm{r}) = C_p(\overline{\mathcal{E}}\mathrm{r})^{\varsigma_p}\,, \quad \varsigma_p = p/3\,, \quad \mathrm{l} \ll \mathrm{r} \ll \mathrm{L}$$

- $p = 2, \quad C_2 \simeq C_k$ — Kolmogorov constant $C_k \approx 1.5$

Models



Exponents dependence ς_p on p



The cascade according to the β model. Notice that with each step the eddies become less and less space-filling

$$\varsigma_p = \frac{p}{3} + (3 - D)(1 - \frac{p}{3})$$

Intermittency

- The intermittency means that statistical properties are dominated by rare spatiotemporal configurations, in which the regions with strong turbulent activity have exotic (fractal) geometry and are embedded into the vast regions with regular (laminar) flow.
- In the turbulence, such a phenomenon is believed to be related to strong fluctuations of the energy flux. Therefore, it leads to deviations from the predictions of the KO41. Such deviations, referred to as “anomalous” or “non-dimensional” scaling, manifest themselves in singular (arguably power-like) dependence of correlation or structure functions on the distances and the integral (external) turbulence scale L. The corresponding exponents are certain nontrivial and nonlinear functions of the order of the correlation function, the phenomenon referred to as “multiscaling.”

Canonical dimensions

- Original action

$$S(\mathbf{v},\mathbf{v}') = \frac{g_0\nu_0^3}{2}\mathbf{v}D\mathbf{v} + \mathbf{v}'[-\partial_t\mathbf{v} + \nu_0\triangle\mathbf{v} - (\mathbf{v}\nabla)\mathbf{v}]$$

$$\nu_0 \longleftrightarrow \nu, \quad g_0 \longleftrightarrow g\mu^{2\epsilon}$$

$$d_{\mathbf{x}}^p = d_t^\omega = -1, \qquad d_{\mathbf{x}}^\omega = d_t^p = 0$$

F	$\mathbf{v}$	$\mathbf{v}'$	$\mathrm{L}^{-1}, \mathrm{l}^{-1}, \mu$	ν_0, ν	$\overline{\mathcal{E}}$	g_0	g
d_F^p	-1	$d+1$	1	-2	-2	2ϵ	0
d_F^ω	1	-1	0	1	3	0	0
d_F	1	$d-1$	1	0	4	2ϵ	0

$$d_F = d_F^p + 2d_F^\omega$$

- Original action

$$S(\mathbf{v},\mathbf{v}') = \frac{g_0\nu_0^3}{2}\mathbf{v}D\mathbf{v} + \mathbf{v}'[-\partial_t\mathbf{v} + \nu_0\triangle\mathbf{v} - (\mathbf{v}\nabla)\mathbf{v}]$$

- Renormalized action

$$S_R(\mathbf{v},\mathbf{v}') = \frac{g\nu^3\mu^{2\epsilon}}{2}\mathbf{v}D\mathbf{v} + \mathbf{v}'[-\partial_t\mathbf{v} + \nu Z_\nu\triangle\mathbf{v} - (\mathbf{v}\nabla)\mathbf{v}]$$

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\epsilon}Z_g, \quad Z_g = Z_\nu^{-3}, \; D_0 = g_0\nu_0^3 = g\nu^3$$

$$Z_\nu = 1 - \frac{cg}{2\epsilon} + O(g^2), \qquad c = \frac{(d-1)S_d}{4(2\pi)^d(d+2)}, \qquad S_d = 2\pi^{d/2}/\Gamma(d/2)$$

Renormalization

- Scaling equations

$$W(e_\lambda) = \lambda^{\Delta_W} W(e), \qquad e_{i\lambda} = \lambda^{\Delta_i} e_i,\ \forall i = 1, \ldots, n$$

$$W(e_1, \ldots, e_n) = |e_n|^\alpha \cdot f\left(\frac{e_1}{|e_n|^{\alpha_1}}, \ldots, \frac{e_{n-1}}{|e_n|^{\alpha_{n-1}}}\right), \alpha = \Delta_W/\Delta_n, \alpha_i = \Delta_i/\Delta_n$$

