Effective Potential and Conformal Symmetry in φ^4

I.V. Anikin
(JINR, Dubna)

DSPIN-23 (in memorial of Prof. A.V. Efremov)
September, 7th 2023
Dubna

(a) $\boxed{V_{eff}(m|\varphi_c) : \left\{ m^2 \text{ in prop.} \Rightarrow m^2 \text{ in vertex} \Rightarrow \text{massless prop.} \Rightarrow \text{CS} \right\}}$

(b) $\boxed{V_{eff}(m|\varphi_c) \Rightarrow \text{massless G-I vacuum integration} \Rightarrow \delta(0)}$

(c) $\boxed{\Gamma^{(l)}[\varphi_c] \sim \int_{\text{UV}} \frac{(d^D k)}{(k^2)^2} \sim \frac{\pi^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \frac{1}{\varepsilon}\Big|_{\varepsilon\to 0}}$

(d) Gorishnii-Isaev's vacuum integration:

$$\boxed{\Gamma^{(l)}[\varphi_c] = \frac{[\lambda_0^{(a)}]^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{n=1}^{\infty} \delta(n-2+\varepsilon) = \frac{[\lambda_0^{(a)}]^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \delta(\varepsilon)} \quad \Longrightarrow$$

$$\boxed{\delta(0) \equiv \lim_{\varepsilon\to 0} \frac{a_{(l)}}{\varepsilon}}$$

where $a_{(l)}$ can be fixed by the PRs.

(e) The contributions of the first four diagrams are fixed, while the last diagram can be related to the corresponding Green function within Braun-Manashov's approach, which is based on the CS use, and it is evaluated order by order.

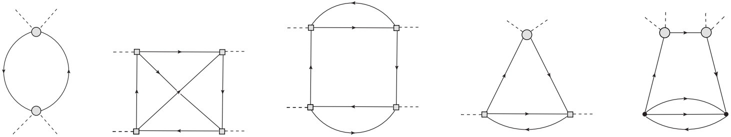

Figure: The diagrams contributing up to the order of $[\lambda]^4$.

We show that the advantages of BM-approach can be used to derive algebraically the evolution of effective potential $\Gamma[\varphi_c]$ at any loop accuracy.

To this purpose, we have to relate the effective potential to the corresponding Green functions which have been studied in Braun-Manashov:2013.

We introduce and define the vacuum $V_{z,x}$-procedure as

$$\boxed{\Gamma^{(n)}[\varphi_c] = \frac{1}{C^{(n)}(D)} V_{z,x}\Big\{ G^{(n)}_{\mathcal{O}}(x_1, x_2; z_1, z_2) \Big\}}$$

- As demonstrated in Braun-Manashov:2013, the certain constraints which are stemmed from the conformal symmetry can be expressed in terms of the deformed generators of the collinear $SL(2)$ subgroup.
- The conformal symmetry is manifested in the critical regime where $\beta(\lambda_*) = 0$.
- S_- and S_0 can be defined at all loops with the help of the evolution kernel.
- The special conformal generator S_+ involves the nontrivial corrections and it can be calculated order by order in perturbation theory.

Provided the generator S_+ is known at the order of $(\ell - 1)$ loop, the corresponding evolution kernel in the physical dimension can be fixed to the ℓ-loop accuracy.
In other words, BM-approach allows us to derive the corresponding anomalous dimensions at the given ℓ-loop accuracy practically without direct calculations but using the algebraic recurrent relations originated from the conformal symmetry properties.

We present the study of the multi-loop effective potential evolution in φ^4-theory using the conformal symmetry.

We demonstrate that the conformal symmetry can still be useful for the effective potential approach even at the presence of the mass parameter.

To this goal, it is necessary to introduce the special treatment of the mass terms as sorts of interaction in an asymptotical expansion of the generating functional. The introduced vacuum $V_{z,x}$-operation is the main tool to the algebraic scheme of anomalous dimension calculations. It is shown that the vacuum $V_{z,x}$-operation transforms the given Green functions to the corresponding vacuum integrations which generate the effective potential.

The generating functional in φ^4 which leads to the effective action/potential has the following form (modulo the normalization constants denoted as *n.c.*):

$$\mathbb{Z}[J] \overset{n.c.}{=} e^{iS_I(\frac{\delta}{\delta J})}\mathbb{Z}_0[J] = \int (\mathcal{D}\varphi)\, e^{iS(\varphi)+i(J,\,\varphi)},$$

$$\mathbb{Z}_0[J] = \mathcal{N} e^{(J,\Delta_F\, J)} = \int (\mathcal{D}\varphi)\, e^{iS_0(\varphi)+i(J,\,\varphi)},$$

where Δ_F implies the Feynman propagator;

$$S(\varphi) = S_0(\varphi; m) + S_I(\varphi)$$

denotes the sum of free and interaction actions.

The stationary phase method applied to $\mathbb{Z}[J]$ gives the following series

$$\mathbb{Z}[J] = e^{iS(\varphi_c)+i(J,\,\varphi_c)} \int (\mathcal{D}\eta) e^{-\frac{i}{2}(\eta,\,\Box\eta)} \exp\Big\{ -i \sum_{n=2}^{4} \frac{[\lambda]_n}{n!} (1, \eta^n) \Big\}$$

$$= e^{iS(\varphi_c)+i(J,\,\varphi_c)} P_\eta \exp\big\{ V(\eta) \big\}\Big|_{\eta=0} \quad \text{with} \quad P_\eta \equiv \exp\Big\{ \frac{1}{2} \big(\frac{\delta}{\delta\eta}, \Delta_F \frac{\delta}{\delta\eta} \big) \Big\}$$

where $\eta = \varphi - \varphi_c$ with

$$\lim_{J\to 0} \varphi_c(x) \equiv \lim_{J\to 0} \langle 0|\varphi|0\rangle^J = \varphi_c = const$$

