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Spin effects in the Sommerfeld-Gamow-Sakharov factor

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Based on the paper
[Yu.D. Chernichenko, O.P. Solovtsova, L.P. Kaptari, Eur. Phys. J. Plus 136 (2021) 302]

It is well known from quantum electrodynamics (QED) that the so-called Coulomb resummation factor plays an important role in describing the system of two charged particles near the threshold [J. Schwinger, Particles, Sources and Fields, Vol. II. p. 397, for example]. The resummation performed on the basis of the nonrelativistic Schrödinger equation with the Coulomb potential

$$V(r) = -\frac{\alpha}{r},$$

where α is the fine structure constant, leads to the the known Sommerfeld (–Gamov–Sakharov) factor, *S*-factor [A. Sommerfeld, Atombau und Spectralinien (Vieweg und Sohn, Brunswick, Deutschland, 1921), Annalen der Physik 403, 257 (1931); G. Gamow, Zeit. Phys. 51, 204 (1928); A.D. Sakharov, Sov. Phys. JETP 18, 631 (1948)] which, for two particles of equal masses, reads as:

$$S_{\rm nr} = \frac{X_{\rm nr}}{1 - \exp(-X_{\rm nr})}, \qquad X_{\rm nr} = \frac{\pi\alpha}{v_{\rm nr}}, \tag{1}$$

where $v_{\rm nr}$ denotes the velocity of each particle in the center-of-mass frame.

At small values of $\alpha/v_{\rm nr}$, the S-factor (1) behaves as

$$S_{\rm nr} \sim 1 + \frac{\pi\alpha}{2v_{\rm nr}}$$

whence it follows that at low velocities the corresponding cross section increases ($2v_{\rm nr}$ is the relative velocity of two nonrelativistic particles).

As originally mentioned by Sommerfeld, this enhancement is directly related to the wave function of the relative motion in the Coulomb field evaluated at the origin by $|\psi(0)|^2$ evaluated at the origin, i.e. to the probability to find two interacting particles close to each other. Such an increase of the S-factor is often referred to as the Sommerfeld enhancement effect. A similar phenomenon was also predicted and then confirmed by Gamow [G. Gamov, Zeit. Phys. 51, 204 (1928)] in nuclear reactions when interacting particles overcome the Coulomb barrier, and by Sakharov [A.D. Sakharov, Sov. Phys. JETP 18, 631 (1948)] in electron-positron pair production.

At present, intensive studies of the Sommerfeld-effect demonstrate the crucial role it plays in understanding a large variety of inelastic processes with heavy particles moving slowly in a thermal environment. A classic example is given by astrophysical nuclear reactions taking place within the electromagnetic plasma of stars (more details of the theoretical consideration can be found, e.g., in Refs. [N. Arkani-Hamed et al. Phys. Rev. D 79 (2009) 015014, T. Binder, K. Mukaida, K. Petraki, Phys. Rev. Lett. 124 (2020) 161102].

Another manifestation of the enhancement of the Sommerfeld–Gamow–Sakharov factor can be clearly seen in abundant creations of di-lepton pairs with in heavy-ion collisions [I. M. Dremin, S. R. Gevorkyan, D. T. Madigozhin, Eur. Phys. J .C **81** (2021) 276].

Within quantum chromodynamics (QCD), an increase in the S-factor may occur in heavy quark and anti-quark pair annihilation into light quarks and gluons within a quark-gluon plasma environment generated in heavy ion collision experiments. Bound states contribute to the annihilation process, and enhancement factors of up to ~ 100 can be encountered, see, e.g., [S. Kim, M. Laine, Phys. Lett. B **795** (2019) 469]. Other examples where the Sommerfeld effect is crucial, include threshold production of heavy states at colliders and partial decay rates when the products have large phase space suppression. It should be noted that most of the theoretical studies of the S-factor have been performed within nonrelativistic approaches originating from the Schrödinger equation.

Coming back to Eq. (1) one shall stress that, in spite of the expansion of the S-factor in a power series $(\alpha/v)^n$ with respect to the coupling constant α reproduces the threshold singularities of the Feynman diagrams, in the threshold-near region, $v \to 0$, the parameter α/v can no longer be adequate to cut off the perturbative series. Consequently, the S-factor should be taken into account in its entirety.

A similar situation also arises also in QCD because in describing a quark-antiquark system near the threshold, $s_{th} = (m_q + m_{\bar{q}})^2$, the expansion parameter α_s/v becomes singular when $v \to 0$ [T. Appelquist, H. D. Politzer, Phys. Rev. Lett. **34**, 43 (1975); Phys. Rev. D **12**, 1404 (1975)]. Here α_s denotes the strong coupling, and the quark velocity v for the case of equal masses, $m_q = m_{\bar{q}} \equiv m$, reads as

$$\boxed{v = \sqrt{1 - \frac{4m^2}{s}}} \qquad (2)$$

where $\sqrt{s}$ is the total energy of the considered two particles in their c.m. system.

For a systematic relativistic analysis of quark-antiquark systems, it is essential from the very beginning to have a relativistic generalization of the S-factor.

Here it is important to note that the problem of accounting for the relativistic effects in a few body system is a more general and longstanding task. Up to now there are no reliable relativistic equations derived from the first principles to describe few body bound and/or continuum states near the threshold. For this reason, one is forced to use phenomenological or quasi-phenomenological formalisms to relativistically describe these systems.