- Scaling equation

$$\left[\sum_e \Delta_e \mathcal{D}_e - \Delta_W\right] W(e) = 0, \qquad \mathcal{D}_e \equiv e\frac{\partial}{\partial e} \equiv e\partial_e$$

Renormalization

- Scaling equations

$$\left[\mathcal{D}_\mu - \mathcal{D}_{\mathbf{x}} + \sum_e d_e^p \mathcal{D}_e - \sum_\phi n_\Phi d_\Phi^p\right] W_{n_\Phi R}(t, \mathbf{x}, e, \mu) = 0$$

$$\left[-\mathcal{D}_t + \sum_e d_e^\omega \mathcal{D}_e - \sum_\phi n_\Phi d_\Phi^\omega\right] W_{n_\Phi R}(t, \mathbf{x}, e, \mu) = 0$$

- Relations for renormalized and unrenormalized functions

$$W_{n_\Phi R}(t, \mathbf{x}, e, \mu) = Z_\Phi^{-n_\Phi} W_{n_\Phi}(t, \mathbf{x}, e_0), \quad \tilde{\mathcal{D}}_\mu W_n(t, \mathbf{x}, e_0) = 0, \quad \tilde{\mathcal{D}}_\mu \equiv \mu \frac{\partial}{\partial \mu}|_{e_0}$$

- RG equation

$$\left[\mathcal{D}_{RG} + \sum_\phi n_\Phi \gamma_\phi\right] W_{n_\Phi R} = 0, \qquad \mathcal{D}_{RG} = \tilde{\mathcal{D}}_\mu = \mathcal{D}_\mu + \beta_g \partial_g - \sum_a \gamma_a \mathcal{D}_a$$

- RG functions

$$\beta = \tilde{\mathcal{D}}_\mu g = -g(2\epsilon + \gamma_g), \quad \gamma_e = \tilde{\mathcal{D}}_\mu ln Z_e, \quad \gamma_\Phi = \tilde{\mathcal{D}}_\mu ln Z_\Phi$$

$$\gamma_g = -\frac{2\epsilon \mathcal{D}_g ln Z_g}{1 + \mathcal{D}_g ln Z_g}, \quad \beta = -\frac{2\epsilon_g}{1 + \mathcal{D}_g ln Z_g}, \quad \gamma_a = -(2\epsilon + \gamma_g)\mathcal{D}_g ln Z_g$$

- Critical scaling: $\beta(g_*) = 0, \quad \beta'(g_*) > 0$

$$\left[-\mathcal{D}_{\mathbf{x}} - \Delta_\omega \mathcal{D}_t + \sum_a \Delta_a \mathcal{D}_a - \sum_\Phi n_\Phi \Delta_\Phi \right] W_{n_\Phi R}(t, \mathbf{x}, e, \mu) \Big|_{IR} = 0$$

- Critical (scaling) exponents

$$\Delta_F = d_F^p + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_?^*$$

- Scaling equations

$$\left[\mathcal{D}_\mu - \mathcal{D}_{\mathbf{x}} - \mathcal{D}_{\mathrm{L}} - 2\mathcal{D}_\nu + n_{\mathbf{v}} - (d+1)n_{\mathbf{v}'}\right] W_{nR}(t,\mathbf{x},g,\nu,\mu) = 0$$

$$\left[-\mathcal{D}_t + \mathcal{D}_\nu - n_{\mathbf{v}} + n_{\mathbf{v}'}\right] W_{nR}(t,\mathbf{x},g,\nu,\mu) = 0, \quad n = n_{\mathbf{v}} + n_{\mathbf{v}'}$$