We use the notations:

$$(a, Kb) = \int dz_1\, dz_2\, a(z_1) K(z_1, z_2) b(z_2)$$

This expansion should actually be considered as an asymptotical series and all inner lines correspond to the scalar *massless* propagators. Besides, the generating function generates the vertices which are

$$(a) \quad \Rightarrow [\lambda]_2\eta^2 \equiv \lambda_0^{(a)}\eta^2 \stackrel{\text{def}}{=} \big(m_0^2 + \lambda_0\varphi_c^2/2\big)\eta^2;$$

$$(b) \quad \Rightarrow [\lambda]_3\eta^3 \equiv \lambda_0^{(b)}\eta^3 \stackrel{\text{def}}{=} \lambda_0\varphi_c\eta^3;$$

$$(c) \quad \Rightarrow [\lambda]_4\eta^4 \stackrel{\text{def}}{=} \lambda_0\eta^4.$$

The mass and coupling constant (charge) are bare ones. It is worth to note that the vertices (*a*) and (*b*) should be treated as effective ones, while (*c*) corresponds to the standard vertex in the φ^4-theory under consideration.

The connected generalizing functional $\mathbb{W}[J]$ is related to the effective action $\Gamma[\varphi]$ as (the Legendre transformations)

$$\Gamma[\varphi] = \mathbb{W}[J] - i(J,\, \varphi).$$

Based on the generating functional and on the Legendre transform, we can readily derive the expression for the effective action/potential. Symbolically, we have

$$\Gamma[\varphi_c] = S(\varphi_c) + \Big\{ n\text{-loop connected diagrams} \Big\},$$

where the term of $\ln \big[(\det \widehat{\Box})^{-1/2}\big]$, which corresponds to the one-loop standard diagram contribution only, does not actually contribute in the massless propagator case.

The second term involves the full set of the connected diagrams which can be grouped as follows:

- *(a)* the standard diagrams in φ^4 with the $[\lambda]^n$-vertices only. The standard vacuum diagrams with $[\lambda]^n$-vertices do not depend on φ_c and, therefore, they can be omitted at the moment.;
- *(b)* the non-standard diagrams of type-*I* with the $[\lambda^{(a)}]^n$-vertices only;
- *(c)* the non-standard diagrams of type-*II* with the $[\lambda^{(b)}]^{2n}$-vertices only;
- *(d)* the diagrams of type-*III* with the mixed vertices as $[\lambda^{(a)}]^{n_1}[\lambda^{(b)}]^{n_2}[\lambda]^{n_3}$.

The singular parts should be eliminated by the corresponding counterterms within the certain renormalization procedure resulting in the appearance of dimensional parameter (scale) μ.
The evolution of effective action/potential with respect to the different scale choice is governed by the corresponding anomalous dimension. That is, we ultimately deal with the following effective action

$$\Gamma[\varphi_c] = \sum_n \Gamma_n[\varphi_c] \equiv \sum_n a_n \Gamma_n(0)\varphi_c^n(x) = \sum_{n=2,4} a_n \Gamma_n(0)\varphi_c^n(x) + ...$$

and

$$\Gamma_n[\varphi_c]\Big|_{\mu_1} = \Gamma_n[\varphi_c]\Big|_{\mu_2} \exp\Big\{ \int_{\mu_2}^{\mu_1} (dt)\gamma_{\Gamma_n} \Big\}.$$

$\Gamma_n(0)$ denotes the 1PI (vertex) Green functions, a_n implies the combinatory factors, see below.

The non-standard diagrams of type-I contribute only to the one-loop approximation. For this type of diagrams, we have the following representation ($D = 4 - 2\varepsilon$)

$$\Gamma^{(I)}[\varphi_c] = \sum_{n=1}^{\infty} \int (d^D k) \frac{[\lambda^{(a)}]^n}{(k^2)^n} = \sum_{n=1}^{\infty} \frac{[\lambda^{(a)}]^n}{\Gamma(n)} \delta\big(n - D/2\big) = \frac{[\lambda^{(a)}]^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \delta(0).$$

The delta-function has been considered within the sequential approach [Antosik:1973] where the singularity/uncertainty of $\delta(0)$ should be treated as the singularity of corresponding meromorphic function, *i.e.*

$$\delta(0) \equiv \lim_{\varepsilon \to 0} [1/\varepsilon].$$

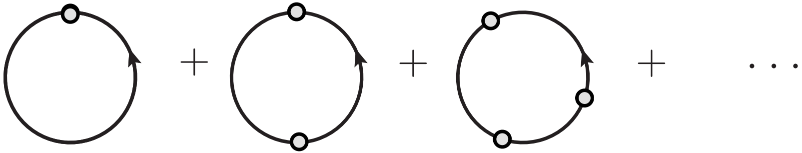

Figure: The diagrams of type-I.

In the vacuum integration series we focus on the ultraviolet divergency only, otherwise the vacuum massless integrations are nullified after the infrared divergency has been included. Then, $\Gamma^{(I)}[\varphi_c]$ receives the only contribution which goes from the following integration [Grozin:2005] (here $D = 4 - 2\varepsilon$)

$$\Gamma^{(I)}[\varphi_c] = [\lambda_0^{(a)}]^2 \int_{\text{UV}} \frac{(d^D k)}{(k^2)^2} \equiv [\lambda_0^{(a)}]^2 \frac{\pi^{D/2}}{\Gamma(D/2)} \int_{\mu^2}^{\infty} d\beta \beta^{D/2-3} =$$

$$[\lambda_0^{(a)}]^2 \frac{\pi^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \frac{\mu^{-2\varepsilon}}{\varepsilon}\Big|_{\varepsilon\to 0} = [\lambda_0^{(a)}]^{2-\varepsilon} \frac{\pi^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \frac{1}{\varepsilon}\Big|_{\varepsilon\to 0},$$

where $\beta = |k|^2$ and μ^2 has been chosen to be equal to $\lambda_0^{(a)}$.