One can mention several approaches defining the relativistic wave functions of few-body systems. One of the approaches is based on the fully covariant and Lorentz-invariant Bethe–Salpeter (BS) formalism [H.A. Bethe, E.E. Salpeter, Phys. Rev. 82 (1951) 309; Phys. Rev. 84 (1951) 1232] (see paper [Yu.D. Chernichenko, O.P. Solovtsova, L.P. Kaptari, Eur. Phys. J. Plus 136 (2021) 302] and references therein).

We use the relativistic quasipotential (RQP) approach to quantum field theory proposed by A. A. Logunov and A. N. Tavkhelidze in Ref. [A. A. Logunov, A. N. Tavkhelidze, Nuov. Cim. A **29**, 380 (1963)], in the form suggested by V. G. Kadyshevsky [V. G. Kadyshevsky, Nucl. Phys. B **6**, 125 (1968)] on the base of covariant Hamiltonian formulation of quantum field theory via a transition from the momentum representation in Lobachevsky space to the three-dimensional relativistic configuration representation. The RQP equations have the structure very similar to the Schrödinger equation; however, the interaction quasipotential is now energy-dependent.

Note here that the relativistic generalization of the S-factor is obviously not unique, since there are numerous ways of expressing the nonrelativistic velocity in terms of the relativistic energy $\sqrt{s}$.

For the first time the relativization of the S-factor (1) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in Refs. [V.S. Fadin, V.A. Khoze, Yad. Fiz. **48**, 487 (1988); V.S. Fadin, V.A. Khoze, A.D. Martin, and A. Chapovsky, Phys. Rev. D **52**, 1377 (1995)] and it consisted in the change $\boxed{v_{\rm nr} \to v}$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$.

Just the same form of the S-factor but with the change $\boxed{v_{\rm nr} \to v/(1+v^2)}$ for the interaction of two particles of equal masses was later suggested in Ref. [A.H. Hoang, Phys. Rev. D **56**, 7276 (1997)].

Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [J.H. Yoon, C.Y. Wong, Phys. Rev. C **61**, 044905 (2000); J. Phys. G: Nucl. Part. Phys. **31**, 149 (2005)].

The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was for the first time presented in Ref. [A.B. Arbuzov, Nuov. Cim. A **107**, 1263 (1994)]. This factor was derived within the framework of a version of the relativistic quantum mechanics, on the basis of the Schrödinger equation for the w. f. $\Psi(t, \mathbf{x}_1, \mathbf{x}_2)$ in a specific frame of reference, $\mathbf{p}_2 = -\mathbf{p}_1 m_2/m_1$, and treating $\mathbf{p}_{1,2}$ as differential operators.

Relativistic S-factor for spinless particles

Case of equal masses

Based on the relativistic quasipotential approach suggested by Kadyshevsky in Ref. [V.G. Kadyshevsky, Nucl. Phys. B **6**, 125 (1968)], Milton and Solovtsov [K.A. Milton, I.L. Solovtsov, Mod. Phys. Lett. A **16**, 2213 (2001)] obtained the following expression (in the case of two spinless particles of $m_1 = m_2 = m$):

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}, \quad X(\chi) = \frac{\pi\,\alpha}{\sinh\chi}, \quad \sinh\chi = \frac{v}{\sqrt{1-v^2}}, \tag{3}$$

where $\tilde{\alpha}$ denotes the coupling constant in the Coulomb-like potential $V(r) = -\tilde{\alpha}\,/r$, χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$ by $2m\cosh\chi = \sqrt{s}$.

The use of (3) in QCD requires the replacement of $\tilde{\alpha} \to 4\alpha_s/3$, where $4/3$ is due to the SU(3) color factor. Note also that the Coulomb quasipotential formally has the same form as the nonrealistic Coulomb potential since its behavior in the momentum Lobachevsky space corresponds to the static quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_s(Q^2)/Q^2$ for which $\bar{\alpha}_s(Q^2) \sim 1/\ln(Q^2)$ plays a role of effective charge [V.I. Savrin, N.B. Skachkov, Lett. Nuovo Cim. **29** (1980) 363]. Thereby, one accumulates a dominant effects induced by the running QCD coupling.

It is important to note that the use of the RQP approach for this task is based on the fact that the BS amplitude $\Phi_{\rm BS}(x)$, $x=(x_0,\mathbf{x})$, which parameterizes the physical quantity, such as the $R(s)$-ratio, $R(s)={\rm Im}\Pi(s)/\pi$, in QCD, is evaluated at $x=0$ [R. Barbieri, P. Christillin, E. Remiddi, Phys. Rev. A **8**, 2266 (1973)], and hence at the relative time $x_0=0$. Consequently, the BS amplitude can be related to the RQP wave function in momentum space, $\Psi_\chi(\mathbf{p})$, and in the configuration representation, $\psi_\chi(\mathbf{r})$, as

$$\Phi_{\rm BS}(x=0)=\frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}}\Psi_\chi(\mathbf{p})=\psi_\chi(\mathbf{r})\Big|_{r=i\lambda}, \tag{4}$$

where $\lambda=1/m$ is the Compton wavelength of the particle of mass m, $d\Omega_{\mathbf{p}}=(m d\mathbf{p})/p_0$ is the invariant space volume in the Lobachevsky space realized on the hyperboloid $p_0^2-\mathbf{p}^2=q_0^2-\mathbf{q}^2=m^2$. According to Eq. (4), the S-factor within the RQP approach is defined through the corresponding wave function in the continuum, $\psi_\chi(\mathbf{r})$, as follows (see [K.A. Milton, I.L. Solovtsov, Mod. Phys. Lett. A **16** (2001)]):

$$S_{\rm RQP}(\chi)=\lim_{r\to i\lambda}|\psi_\chi(\mathbf{r})|^2. \tag{5}$$

The nonrelativistic replacement of $\Phi_{\rm BS}(x)$ by the wave function, which obeys the Schrödinger equation with the Coulomb potential, leads to formula (1).