- Relations for renormalized and unrenormalized functions

$$W_{nR}(t,\mathbf{x},g,\nu,\mu) = W_n(t,\mathbf{x},g_0,\nu_0)$$

- RG equation

$$\left[\mathcal{D}_\mu + \beta_g \partial_g - \gamma_\nu \mathcal{D}_\nu\right] W_{nR} = 0$$

- RG functions

$$\beta = \tilde{\mathcal{D}}_\mu g = -g(2\epsilon - 3\gamma_\nu), \quad \gamma_\nu = -\frac{2\epsilon \mathcal{D}_g ln Z_\nu}{1 - 3\mathcal{D}_g ln Z_\nu}$$

- Critical scaling: $\beta(g_*) = 0, \quad \beta'(g_*) > 0, \quad \gamma_\nu^* = 2\epsilon/3$

$$\left[-\mathcal{D}_{\mathbf{x}} - \Delta_\omega \mathcal{D}_t - \mathcal{D}_{\mathrm{L}} - n_{\mathbf{v}} \Delta_{\mathbf{v}} - n_{\mathbf{v}'} \Delta_{\mathbf{v}'}\right] W_{nR}(t, \mathbf{x}, \mu)\Big|_{IR} = 0$$

- Critical (scaling) exponents

$$\Delta_{\mathbf{v}} = 1 - \gamma_\nu^*, \quad \Delta_{\mathbf{v}'} = d - 1 + \gamma_\nu^*, \quad \Delta_\omega = 2 - \gamma_\nu^*$$

Scaling

- large-scale ($r/l >> 1$) asymptotics:

$$[r\frac{\partial}{\partial r} + L\frac{\partial}{\partial L} - \Delta_p]S_p(r) = 0$$

- solution:

$$S_p(r) = (\varepsilon r)^{\Delta_p} f_p(r/L)$$

- $\Delta_p = p[2\epsilon/3 - 1] \quad \Rightarrow \quad \Delta_p = p/3$ Kolmogorov exponents!
- result:
- validity of Kolmogorov second hypothesis — YES
- validity of first hypothesis — we don't know yet
- RG gives a receipt how to pass from molecular constants to the effective variables

Operator product expansion

- have we chance to find scaling fuction f_p?
- Wilson operator product expansion:

$$f_p(r/L) = \sum_{i=1}^{\infty} C_i^p (r/L)^{\Delta_i^p}, \; r/L << 1$$

- $\Delta_i^p > 0 \Rightarrow f_p$ is a regular function at $L \to \infty \Rightarrow$ corrections to the leading anomalies calculated in the framework of canonical RG approach
- in the theory of turbulence the situation is quite different: strong space-time gradients of velocity fluctuations $\mathbf{v}$ lead to the appearance of quantities - dangerous composite operators - with negative exponents Δ_i^p

Anomalous multiscaling

- Dangerous composite operators in the SNS model occur only for finite values of the RG expansion parameter ϵ
- Dangerous operators enter into the operator product expansions in the form of infinite families with the spectrum of critical dimensions unbounded from below, and the analysis of the large L behaviour implies the summation of their contributions
- This is clearly not a simple problem and it requires considerable improvement of the present technique
- consequence: anomalous multiscaling $\Leftrightarrow$ intermittency $\Leftrightarrow$ multifractality
- this is an open problem in theory of developed turublence

Passive scalar

- Advection of passive scalar field by turbulent flow

$$\partial_t \theta + (\mathbf{v}\cdot\nabla)\theta = \nu_0 \triangle\theta + f \qquad \langle f(x) f(x')\rangle = \delta(t-t')\, C(|\mathbf{x}-\mathbf{x}'|/L)$$

$$D_{ij}(x,x') = \frac{\delta(t-t')}{(2\pi)^d} D_0 \int d\mathbf{k}\, P_{ij}(\mathbf{k})(k^{-d-\epsilon} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}$$