On the other hand, let us calculate the series related to $\Gamma^{(I)}[\varphi_c]$ with the help of the vacuum integration technique [Gorishnii-Isaev:1984]. We obtain that

$$\Gamma^{(I)}[\varphi_c] = \sum_{n=1}^{\infty} \int (d^D k) \frac{[\lambda_0^{(a)}]^n}{(k^2)^n} = \frac{[\lambda_0^{(a)}]^{D/2}}{\Gamma(D/2)} \sum_{n=1}^{\infty} \delta\big(n - D/2\big)$$

$$= \frac{[\lambda_0^{(a)}]^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{n=1}^{\infty} \delta\big(n - 2 + \varepsilon\big) = \frac{[\lambda_0^{(a)}]^{2-\varepsilon}}{\Gamma(2-\varepsilon)} \delta\big(\varepsilon\big)$$

It involves the singular generated function (distribution) $\delta(\varepsilon)$ that is the well-defined functional on the finite ϕ-function space with the integration measure $d\mu(\varepsilon) = d\varepsilon\,\phi(\varepsilon)$. Nonetheless, in many cases it is not convenient, from the technical viewpoint, to introduce the space with the measure $d\mu(\varepsilon)$.

Both eqns. should be equivalent (these equations are merely different representations of the given diagram), it hints to use the sequential approach [Antosik:1973] to the delta-function and, as consequence, to the treatment of $\delta(0)$-singularity/uncertintity. In other words, we may say that $\delta(0)$ is only a symbol of the limit given by $\lim_{\varepsilon\to 0}[1/\varepsilon]$.

The non-standard diagrams of type-*II* are given by the following set. One can see that among all these diagrams the non-zero contribution is stemmed from the three-loop box-like diagram that reads [All *G*-functions are determined as in Grozin:2005.]

$$\Gamma^{(II)}[\varphi_c] = G(1,1,1,1,1)\,\frac{[\lambda^{(b)}]^4}{3\,\Gamma(2+2\varepsilon)}\delta(0).$$

Indeed, the general structure of the sum can be presented as

$$\Gamma^{(II)}[\varphi_c] \sim \sum_{n=1}^{\infty}[\lambda^{(b)}]^{2n}\,\delta\big(3n-(n+1)D/2\big).$$

It shows that the only contribution originates from the case of $n=2$ that gives $\delta(6-3D/2) \sim \delta(3\varepsilon)$.

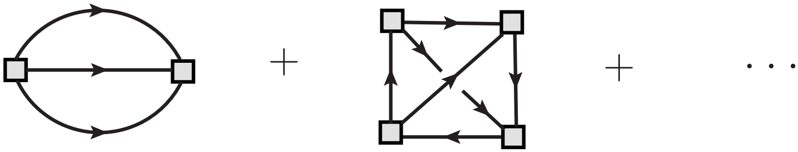

Figure: The diagrams of type-*II*.

The mixed diagrams of type-*III* can be aggregated into two classes.

The first class of diagrams with $n_1 = n,\ n_2 = 2,\ n_3 = 0$ leads to the two-loop contributions which are given by

$$
\begin{aligned}
\Gamma_1^{(III)}[\varphi_c] &= [\lambda^{(b)}]^2 \sum_{n=1}^{\infty} \int (d^D k) \frac{[\lambda^{(a)}]^n}{(k^2)^{n+1}} \int \frac{(d^D \ell)}{\ell^2 (\ell - k)^2} \\
&\sim [\lambda^{(b)}]^2 \sum_{n=1}^{\infty} [\lambda^{(a)}]^n \delta\big(n + 3 - D\big) \\
&= [\lambda^{(b)}]^2\, G(1,1) \frac{[\lambda^{(a)}]^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \delta(0)
\end{aligned}
$$

The second class of diagrams with $n_1 = n,\ n_2 = 0,\ n_3 = 2$ can be presented in the form of three-loop integration as

$$
\begin{aligned}
\Gamma_2^{(III)}[\varphi_c] &= [\lambda]^2 \sum_{n=1}^{\infty} \int (d^D k) \frac{[\lambda^{(a)}]^n}{(k^2)^{n+1}} \int \frac{(d^D \ell)}{\ell^2} \int \frac{(d^D p)}{p^2 (k+p-\ell)^2} \\
&\sim [\lambda]^2 \sum_{n=1}^{\infty} [\lambda^{(a)}]^n \delta\big(n + 4 - 3D/2\big) \\
&= [\lambda]^2\, G(1,1)\, G(1,\varepsilon)\, \frac{[\lambda^{(a)}]^{2-3\varepsilon}}{\Gamma(2-\varepsilon)}\, \delta(0).
\end{aligned}
$$

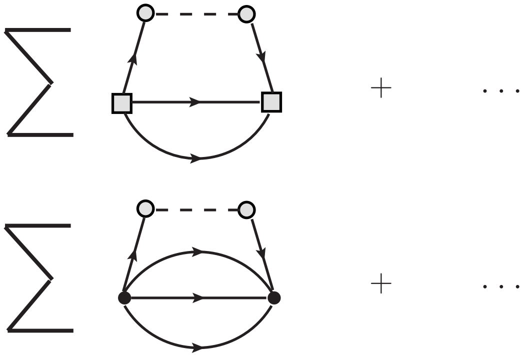

Figure: The diagrams of type-*III*: the first panel corresponds to the first class, the second – the second class.