One can seen that the shape of the S-factor in the relativistic case remains the same as in Eq. (1), albeit with a replacement

$$v_{\rm nr} \to \frac{v}{\sqrt{1-v^2}} \qquad X_{\rm nr} \sim \frac{1}{v_{\rm nr}}, \qquad X_{\rm RPQ} \sim \frac{\sqrt{1-v^2}}{v}$$

The presence of the square root $\sqrt{1-v^2}$ in the denominator is essential since it provides the correct expected relativistic limit ($v \to 1$) of the S-factor. Notice that the relativistic limits of the S-factors in Refs. [J.H. Yoon, C.Y. Wong, Phys. Rev. C **61**, 044905 (2000); J. Phys. G: Nucl. Part. Phys. **31**, 149 (2005)] differ substantially from the relativistic limit ($v \to 1$) of the S-factor corresponding to Eq. (3) (see [O.P. Solovtsova, Yu.D. Chernichenko, Phys. At. Nucl. **73**, 1612 (2010)] for more detail).

A relativistic Coulomb-like resummation factor for arbitrary masses and orbital moment $\ell \geq 1$, called the L-factor, was presented in Ref. [O. P. Solovtsova, Yu. D. Chernichenko, Theor. Math. Phys. **166**, 194 (2011)]. Applications of the factor (3) for describing some hadronic processes can be found in Refs. [K.A. Milton, I.L. Solovtsov, O.P. Solovtsova, Phys. Rev. D **64**, 016005 (2001); Mod. Phys. Lett. A **21**, 1355 (2006)]. Also, the relativistic S-factor (3) has been applied to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron.

Case of arbitrary masses

The wave function $\Psi_{q'}(\mathbf{p}')$ within the RQP approach [V.G. Kadyshevsky, M.D. Mateev, R.M. Mir-Kasimov, Yad. Fiz. **11** (1970) 692] for two the relativistic particles of arbitrary masses m_1 and m_2 is found from the equation ($c = \hbar = 1$)

$$\left(2E_{q'} - 2E_{p'}\right) \Psi_{q'}(\mathbf{p}') = \frac{2\mu}{m'(2\pi)^3} \int d\Omega_{\mathbf{k}'} \widetilde{V}\left(\mathbf{p}', \mathbf{k}'; E_{q'}\right) \Psi_{q'}(\mathbf{k}'), \qquad (6)$$

where

$$d\Omega_{\mathbf{k}'} = \frac{m' \, d\mathbf{k}'}{E_{k'}}$$

is the relativistic three-dimensional volume element in the Lobachevsky space, $E_{k'} = \sqrt{m'^{\,2} + \mathbf{k}'^{\,2}}$, $m' = \sqrt{m_1 m_2}$; $\mu = m_1 m_2/(m_1 + m_2)$ is the usual reduced mass.

Equation (6) represents a relativistic generalization of the Schrödinger equation in the spirit of Lobachevsky geometry which is realized on the upper half of the mass hyperboloid $E_{k'}^2 - \mathbf{k}'^{\,2} = m'^{\,2}$. This equation describes the scattering over the quasipotential $\widetilde{V}\left(\mathbf{p}', \mathbf{k}'; E_{q'}\right)$ of an effective relativistic particle having mass m' and a relative 3-momentum $\mathbf{k}'$, emerging instead of the system of two particles and carrying the total c. m. energy of the interacting particles, $\sqrt{s}$, proportional to the energy $E_{k'}$ of one effective relativistic particle of mass m'

$$\sqrt{s} = \sqrt{m_1{}^2 + \mathbf{k}^2} + \sqrt{m_2{}^2 + \mathbf{k}^2} = \frac{m'}{\mu}\sqrt{m'^2 + \mathbf{k}'^2}\,.$$

Corresponding to this case the S-factor can be written as [O.P. Solovtsova, Yu.D. Chernichenko, Phys. At. Nucl. **73**, 1612 (2010)]

$$S_{\rm uneq}(u) = \frac{X_{\rm uneq}(u)}{1 - \exp\left[-X_{\rm uneq}(u)\right]}; \quad X_{\rm uneq}(u) = \frac{\pi\tilde{\alpha}\sqrt{1-u^2}}{u}, \tag{7}$$

where the subscript "uneq" indicates the quantities related to the case of unequal masses, and the velocity u is determined by the expression

$$u = \sqrt{1 - \frac{4m'^2}{s - (m_1 - m_2)^2}}. \tag{8}$$

Here $m' = \sqrt{m_1 m_2}$ is the mass of an effective particle emerging instead of the system of two particles. In the case when $m_1 = m_2 = m$, the velocity (8) reduces to the velocity v, Eq. (2).