- Solution of basic RG equation for SF $\mathcal{S}_{2p}$

$$\mathcal{S}_{2p}(r) \sim r^{p(2-\epsilon)} f_{2p}(r/L), \quad r/l >> 1$$

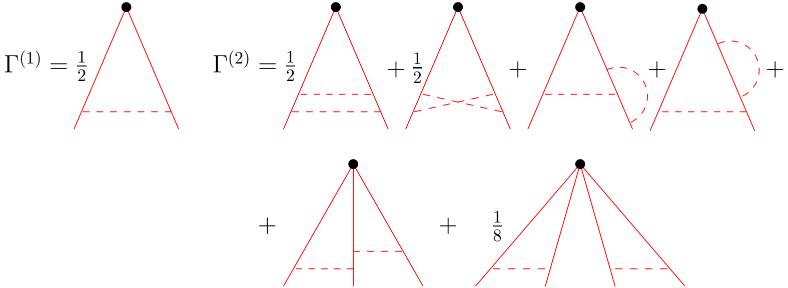
- Operator product expansion

$$f_{2p}(r/L) = \sum_{F_k} C_{F_k}(r/L)\,(r/L)^{\Delta_k}, \;\; r/L \to 0$$

- The leading composite operators

$$F_k = (\partial_i \theta \partial_i \theta)^k \qquad \Delta_k = -k\gamma_\nu^* + \gamma_{F_k}^*$$

- anomalous scaling $\Delta_k < 0$
- Multiplicative renormalization of F: $F_s = Z_{sl} F_l^R$
- Matrix of renormalization constants Z_{sl} are determined up to two loop approximation by the divergent parts of Feynman diagrams:



Critical dimensions

- Anomalous $\gamma^*_{F_k}$ and critical dimensions Δ_k are determined via eigenvalues of matrix Z
- Complete two-loop calculation of the critical dimensions of the composite operators F_k for arbitrary values of k, d:

$$\Delta_k = \Delta_k^{(1)}\epsilon + \Delta_k^{(2)}\epsilon^2$$

$$\Delta_k^{(1)} = \frac{-k(k-1)}{(d+2)} \qquad \Delta_k^{(2)} < 0$$

  - all Δ_s are negative already at small ϵ
  - Infinity series of OPE for SF S_{2p} truncate at p and CO F_p gives leading singular contribution Δ_p into asymptotic behaviour of scaling function at $r/L \ll 1$.

$$S_{2p}(r) \sim r^{p(2-\epsilon)}(r/L)^{\Delta_p}, \quad l << r << L$$

- MHD turbulence ($A = 1$), Linearized NS equation ($A = -1$), advection of passive scalar ($A = 0$)

$$\begin{aligned} \partial_t \mathbf{v} &= \nu_0 \triangle \mathbf{v} - (\mathbf{v} \cdot \nabla \mathbf{v} - \nabla \mathcal{P} + (\mathbf{b} \cdot \nabla)\mathbf{b} + \mathbf{f} \\ \partial_t \mathbf{b} &= \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v} \cdot \nabla)\mathbf{b} + A(\mathbf{b} \cdot \nabla)\mathbf{v} \end{aligned}$$

$$D_{ij}(x,x') = \delta(t-t') \int \frac{d^d\mathbf{k}}{(2\pi)^d} D(\mathbf{k}) R_{ij} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \quad R_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k})$$

- Tensorial structure

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2, \quad H_{ij}(\mathbf{k}) = i\rho\,\varepsilon_{ijl} k_l / k, \quad |\rho| \leq 1, \quad \rho \sim \langle \mathbf{v} rot \mathbf{v} \rangle$$

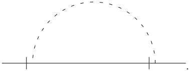
- Field-theoretic action

$$\begin{aligned} S(\Phi) &= \frac{1}{2}\mathbf{v}' D \mathbf{v}' + \mathbf{v}'[-\partial_t \mathbf{v} + \nu_0 \triangle \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{b}] \\ &+ \mathbf{b}'[-\partial_t \mathbf{b} + \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v} \cdot \nabla)\mathbf{b} + A(\mathbf{b} \cdot \nabla)\mathbf{v}] \end{aligned}$$