- We are only restricted by the order of $[\lambda]^4$ for the connected diagrams. Though, the main features of calculations have a rather general character.
- The contributions of $\Gamma^{(I)}[\varphi_c]$, $\Gamma^{(II)}[\varphi_c]$ and $\Gamma_1^{(III)}[\varphi_c]$ are uniquely fixed. That is, they only contribute to the definite order of $[\lambda]^k$ ($k = 2, 4, 3$ respectively).
- In contrast to these contributions, $\Gamma_2^{(III)}[\varphi_c]$ can involve the higher order of $[\lambda]^k$ with $k \geq 4$.

As usual, the singular parts of the diagram contributions given by $\Gamma^{(i)}[\varphi_c]$ generate the corresponding Z-factor needed for the mass and charge renormalizations [Anikin:2023]. The anomalous dimensions are determined through the coefficients $c_1(\lambda)$ at the $1/\varepsilon$-singularities. In the simplest case of lowest loop accuracy, it is not difficult to calculate the anomalous dimensions immediately. However, the highest loop (multi-loop) accuracy demands rather a lot of works.

We have found that the contribution of diagram given by $\Gamma_2^{(III)}[\varphi_c]$ to the anomalous dimension can be computed practically algebraic based on the known anomalous dimension of the corresponding non-local operator Green function $G^{(2)}_{\mathcal{O}}$ computed within Braun-Manashov's approach Braun:2013. It can be implemented due to the vacuum $V_{z,x}$-operation Anikin:2023.

Let us discuss a formal transformation of action/potential with masses to the massless (conformal-invariant) object. We remind that the effective potential is a part of the effective action which does not involve the derivatives over fields. Therefore, if $\varphi_c = const$, the effective action is equivalent to the effective potential modulo $V \times T \sim \delta^{(D)}(0)$.
In the case of $J \neq 0$, we consider the effective action/potential given by the one-particle-irreducible (1PI) Green functions as

$$\Gamma[\varphi_c] = \sum_N \int (dx)_n \, \Gamma_n(x_1, ..., x_n) \, \varphi_c(x_1)...\varphi_c(x_n),$$

where the 1PI Green functions in x-space are transforming to the corresponding Green functions in p-space with the nullified external momenta giving the vacuum diagrams, *i.e.*

$$\Gamma_n(x_1, ..., x_n) \stackrel{\mathcal{F}}{=} \Gamma_n(p_1, ..., p_n)\Big|_{p_i=0} \equiv \Gamma_n(0),$$

where $\stackrel{\mathcal{F}}{=}$ denotes the Fourier transform.

As above-mentioned, the theory under our discussion contains masses (or massive parameters) that destroy the conformal symmetry even at the classical level. Since we adhere the approach with small mass and coupling constant, it is legitimated to include the massive parameters in the vertices forming the effective interactions. As a result, the scalar propagators in diagrams describing interactions are massless ones. We emphasize that this diagram technique is absolutely equivalent to the usual technique with the standard λ-interaction vertex in $\{\varphi^4\}_D$ and with the massive propagators.

Further, we focus on the simplest vacuum diagram with one $\lambda^{(a)}$ vertex and one massless scalar propagator (this is the so-called tadpole-like contribution). If we now remove the dimensionful vertices by the corresponding differentiation, we can get the conformal invariant object determined by the massless scalar propagator, *i.e.*

$$\frac{d\Gamma_1^{(a)}(0)}{d\lambda^{(a)}} = \Gamma_1^{(\eta^2)}(0) \stackrel{\mathcal{F}}{=} \Delta_F(0).$$

The other illustrative example is provided by the Green function $\Gamma_3^{(a)(b)}(0)$ which corresponds to the vacuum diagram with one $\lambda^{(a)}$ and two $\lambda^{(b)}$ vertices. The loop integration of this diagram reminds the 2-loop diagram in the massless $\{\varphi^4\}_D$ case. If we again remove the dimensionful vertices, we obtain

$$\frac{d^3\Gamma_3^{(a)(b)}(0)}{d\lambda^{(a)}d\lambda^{(b)\,2}} = \Gamma_3^{(\eta^2)(\eta^3)}(0),$$

where $\Gamma_3^{(\eta^2)(\eta^3)}(0)$ is the conformal invariant object as well.

We introduce the vacuum $V_{z,x}$-procedure which transforms the usual Green functions to the vacuum integrations. It reads

$$\Gamma^{(n)}[\varphi_c] = \frac{1}{C^{(n)}(D)} V_{z,x}\Big\{ G^{(n)}_{\mathcal{O}}(x_1, x_2; z_1, z_2)\Big\},$$

where $C^{(n)}(D)$ denotes the combination of Γ-functions and

$$V_{z,x}\Big\{ G^{(n)}_{\mathcal{O}}(x_1, x_2; z_1, z_2)\Big\} \stackrel{\text{def}}{=}$$
$$[\lambda^{(a)}]^{3D/2-4} \int d^D z_1\, d^D z_2 \Delta_F(z_1 - z_2)$$
$$\times\Big[\int d^D x_1 d^D x_2 \delta(x_1 - x_2)\widehat{\Box}_{x_2} \Big\{ G^{(n)}_{\mathcal{O}}(x_1, x_2; z_1, z_2)\Big\}\Big].$$