The functions $X_{\rm uneq}$ in (7) can be expressed as

$$X_{\rm uneq} = \frac{2\pi\tilde{\alpha}}{u'_{\rm rel}}, \tag{9}$$

where the relative velocity of an effective relativistic particle with mass m' is

$$u'_{\rm rel} = \frac{2u}{\sqrt{1-u^2}}. \tag{10}$$

As within the RQP approach, the dependence between the energy of the relative motion and the relativistic relative velocity $\mathbf{v}$ is given by expression [V.G. Kadyshevsky, M.D. Mateev, R.M. Mir-Kasimov, Yad. Fiz. **11** (1970) 692]:

$$\frac{\mathbf{k}'^2}{2\mu} = \mu\left(\frac{1}{\sqrt{1-\mathbf{v}^2}} - 1\right), \tag{11}$$

that has llowed to enter the concept of an effective relativistic particle, emerging instead of the system of two particles and having mass m', the relative 3-momentum $\mathbf{k}'$ and carrying the total c. m. energy of interacting particles, $\sqrt{s}$. Notice that the relative 3-momentum $\mathbf{k}'$ of an effective relativistic particle, according to the expression (11), is invariant of the Loretz transformations.

Recall that within the RQP approach the modulus of the relative relativistic velocity $|\mathbf{v}|$ of two particles can be expressed in terms of their total c.m. energy $\sqrt{s}$ as

$$|\mathbf{v}| = 2\sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}}\left[1+\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}\right]^{-1}, \tag{12}$$

that is exactly the same as in, e.g., Ref. [A.B. Arbuzov, Nuov. Cim. A **107**, 1263 (1994)]. From Eqs. (8) and (12) one can infer that

$$|\mathbf{v}| = \frac{2u}{1+u^2}. \tag{13}$$

It is obvious that the relativistic limit $|\mathbf{v}| \to 1$, of the S-factor considered in Ref. [A.B. Arbuzov, Nuovo Cim. A **107**, 1263 (1994)]

$$S(|\mathbf{v}|) = \frac{X(|\mathbf{v}|)}{1 - \exp\left[-X|\mathbf{v}|\right)]}, \qquad X(|\mathbf{v}|) = \frac{2\pi\tilde{\alpha}}{|\mathbf{v}|}, \tag{14}$$

differs from the S-factor defined by (7) for which the relativistic limit $u \to 1$. However, for small values of u the S-factors defined by Eqs. (7) and (14) provide the same result.

Note that the behavior of the S-factor at intermediate values of v may be very important. For example, for the quantities which defined through the function $R(s)$, the normalized hadronic cross-section in e^+e^- annihilation, and integrated with some other functions:

$$a_\ell^{\rm had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_0^\infty \frac{ds}{s}\, K_\ell(s)\, R_{e^+e^-}(s)\;, \quad \Delta\alpha_{\rm had}^{(5)}(s) = -\frac{\alpha(0)}{3\pi}\, s\, \mathcal{P} \int\limits_0^\infty \frac{ds'}{s'}\, \frac{R_{e^+e^-}(s')}{s'-s}.$$

Relativistic S-factor for spinor particles

Case of equal masses

The S-factor corresponding two spin-1/2 particles (quarks) with equal masses, interacting via a Coulomb-like potential, read as [Yu.D. Chernichenko, O.P. Solovtsova, L.P. Kaptari, Eur. Phys. J. Plus 136 (2021) 302]

$$S_{\mathrm{eq,s}}(\chi) = \frac{X_{\mathrm{eq,s}}(\chi)}{1-\exp\left[-X_{\mathrm{eq,s}}(\chi)\right]} \tag{15}$$

$$\times e^{-\pi\tilde{\rho}}\left|\Gamma(2+i\tilde{\rho})\,{}_2F_1(1+iB,-i\tilde{\rho};2;1-e^{-2\chi})\right|^2,$$

where $\Gamma(2+i\tilde{\rho})$ is the familiar Euler gamma function, ${}_2F_1(1+iB,-i\tilde{\rho};2;1-e^{-2\chi})$ is the hypergeometric function and the subscript "s" to spinors. The quantities $X_{\mathrm{eq,s}}(\chi)$ and $\tilde{\rho}$, B are defined as

$$X_{\mathrm{eq,s}}(\chi) = \frac{\pi\tilde{\alpha}(a\cosh^2\chi+b)}{2\sinh\chi}, \quad \tilde{\rho} = \frac{\tilde{\alpha}a\cosh\chi}{4}, \quad B = \frac{\tilde{\alpha}(a\cosh^2\chi+b)}{4\sinh\chi}, \tag{16}$$

where the parameters a and b for different total spin of the system are:

$$\begin{aligned}
a &= 1\,, & b &= 0\,, & &\text{for pseudoscalar } (\hat{O}=\gamma_5),\\
a &= 1/2\,, & b &= 1/4\,, & &\text{for vector } (\hat{O}=\gamma_\mu),\\
a &= -1/2\,, & b &= 3/4\,, & &\text{for pseudovector } (\hat{O}=\gamma_5\gamma_\mu)\,.
\end{aligned} \tag{17}$$

The quantity $X_{\rm eq,s}(\chi)$ can be expressed in terms of the velocity in (2) as

$$X_{\rm eq,s}(v) = \frac{\pi\tilde{\alpha}_s(a+b-bv^2)}{2v\sqrt{1-v^2}}.$$

The relativistic limit, $\chi \to +\infty \, (v \to 1)$, of the S-factor (15) is

$$S_{\rm eq,s}(\chi)|_{\chi\gg 1} \simeq \frac{2\pi(B-\tilde{\rho})}{1-\exp[-2\pi(B-\tilde{\rho})]}|_{\chi\gg 1} \simeq$$

$$\simeq \frac{\pi\tilde{\alpha}_s}{4}(a+2b)e^{-\chi} \xrightarrow[\chi\to+\infty]{} 1+0.$$