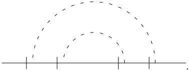
- Two-loop calculations

$$\Sigma^{b'b} = \Gamma^{(1)} + \Gamma^{(2)} = \Gamma^{(1)} + \sum_{l=1}^{8} s_l \Gamma_l^{(2)}$$

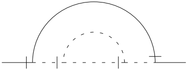
$\Gamma^{(1)} =$



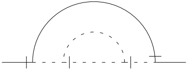
$\Gamma_1^{(2)} =$



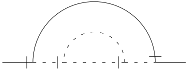
$\Gamma_2^{(2)} =$



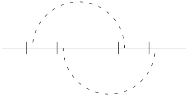
$\Gamma_3^{(2)} =$



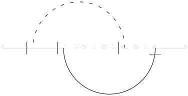
$\Gamma_4^{(2)} =$



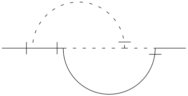
$\Gamma_5^{(2)} =$



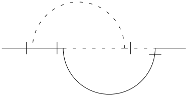
$\Gamma_6^{(2)} =$



$\Gamma_7^{(2)} =$



$\Gamma_8^{(2)} =$



Turbulent Prandtl number

- Dyson equations for response fuctions

$$\langle \mathbf{v}(\mathbf{k},\omega)\mathbf{v}'(-\mathbf{k},0)\rangle^{-1} = (G^{\mathbf{v}\mathbf{v}'})^{-1}(\mathbf{k},\omega) = [-i\omega + \nu_0 k^2 + \Sigma^{\mathbf{v}'\mathbf{v}}(\omega,\mathbf{k})]$$
$$\langle \mathbf{b}(\mathbf{k},\omega)\mathbf{b}'(-\mathbf{k},0)\rangle^{-1} = (G^{\mathbf{b}\mathbf{b}'})^{-1}(\mathbf{k},\omega) = [-i\omega + u_0\nu_0 k^2 + \Sigma^{\mathbf{b}'\mathbf{b}}(\omega,\mathbf{k})]$$

- Inverse turbulent Prandtl number - ration of response functions

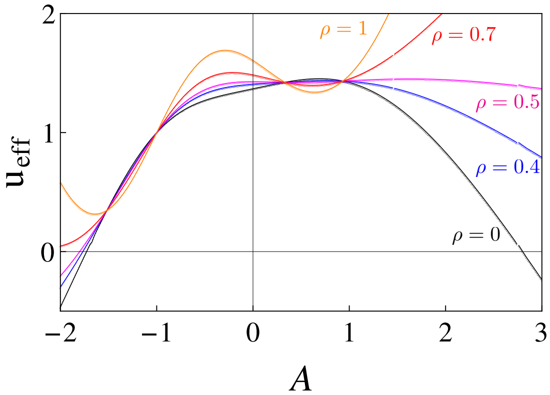
$$u_{eff} = \frac{G^{\mathbf{b}\mathbf{b}'}(\mathbf{k},\omega=0)}{G^{\mathbf{v}\mathbf{v}'}(\mathbf{k},\omega=0)}, \quad u_{eff} \to u^*_{eff}$$

- Two-loop precision $d=3$, $\rho \neq 0$

$$u^*_{eff}(A) = u^{(0)}(A) + \rho^2 u^{(\rho)}(A)$$

Turbulent Prandtl number

- Inverse Prandtl number - dependence on A



Obr.: Inverse turbulent Prandtl number u_{eff} as a function of A shown for fixed values of ρ.