In the interaction representation, the non-local operator Green function $G^{(n)}_{\mathcal{O}}(x_1, x_2; z_1, z_2)$ to $[\lambda]^n$-order reads

$$G^{(n)}_{\mathcal{O}}(x_1, x_2; z_1, z_2) = \langle 0| T\eta(x_1)\eta(x_2)\,\mathcal{O}(z_1, z_2)\Big([\lambda]\int d^D y \eta^4(y)\Big)^n|0\rangle,$$

with

$$\mathcal{O}(z_1, z_2) = \eta(z_1)\eta(z_2).$$

On the other hand, $G_{\mathcal{O}}^{(n)}(x_1, x_2; z_1, z_2)$ can be written as

$$G_{\mathcal{O}}^{(n)}(x_1, x_2; z_1, z_2) = \langle \mathcal{O}(z_1, z_2) \rangle^{(n)} \Big|^{\eta(z_{12}^{\alpha_1}) \to \Delta_F(x_1 - z_{12}^{\alpha_1})}_{\eta(z_{21}^{\alpha_2}) \to \Delta_F(z_{21}^{\alpha_2} - x_2)},$$

where the correlator of non-local operator is defined as

$$\langle \mathcal{O}(z_1, z_2) \rangle^{(n)} = \langle 0 | T \mathcal{O}(z_1, z_2) \Big([\lambda] \int d^D y \eta^4(y) \Big)^n | 0 \rangle.$$

We emphasize that $\langle \mathcal{O}(z_1, z_2) \rangle$ is now the BM-like object which we need for our consideration.

For the sake of simplicity, our consideration begins with the $[\lambda]^2$-order, *i.e.* $n = 2$. In this case, we can write as

$$\Gamma^{(2)}[\varphi_c] = V_{z,x}\Big\{\langle\mathcal{O}(z_1, z_2)\rangle^{(2)}\Big|^{\eta(z_{12}^{\alpha_1})\to\Delta_F(x_1 - z_{12}^{\alpha_1})}_{\eta(z_{21}^{\alpha_2})\to\Delta_F(z_{21}^{\alpha_2} - x_2)}\Big\}.$$

For the non-local correlator $\mathcal{O}(z_1, z_2)$, one can calculate the anomalous dimension using the Braun-Manashov method.

We are going over to the discussion of the evolution equation for the effective potential.
Let us now consider the diagram presented by the loop integration $\Gamma^{(2)}[\varphi_c]$.
The conformal BM-object can be obtain by the differentiation as

$$\overline{\Gamma^{(2)}[\varphi_c]} = \frac{\partial^2 \Gamma^{(2)}[\varphi_c]}{\partial \lambda^{(a)\,2}} = \overline{V}_{z,x} \Big\{ \langle \mathcal{O}(z_1, z_2) \rangle^{(2)} \Big|^{\eta(z_{12}^{\alpha_1}) \to \Delta_F(x_1 - z_{12}^{\alpha_1})}_{\eta(z_{21}^{\alpha_2}) \to \Delta_F(z_{21}^{\alpha_2} - x_2)} \Big\}.$$

$\mathcal{O}(z_1, z_2)$ can be treated as a subject of BM-approach. First, we calculate the anomalous dimension of $\mathcal{O}(z_1, z_2)$ at the order of $[\lambda]^2$, we have

$$\langle \mathcal{O}(z_1, z_2)\rangle^{(2)} \Rightarrow \frac{1}{\varepsilon}\left[\mathbb{H}^{(2)}_{12}\mathcal{O}\right](z_1, z_2).$$

Then, we apply the $V_{z,x}$-operation to get the coefficient c_2^{Γ} at $1/\varepsilon^2$-singularity in the effective potential $\Gamma^{(2)}[\varphi_c]$, *i.e.* it reads

$$c_1^{\mathcal{O}} = \left[\mathbb{H}^{(2)}_{12}\mathcal{O}\right] \stackrel{V_{z,x}}{\Longrightarrow} c_2^{\Gamma}.$$

The last step is to use the corresponding pole relations written for the effective potential in order to obtain the anomalous dimension (evolution kernel) for $\Gamma[\varphi_c]$, we have

$$c_1^{\Gamma} = \mathrm{P}(c_2^{\Gamma}) \equiv \left[\mathbb{H}^{(2)}\Gamma[\varphi_c]\right].$$

The operator P is entirely defined by the pole relations. In its turn, the pole relations are stemmed from the $\mu\partial_\mu$-differentiation of the effective potential Z-factors, Z_m and Z_λ.

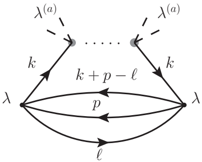

Figure: The diagram with $[\lambda]^2[\lambda^{(a)}]^n$-vertices.

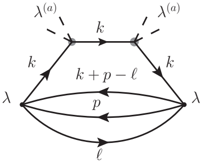

Figure: The diagram with $[\lambda]^2[\lambda^{(a)}]^2$-vertices.

We study the important consequences of the pole relations that not only relate the different coefficient c_i, but they can fix the arbitrary constants $a_{(i)}$. We begin with the schematic derivation of the pole relations. The pole relations for $\Gamma[\varphi_c]$ are stemmed from the $\mu\partial_\mu$-differentiation of the effective potential Z-factors, Z_m and Z_λ, defined as

$$\Gamma_0[\varphi_c] = Z^{\Gamma[\varphi_c]}\Gamma[\varphi_c], \quad Z^{\Gamma[\varphi_c]} = 1 + \sum_{n=1}^{\infty} \frac{C_n([\lambda])}{\varepsilon^n}.$$

Having calculated $\mu\partial_\mu$-derivative of Z-factor, we obtain that

$$\Big\{1+\sum_{n=1}^{\infty}\frac{C_n([\lambda])}{\varepsilon^n}\Big\}\gamma_{\Gamma[\varphi_c]}=\beta_\lambda([\lambda])\partial_\lambda\sum_{n=1}^{\infty}\frac{C_n([\lambda])}{\varepsilon^n}\ \text{ with }\ \gamma_{\Gamma[\varphi_c]}\equiv\mu\partial_\mu\ln Z^{\Gamma[\varphi_c]}$$

and, as a consequence, we have the following pole relations

$$\begin{aligned}\text{at }\varepsilon^0 &: \quad \gamma_{\Gamma[\varphi_c]}=-\lambda\partial_\lambda C_1(\lambda),\\ \text{at }\varepsilon^{-1} &: \quad C_1(\lambda)\gamma_{\Gamma[\varphi_c]}=-\lambda\partial_\lambda C_2(\lambda)+\beta_4\partial_\lambda C_1(\lambda),\quad \text{etc.}\end{aligned}$$