In the nonrelativistic limit, $\chi \to +0 \;\; (v \to 0)$, we have the expression

$$S_{\rm eq,s}(\chi)|_{\chi\to+0} \simeq \frac{\pi\tilde{\alpha}_s(a+b)/2\sinh\chi}{1-\exp[-\pi\tilde{\alpha}_s(a+b)/2\sinh\chi]}\,\frac{\pi\tilde{\alpha}_s a/2}{\exp(\pi\tilde{\alpha}_s a/2)-1}\times$$

$$\times\left(1+\frac{\tilde{\alpha}_s^2 a^2}{16}\right)|{}_1F_1(-i\tilde{\alpha}_s a/4;2;i\tilde{\alpha}_s(a+b)/2)|^2 ,$$

where ${}_1F_1(\alpha;\beta;z)$ is the confluent hypergeometric function.

We note also that, at $a=0$ and $b=2/m$, the relativistic S-factor (15) goes over to the spinless S-factor (3), which in the nonrelativistic limit ($v \ll 1$) reproduces the well-known nonrelativistic result.

Case of arbitrary masses

The RQP equation has the following form [Yu.D. Chernichenko, L.P. Kaptari, O.P. Solovtsova, Eur. Phys. J. Plus. **136** (2021) 302]

$$\int_0^\infty d\chi \left(\cosh\chi' - \cosh\chi\right) \sin\rho\chi \int_0^\infty d\rho' \sin(\rho'\chi)\varphi_0(\rho',\chi') = \tag{18}$$

$$= -\frac{\tilde{\alpha}'_s}{\rho} \int_0^\infty d\chi \hat{A}(\cosh\chi) \sin\rho\chi \int_0^\infty d\rho' \sin(\rho'\chi)\varphi_0(\rho',\chi'),$$

where the rapidity χ' connects with $\sqrt{s}$ as

$$\sqrt{s} = 2m'g'\cosh\chi', \tag{19}$$

$\rho = r/\lambda'$ and $\lambda' = 1/m'$ is the Compton wavelength associated with the effective relativistic particle having mass m', $\tilde{\alpha}'_s = m'\alpha_s$, and the operator $\hat{A}$ is given by the expression

$$\hat{A}\left(\cosh\chi'\right) = \frac{1}{4}\left[a'\cosh^2\chi' + b'\right]$$

with spin parameters a' and b' that depends not only on the spin of particles (quarks) but also their masses across the factor g':

$$g' = \frac{m'}{2\mu} = \frac{m_1 + m_2}{2\sqrt{m_1 m_2}}, \tag{20}$$

Case of arbitrary masses

$$a' = g'^2, \quad b' = 1 - a' \quad \text{for } \hat{O} = \gamma_5 \,(\text{pseudoscalar}), \tag{21}$$

$$a' = \frac{1}{2} g'^2, \; b' = \frac{3}{4} - a' \quad \text{for } \hat{O} = \gamma_\mu \,(\text{vector}),$$

$$a' = -\frac{1}{2} g'^2, \; b' = \frac{1}{4} - a' \quad \text{for } \hat{O} = \gamma_5 \gamma_\mu \,(\text{pseudovector}).$$

If $m_1 = m_2 = m$, the parameters a', b' in (21) coincide with their analogues in (17).

In the case when the interaction vanishes, $\alpha_s \to 0$, the solution $\varphi_0(\rho, \chi')$ should reproduce the known free wave function

$$\lim_{\alpha_s \to 0} \varphi_0(\rho, \chi') \xrightarrow[\rho \to \infty]{} \frac{\sin(\rho \chi')}{\sinh \chi'}. \tag{22}$$

The resulting solution of Eq. (18) with the boundary condition (22) can be presented in the form [Yu. D. Chernichenko, Phys. At. Nucl. **84**, 339 (2021)]

$$\varphi_0(\rho, \chi') = 2\pi C_0(\chi') e^{iB'\chi' - \chi' + i(\rho - \tilde{\rho}')\chi'} (\rho - \tilde{\rho}') \times \tag{23}$$

$$\times {}_2F_1\left(1 - iB', 1 - i(\rho - \tilde{\rho}'); 2; 1 - e^{-2\chi'}\right),$$

where the real-valued factor $2\pi C_0(\chi')$ gives by the expression

$$|2\pi C_0(\chi')|^2 = e^{\pi B'}|\Gamma(1 - iB')|^2, \tag{24}$$

the parameters a' and b' are defined in Eq. (21), and the parameters $\tilde{\rho}'$ and B' are given as

$$\tilde{\rho}' = \frac{\tilde{\alpha}'_s a' \cosh \chi'}{4}, \quad B' = \frac{\tilde{\alpha}'_s (a' \cosh^2 \chi' + b')}{4 \sinh \chi'}. \tag{25}$$