Turbulent Prandtl number

- Inverse Prandtl number with two-loop precision

A	-1.0	-0.5	0	$+0.5$	$+1.0$	$+1.5$
$u^{(0)}(A)$	$+1.000$	$+1.2705$	$+1.3685$	$+1.4436$	$+1.4205$	$+1.2145$
$u^{(\rho)}(A)$	0.000	$+0.3587$	$+0.2376$	-0.0854	$+0.0623$	$+0.9444$

$$u^*_{eff}(A,\rho) = Pr^{-1}(A,\rho) = u^{(0)}(A) + \rho^2 u^{(\rho)}(A)$$

- Prandtl number in advection-diffusion problem, helicity accelerates spread of admixture in turbulent enviroment

$$Pr(0,0) = 0.73, \qquad Pr(0,1) = 0.62$$

- Non-helical turbulence – experimental status

$$Pr_{exp}(0,0) = 0.7 - 0.9,$$

averaged value ≈ 0.8 is recommended by

K.-A. Chang and E.A. Cowen, J. Eng. Mech. 128, 1082 (2002)

Instabilities in helical turbulent electrically conducting fluids

- additional linear divergence in response function of magnetic field
- appearance *curl* term in self - energy diagrams Σ

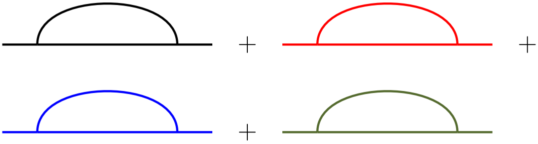
- $\Sigma_{ij}^{\mathbf{b}\mathbf{b}'}(\mathbf{p}) \sim \varepsilon_{isl} p_s T_{lj} \qquad T_{lj} \sim \Lambda \delta_{lj}$
- response function $\quad \langle \mathbf{b}(\mathbf{k}, t)\mathbf{b}'(-\mathbf{k}, 0)\rangle \sim \theta(t) e^{-\nu u(k^2 - \alpha \Lambda k)t}$
- Λ UV cut-off $\sim l^{-1}$, l is Kolmogorov dissipation length, connected with L $L = Re^{3/4} l$
- exponential growth of fluctuations $\mathbf{b}$ in the region of large space scales

Elimination of instability

- shift $\mathbf{b} \to \mathbf{b} + \mathbf{B}, \qquad \mathbf{B} \equiv \langle \mathbf{b} \rangle \neq 0$

$$
\begin{aligned}
S(\Phi) &= \frac{1}{2}[\mathbf{v}' D^{\mathbf{v}} \mathbf{v}' + \mathbf{b}' D^{\mathbf{b}} b'] + \mathbf{v}'[-\partial_t \mathbf{v} + \nu Z_1 \triangle \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} + \\
&+ Z_3(\mathbf{b} \cdot \nabla)\mathbf{b}] + \mathbf{b}'[-\partial_t \mathbf{b} + \nu u Z_2 \triangle \mathbf{b} - (\mathbf{v} \cdot \nabla)\mathbf{b} + A(\mathbf{b} \cdot \nabla)\mathbf{v}] + \\
&+ Z_3 \mathbf{v}'(\mathbf{B} \cdot \nabla)\mathbf{b} + A\mathbf{b}'(\mathbf{B} \cdot \nabla)\mathbf{v}
\end{aligned}
$$

- new cross propagators $< \mathbf{vb}' >$, $< \mathbf{bv}' >$, $< \mathbf{bv} >$, $< \mathbf{bb} >$ appear, all propagators are more complicated and depend on uniform field $\mathbf{B}$



- self-energy $\Sigma_{ij}^{\mathbf{bb}'}(\mathbf{p}) \sim \varepsilon_{isl} p_s T_{lj}$
- $T_{lj} = a\Lambda\delta_{lj} - b|\mathbf{B}|(\delta_{lj} + e_l e_j)$, $\mathbf{e} \equiv \mathbf{B}/|\mathbf{B}|$

$$|\mathbf{B}| = \frac{a\Lambda}{b} = \sqrt{\frac{1}{\pi|A|}}\frac{\Gamma(d/2+3/2)}{\Gamma(d/2+1)} u_* \nu\,\Lambda, \qquad A \neq 0, -1$$