From one hand, it gives the definition of the P_Γ-operator. At the same time, from the other hand, the pole relations allow us to fix the uncertainties associated with the $\delta(0)$-singularity. Let us first write down the charge and massive terms of $\Gamma[\varphi_c]$ in the form of

$$\Gamma_{2,4}[\varphi_c] = \begin{pmatrix} m_0^2\varphi_c^2/2 \\ \mu^{2\varepsilon}\lambda_0\varphi_c^4/4! \end{pmatrix} \Big\{ 1 + \begin{pmatrix} d_{(I)}^{\{m\}} \\ d_{(I)}^{\{\lambda\}} \end{pmatrix} \lambda\, Z_\lambda(\lambda)$$

$$+ \begin{pmatrix} d_{(III,A)}^{\{m\}} \\ d_{(III,A)}^{\{\lambda\}} \end{pmatrix} \lambda^2\, Z_\lambda^2(\lambda)\, G(1,1)$$

$$+ \begin{pmatrix} d_{(III,B)}^{\{m\}} \\ d_{(III,B)}^{\{\lambda\}} \end{pmatrix} \lambda^3\, Z_\lambda^3(\lambda)\, G(1,1) G(1,2-D/2) \Big\} \frac{\delta(0)}{\Gamma(2)}$$

$$\equiv \begin{pmatrix} Z_m^{-1}(\lambda)\; m_0^2\varphi_c^2/2 \\ Z_\lambda^{-1}(\lambda)\; \mu^{2\varepsilon}\lambda_0\varphi_c^4/4! \end{pmatrix},$$

where $d_{(i)}^{\{m;\lambda\}}$ denote the numerical coefficients associated with the massive and charge terms of the given diagrams and the charge has been re-expressed via the renormalized quantity in the diagram contributions which are forming the Z^{-1}-factor.

It is convenient to rewrite the above-mentioned equation as

$$\Gamma[\varphi_c] = \sum_{i=(I)\ldots} \lambda^{n_{(i)}} Z_\lambda^{n_{(i)}}(\lambda)\, F^{(i)}(\Gamma;\varepsilon)\delta(0),$$

and

$$\begin{aligned}
F^{(I)}(\Gamma;\varepsilon) &= a_0 + a_1\varepsilon + a_2\varepsilon^2 + o(\varepsilon^3),\\
F^{(III,A)}(\Gamma;\varepsilon) &= \frac{b_{-1}}{\varepsilon} + b_0 + b_1\varepsilon + b_2\varepsilon^2 + o(\varepsilon^3),\\
F^{(III,B)}(\Gamma;\varepsilon) &= \frac{c_{-1}}{\varepsilon} + c_0 + c_1\varepsilon + c_2\varepsilon^2 + o(\varepsilon^3).
\end{aligned}$$

We here take into account the possibility of dimensional extension for the pre-delta functions mentioned above.

At the order of $[\lambda]^2$, focusing on the $1/\varepsilon^2$- and $1/\varepsilon$-singularities, the pole relations generate the following relation

$$C_{22}^{\{\lambda\}} = \left(C_{11}^{\{\lambda\}}\right)^2$$

which leads to the relation given by

$$a_{(III,A)}^{\{\lambda\}} b_{-1} = \left(a_{(I)}^{\{\lambda\}}\right)^2 a_0^2,$$

where b_1 and a_0 are known from the direct calculations, while $a_{(III,A)}^{\{\lambda\}}$ and $a_{(I)}^{\{\lambda\}}$ have to be determined. Without loosing the generality, one can normalize the effective action/potential in order to get $a_{(I)}^{\{\lambda\}} = 1$ for the diagram of I-type. Hence, the constant $a_{(III,A)}^{\{\lambda\}}$ can be readily fixed.

In the similar way, the pole relations for Z_m-factor give

$$2C_{22}^{\{m\}} = \left(C_{11}^{\{m\}}\right)^2 + C_{11}^{\{\lambda\}} C_{11}^{\{m\}}$$

and, hence, the uncertainty fixing relation takes the form of

$$2a_{(III,A)}^{\{m\}} b_{-1} = a_{(I)}^{\{m\}} \left(a_{(I)}^{\{\lambda\}} + a_{(I)}^{\{m\}}\right) a_0^2.$$

The coefficients $a_{(i)}^{\{m\}}$ and $a_{(i)}^{\{\lambda\}}$ have been chosen to be different ones. However, there is an extra condition which can re-express one coefficient from another. Based on the stationary method, we have the functional extremum condition as $\delta\Gamma[\varphi_c]/\delta\varphi_c = 0$ that leads to $m^2 + \lambda\varphi_c^2/6 = 0$. As a result, the coefficients $a_{(i)}^{\{m\}}$ and $a_{(i)}^{\{\lambda\}}$ cannot be independent ones.

As the next step, concentrating on the order of $[\lambda]^3$ we can readily calculate the coefficient giving the anomalous dimension. We obtain that

$$C_{12}^{\{\lambda\}} = \frac{3}{7}\,\frac{C_{23}^{\{\lambda\}}}{C_{11}^{\{\lambda\}}}$$

which defines also the operation P_{Γ}.