The S-factor for a system of two relativistic spinor particles of arbitrary masses define by the expression [Yu.D. Chernichenko, L.P. Kaptari, O.P. Solovtsova, Eur. Phys. J. Plus. **136** (2021) 302]

$$S_{\text{uneq,s}}(\chi') = \lim_{\rho \to i} \left| e^{-\pi\tilde{\rho}'/2}\Gamma(1 + i\tilde{\rho}')\frac{\varphi_0(\rho, \chi')}{\rho} \right|^2, \tag{26}$$

and finally we get

$$S_{\text{uneq,s}}(\chi') = \frac{X_{\text{uneq,s}}(\chi')}{1 - \exp\left[-X_{\text{uneq,s}}(\chi')\right]} \tag{27}$$

$$\times e^{-\pi\tilde{\rho}'}\left|\Gamma(2 + i\tilde{\rho}')\,{}_2F_1(1 + iB', -i\tilde{\rho}'; 2; 1 - e^{-2\chi'})\right|^2$$

with

$$X_{\text{uneq,s}}(\chi') = 2\pi B' = \frac{\pi\tilde{\alpha}'_s (a' \cosh^2 \chi' + b')}{2 \sinh \chi'} \tag{28}$$

which can be expressed in terms of the velocity u, Eq. (8), and the relative velocity $u'_{\rm rel}$ of an effective relativistic particle with mass m', Eq. (10), as

$$X_{\rm uneq,s}(u) = \frac{\pi\tilde{\alpha}'_s\sqrt{1-u^2}}{2g'u}\left[g'^2\left(a'+b'\right)+\frac{a'u^2}{1-u^2}\right] = \\ = \frac{\pi\tilde{\alpha}'_s}{g'u'_{\rm rel}}\left[g'^2\left(a'+b'\right)+\frac{a'}{4}u'^2_{\rm rel}\right].$$

It should be stressed that since in RQP approach both the argument ($r = |\mathbf{r}|$) of the Coulomb-like potential $V(r) = -\tilde{\alpha}\,/r$ and the relative velocity $|\mathbf{v}|$ are relativistic invariants [V. G. Kadyshevsky, R. M. Mir-Kasimov, N. B. Skachkov, Sov. J. Part. Nucl. **2**, 69 (1972)], the S-factor (27) is manifestly relativistic invariant too. Therefore, due to the Eq. (13), the velocity u, Eq. (8), and the relative velocity $u'_{\rm rel}$ of the effective particle, Eq. (10), are also relativistic invariants. Thus, instead of the previously used relative velocity $|\mathbf{v}|$, an appropriate parameter in the S-factor (27) is the relative velocity (10) of an effective particle, which has the same properties as the considered two-particle system.

Limiting cases

In the nonrelativistic limit (at the threshold) we have expression

$$S_{\mathrm{uneq,s}}(\chi')|_{\chi'\to+0} \simeq \frac{\pi\tilde{\alpha}'_s(a'+b')/2\sinh\chi'}{1-\exp[-\pi\tilde{\alpha}'_s(a'+b')/2\sinh\chi']}\times$$

$$\times\frac{\pi\tilde{\alpha}'_s a'/2}{\exp(\pi\tilde{\alpha}'_s a'/2)-1}\left(1+\frac{\tilde{\alpha}'^2_s a'^2}{16}\right)\left|{}_1F_1(-i\tilde{\alpha}'_s a'/4;2;i\tilde{\alpha}'_s(a'+b'))/2\right|^2$$

which for $m_1 = m_2 = m$ it coincides with the equal-masses S-factor (15).

In the relativistic limit, we have expression

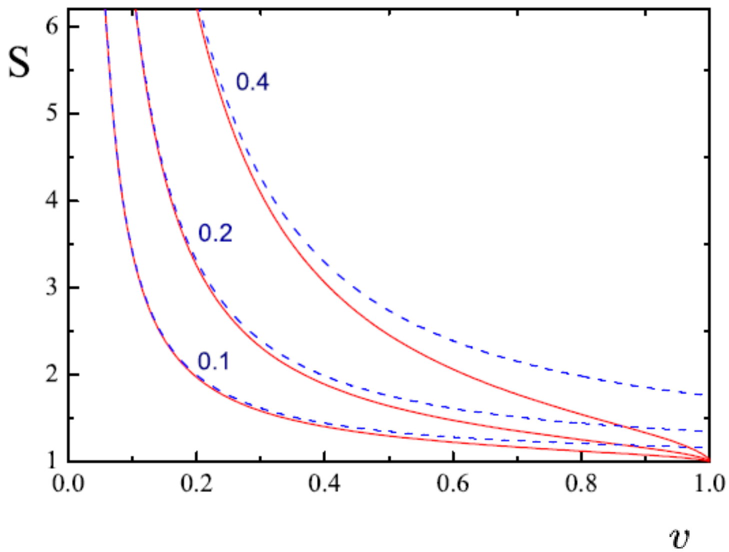
$$S_{\mathrm{uneq,s}}(\chi')|_{\chi'\gg1} \simeq \frac{2\pi(B'-\tilde{\rho}')}{1-\exp[-2\pi(B'-\tilde{\rho}')]}|_{\chi'\gg1} \simeq \tag{29}$$

$$\simeq \frac{\pi\tilde{\alpha}'_s}{4}(a'+2b')e^{-\chi'} \xrightarrow[\chi'\to+\infty]{}$$

$$\xrightarrow[\chi'\to+\infty]{} \begin{cases} 1+0 & \text{for } \hat{O}=\gamma_5 \text{ (pseudoscalar) and } 1\le g'\le\sqrt{2}, \\ 1-0 & \text{for } \hat{O}=\gamma_5 \text{ (pseudoscalar) and } g'>\sqrt{2}; \\ 1+0 & \text{for } \hat{O}=\gamma_\mu \text{ (vector) and } 1\le g'\le\sqrt{3}, \\ 1-0 & \text{for } \hat{O}=\gamma_\mu \text{ (vector) and } g'>\sqrt{3}; \\ 1+0 & \text{for } \hat{O}=\gamma_5\gamma_\mu \text{ (pseudovector) and } g'\ge 1. \end{cases}$$

This expressions are valid at all values of the spin parameters a' and b' in (21).