- real space dimension $d = 3$

$$|\mathbf{B}| = \frac{8}{3\pi\sqrt{|A|}} u_* \nu\,\Lambda$$

Alfven waves and corrections

- linearized MHD equations in polarised medium

$$\begin{aligned} \partial_t \mathbf{v} &= \nu k^2 \mathbf{v} + i\gamma \mathbf{b} \\ \partial_t \mathbf{b} &= u\nu k^2 \mathbf{v} + i\gamma \mathbf{v} - i\mu[\mathbf{k} \times \mathbf{e}](\mathbf{eb}) \end{aligned}$$

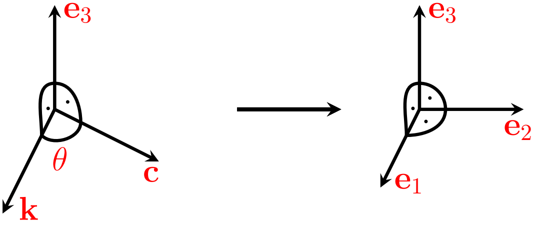
$$\mathbf{v} \equiv \mathbf{v}(t, \mathbf{x}) = \mathbf{v}(t) e^{i\mathbf{kx}}, \qquad \mathbf{b} \equiv \mathbf{b}(t, \mathbf{x}) = \mathbf{b}(t) e^{i\mathbf{kx}}$$

- solution in inviscid medium ($\nu = 0$) without exotic term ($\mu = 0$): Alfven waves $\quad \mathbf{v}(t) \sim \mathbf{b}(t) \sim e^{-i\omega t}, \quad \omega \equiv \gamma$

Exotic perturbations

- solutions with exotic term ($\mu \neq 0$)
- orthonormal basis of three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{e}_1 \equiv \mathbf{k}/|\mathbf{k}|, \quad \mathbf{e}_2 \equiv (\mathbf{e} - \mathbf{e}_1 \cos\theta)/\sin\theta, \quad \mathbf{e}_3 \equiv [\mathbf{e}_1 \times \mathbf{e}_2] = [\mathbf{e}_1 \times \mathbf{e}]/\sin\theta$$



Exotic perturbations

- velocity and magnetic vector fields are perpendicular to wave vector $\mathbf{k}$ or, equivalently to basis vector $\mathbf{e}_1$

$$\mathbf{v}(t) = v_2(t)\mathbf{e}_2 + v_3(t)\mathbf{e}_3, \quad \mathbf{b}(t) = b_2(t)\mathbf{e}_2 + b_3(t)\mathbf{e}_3$$

- modes $v_i(t), b_i(t)$ satisfy the equations:

$$\begin{aligned} \partial_t v_2(t) &= i\gamma b_2(t), \qquad \partial_t b_2(t) = i\gamma v_2(t) \\ \partial_t v_3(t) &= i\gamma b_3(t), \qquad \partial_t b_3(t) = i\gamma v_3(t) + 2i\lambda b_2(t) \end{aligned}$$

- solutions:

$$\begin{aligned} b_2(t) &= -v_2(t) = b_2 e^{-i\gamma t} \\ b_3(t) &= [b_3 + i\lambda b_2\, t]e^{-i\gamma t}, \qquad v_3(t) = [-b_3 - \lambda/\gamma b_2 - i\lambda b_2\, t]e^{-i\gamma t} \end{aligned}$$

Physical interpretation

- Generation of uniform magnetic field -turbulent dynamo
- Turbulent dynamo - result of spontaneous symmetry breaking
- Accompanying physical effect: Appearance of disturbances in Alfven waves perpendicular to spontaneous field **B** which leads to their linear growth in time
- It produces long-lived pulses $t\exp(-u\nu k^2 t)$ - like goldstone boson