We now present the generation of $V_{z,x}$-operation to the higher order of $[\lambda]$. Let $G^{(n\geq 2)}_{\mathcal{O}}(x_1, x_2; z_1, z_2)$ be the non-local operator Green function corresponding to the higher order of $[\lambda]$. Focusing on the singular part of this function, we have

$$G^{(n\geq 2)\,\text{sing.}}_{\mathcal{O}}(x_i; z_j) = \sum_k G^{(n\geq 2)}_{\mathcal{O}}(x_i; z_j|1/\varepsilon^k)$$
$$\Rightarrow \frac{c^G_k}{\varepsilon^k} + \frac{c^G_{k-1}}{\varepsilon^{k-1}} + ... + \frac{c^G_1}{\varepsilon} + c^G_0 + o^G(\varepsilon).$$

In ε-expansion, the prefactor $C^{(n\geq 2)}(D)$ being the combination of Γ-functions has a form of series as

$$C^{(n\geq 2)}(D) = 1 + o_1(\varepsilon),$$

where $o_1(\varepsilon)$ implies the certain series over ε depending on the order but the exact form of series is irrelevant for our consideration, see below.

With these, for an arbitrary order, it takes the following form

$$\overline{\Gamma}^{(k\geq 2)}[\varphi_c] = \frac{1}{C^{(k\geq 2)}(D)}\, V_{z,x}\Big\{ G_{\mathcal{O}}^{(n\geq 2)}(x_i; z_j)\Big\}$$

or, in other words, we have

$$\frac{c^{\Gamma}_{k+1}}{\varepsilon^{k+1}} + \frac{c^{\Gamma}_{k}}{\varepsilon^{k}} + ... + \frac{c^{\Gamma}_{1}}{\varepsilon} + c^{\Gamma}_0 + o^{\Gamma}(\varepsilon) =$$

$$\Big\{1 + o_1(\varepsilon)\Big\}\, V_{z,x}\Big\{ \frac{c^{G}_{k}}{\varepsilon^{k}} + \frac{c^{G}_{k-1}}{\varepsilon^{k-1}} + ... + \frac{c^{G}_{1}}{\varepsilon} + c^{G}_0 + o^{G}(\varepsilon)\Big\}$$

$$\equiv \Big\{1 + o_1(\varepsilon)\Big\}\Big\{ \frac{c^{VG}_{k+1}}{\varepsilon^{k+1}} + \frac{c^{VG}_{k}}{\varepsilon^{k}} + ... + \frac{c^{VG}_{1}}{\varepsilon} + c^{VG}_0 + o^{VG}(\varepsilon)\Big\}.$$

Concentrating on the highest singular terms one can see that

$$c^{\Gamma}_{k+1} = c^{VG}_{k+1}.$$

Of course, such a simple relation is valid only the highest singular terms due to the universal form of C_D. For the other singular terms, one needs the exact form of expansion including the finite terms with respect to ε.
If the anomalous dimension of $G^{(n\geq 2)}_{\mathcal{O}}(x_i; z_j)$, *i.e.* the coefficient c^G_1, is somehow known, we use the pole relations to transform the coefficient c^G_1 to the coefficient c^G_k at the highest singular term and, then, we immediately get the highest singular term of $\overline{\Gamma}^{(k\geq 2)}[\varphi_c]$ with the help of $V_{z,x}$-operation.

Afterwards, we again use the pole relations for $\overline{\Gamma}^{(k\geq 2)}[\varphi_c]$ to derive the coefficient c_1^{Γ}. That is, we have the following chain of operations:

$$c_1^G \stackrel{\mathrm{P}_G}{\Longrightarrow} c_k^G \stackrel{V_{z,x}}{\Longrightarrow} c_{k+1}^{\Gamma} \stackrel{\mathrm{P}_{\Gamma}}{\Longrightarrow} c_1^{\Gamma}.$$

As a result, we can derive the anomalous dimension for the effective potential provided we know the anomalous dimension of the corresponding non-local operator Green function. It is important that this procedure is almost algebraical one which is very useful for the higher order of corrections.

We work in the frame at the critical point, *i.e.* $\lambda = \lambda_*$ and $\beta(\lambda_*) = 0$. The symmetry is extended to the dilatation and the space-time inversion forming the collinear $SL(2)$ subgroup of the conformal group.

The collinear conformal algebra can be realized by the standard way with the help of operators $\mathbb{L}_\pm$ and $\mathbb{L}_0$.

A non-local operator can be considered as a generalizing function for a local operator. For the renormalized operator we write that

$$[\mathcal{O}](z_1, z_2) = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\, [\mathcal{O}]_{Nk}$$

where $\Psi_{Nk}(z_1, z_2)$ – homogeneous polynomials of degree $N + k$, $(z_i\partial_{z_i} + N - k)\Psi_{Nk}(z_1, z_2) = 0$.

Instead of the generators $\mathbb{L}_i$ which act on the operator fields, one can introduce the operators S_α which act on the coefficient functions $\Psi_{Nk}(z_1, z_2)$ (the adjoint representation).

The generators S_α also obey the standard commutation relations, we have

$$[S_\pm, S_0] = \mp S_\pm, \quad [S_+, S_-] = 2S_0$$

with the following realization on the space of homogeneous polynomials

$$S_- \Psi_{Nk}(z_i) = -\Psi_{Nk-1}(z_i),$$
$$S_0 \Psi_{Nk}(z_i) = (j_N + k)\Psi_{Nk}(z_i),$$
$$S_+ \Psi_{Nk}(z_i) = (k+1)(2j_N + k)\Psi_{Nk+1}(z_i)$$

with

$$S_\alpha = S_\alpha^{(0)} + \Delta S_\alpha.$$

The operators S_α within a free theory have the forms of

$$S_-^{(0)} = -\sum_i \partial_{z_i}, \quad S_0^{(0)} = \sum_i z_i \partial_{z_i} + 2j, \quad S_+^{(0)} = \sum_i (z_i^2 \partial_{z_i} + 2j z_i),$$

while the interaction modifies the operators S_α by adding extra terms as

$$\Delta S_- = 0, \quad \Delta S_0 = -\varepsilon + \frac{1}{2}\mathbb{H}(\lambda_*), \quad \Delta S_+ = \sum_i z_i \Big(-\varepsilon + \frac{\lambda_*}{2}\mathbb{H}^{(1)}\Big) + O(\varepsilon^2),$$

where $\mathbb{H}$ denotes the anomalous dimension (or Hamiltonian).