Illustrations:
spinless particles

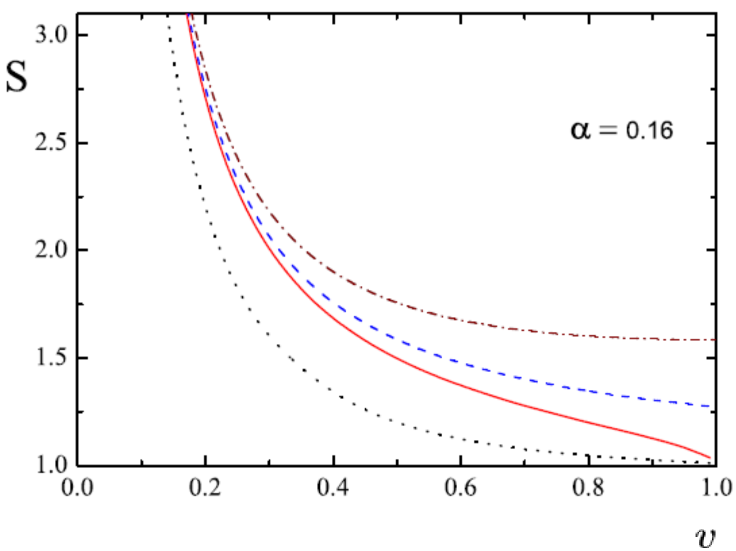


$$S_{\rm nr}=\frac{X_{\rm nr}}{1-\exp(-X_{\rm nr})}, \qquad X_{\rm nr}=\frac{\pi\alpha}{v_{\rm nr}};$$