Following the B-M approach,

$$[\mathbb{H}, S_\alpha^{(0)}] \neq 0$$

beyond the leading order in the interacting theory.

However, the generators S_α can be defined as a sum

$$S_\alpha^{(0)} + \Delta S_\alpha$$

which satisfy the canonical $sl(2)$ commutation relations for the theory at the critical coupling in non-integer dimensions.

The commutation relations impose certain self-consistency relations on the corrections ΔS_α.

Having expanded the relation $[S_+, \mathbb{H}(\lambda_*)] = 0$ in powers of λ_*, the following relations take finally the form of

$$[S_+^{(0)}, \mathbb{H}^{(1)}] = 0, \quad [S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, \Delta S_+^{(1)}],$$
$$[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, \Delta S_+^{(2)}] + [\mathbb{H}^{(2)}, \Delta S_+^{(1)}] \quad \text{etc.},$$

where

$$\Delta S_+ = \sum_{k=1}^{\infty} \lambda_*^k \Delta S_+^{(k)}, \quad \mathbb{H} = \sum_{k=1}^{\infty} \lambda^k \mathbb{H}^{(k)}.$$

The relations show that if the anomalous dimension $\mathbb{H}^{(k)}$ is known at the given ℓ-loop accuracy together with the representation for the corresponding deformed operator $\Delta S_+^{(m)}$, the anomalous dimension $\mathbb{H}^{(k+1)}$ at the given $(\ell + 1)$-loop accuracy can be derived almost algebraically.

For our goal, it means that with the help of $V_{z,x}$-operation we can also readily derive the evolution kernel for the effective potential $\Gamma[\varphi_c]$.

Let us suppose that the one-loop evolution kernel $\mathbb{H}^{(1)}$ has been someway calculated. Using the mentioned recurrent relations, after some algebra one can derive that the evolution kernel for the two-loop accuracy takes a form of [Braun-Manashov:2013]

$$\mathbb{H}^{(2)} = \mathcal{H}^{+} + \mathbb{F}\big(\mathcal{H}^{(d)}, \mathcal{V}^{(d,1)}\big)$$

where $\mathbb{F}(...)$ implies the combinations which do not finally contribution to the vacuum integrations, and

$$\mathcal{H}^{+} \equiv \big[\mathcal{H}^{+}\mathcal{O}\big](z_1, z_2) = \int_0^1 d\alpha_1 \int_0^{\bar{\alpha}_1} \frac{d\alpha_2}{1-\alpha_{12}} \mathcal{O}(z_{12}^{\alpha_1}, z_{21}^{\alpha_2}),$$

where $\alpha_{12...n} = \alpha_1 + ... + \alpha_n$. Due to the relative simplicity, it can be directly calculated without the usage of the recurrent relations. However, in the case of higher loop corrections, the recurrent relations are very useful because they replace rather complicated direct calculations by the almost algebraical calculations.

Next, we apply our $\overline{V}_{z,x}$-operation in order to get the coefficient $c_2^{\overline{\Gamma}}$ for the effective potential. We have the following

$$c_2^{\overline{\Gamma}} = \overline{V}_{z,x}\Big\{ \big[\mathcal{H}^+\mathcal{O}\big](z_1, z_2)\Big|^{\eta(z_{12}^{\alpha_1})\to\Delta_F(x_1 - z_{12}^{\alpha_1})}_{\eta(z_{21}^{\alpha_2})\to\Delta_F(z_{21}^{\alpha_2} - x_2)}\Big\}.$$

It now remains to insert this representation into the corresponding equation to derive the needed anomalous dimension for the effective potential.

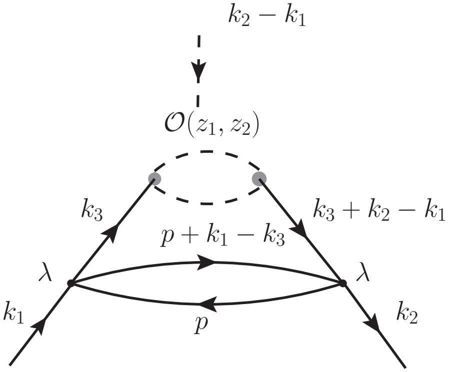

Figure: The diagram of $G^{(2)}_{\mathcal{O}}(x_1, x_2; z_1, z_2)$ at the order of $[\lambda]^2$.

- We have outlined a new approach to calculate the multi-loop effective potential evolutions in φ^4-theory using the conformal symmetry.
- We have demonstrated that the conformal symmetry can be applied for the effective potential approach even at the presence of the mass parameter. Within the stationary phase method, it becomes possible if one introduces the special treatment of the mass terms as a kind of interaction in an asymptotical expansion of the generating functional.
- It has been shown that the multi-loop evolution equations (anomalous dimensions) of the effective potential can be derived using the corresponding results of BM-approach with the help of the original vacuum $V_{z,x}$-operation [Anikin:2023]. This operation leads to the almost algebraic scheme of the anomalous dimension calculations. It is also demonstrated the important role of P-operator stemmed from the use of pole relations for Z-factors.

The proposed approach should be also considered as an alternative way to calculate the effective potential within the massive φ^4-like models.