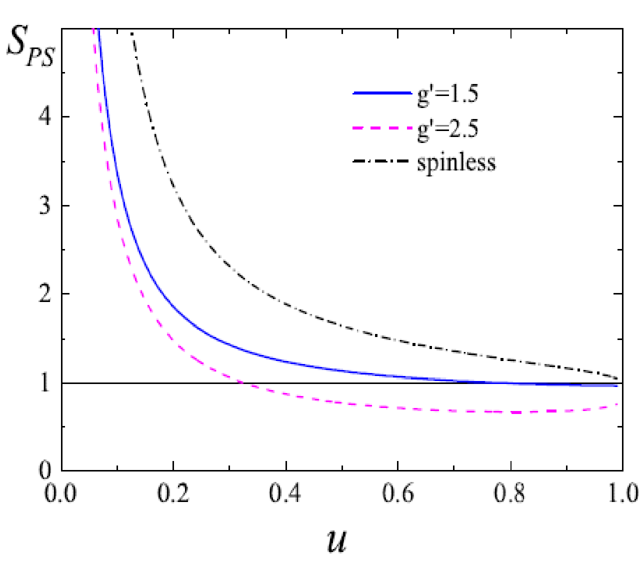
$$v=\sqrt{1-\frac{4m^2}{s}}$$

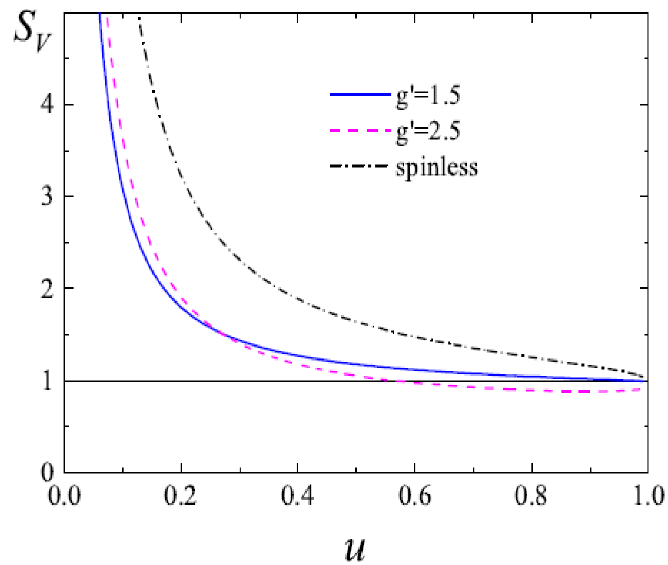
$$v_{\rm nr}\to v$$

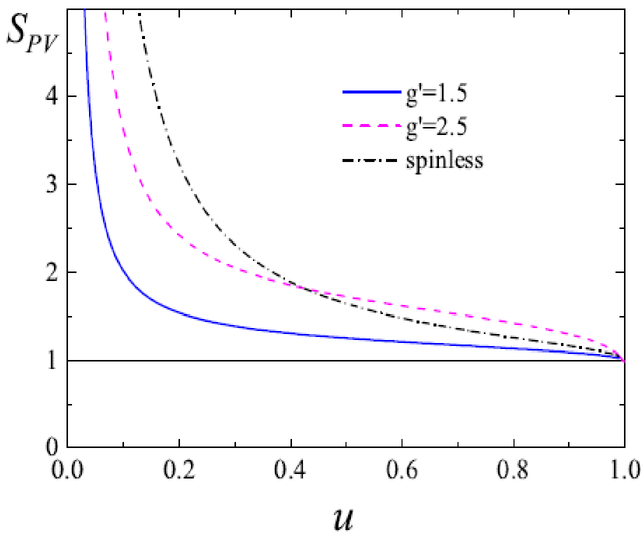
$$v_{\rm nr}\to\frac{v}{\sqrt{1-v^2}}$$



Illustrations: spin particles





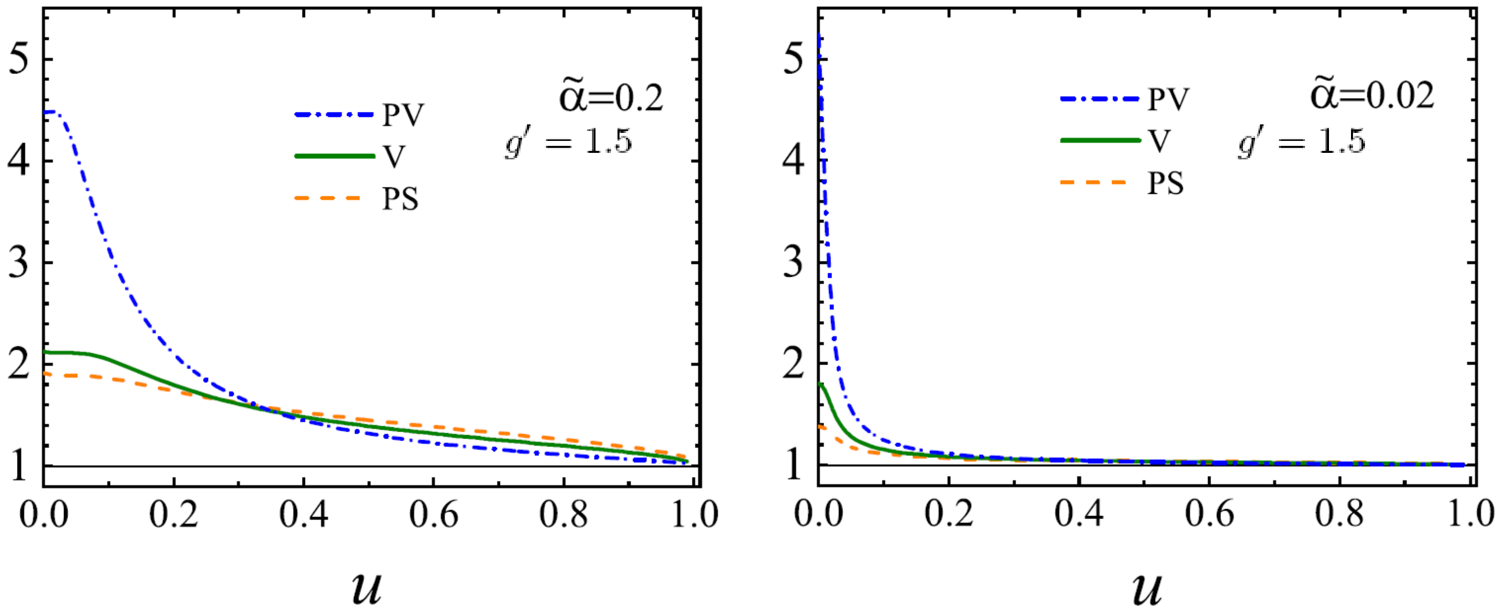


$$u = \sqrt{1 - \frac{4m'^2}{s - (m_1 - m_2)^2}}.$$

$$g' = \frac{m'}{2\mu} = \frac{m_1 + m_2}{2\sqrt{m_1 m_2}};$$

$$m' = \sqrt{m_1 m_2}$$

$$X_{\text{uneq}} = \frac{2\pi\tilde{\alpha}}{u'_{\text{rel}}} \qquad u'_{\text{rel}} = \frac{2u}{\sqrt{1-u^2}}$$



The ratio of the S-factors of spinless systems to the spinor factors as a function of the velocity u: solid curve corresponds to vector, dashed to pseudoscalar and dot-dashed curve to pseudovector systems.

Conclusion

A comparison of the spinor factors with the spinless case persuades us that the spin effects play a significant role in the Sommerfeld effect making it, at small and moderate velocities, systematically smaller than the S-factor for spinless systems. Thus, one can predict a decrease of the cross sections in a large kinematic range of the relative velocity, the effect being amplified in the threshold-near region.

Another interesting observation is that at large velocities, $u \gtrsim 0.4 - 0.5$, the pseudoscalar and vector factors for certain asymmetric systems cross the unity from above and approach the ultrarelativistic limit from below. Since in this region the cross sections become smaller in comparison with the main cross section determined by the corresponding Feynman diagrams, this circumstance can be, in some sense, referred to as the 'anti'-Sommerfel effect, contrarily to the 'true' Sommerfeld effect discussed above. The 'anti'-Sommerfeld effect increases with the mass asymmetry, g', of the system and can cause new impacts in the cross section of processes with highly asymmetric relativistic particles.

Since the new expression for the relativistic S-factor is obtained within a covariant method and has a correct connection with the Bethe-Salpeter function, one can expect that this expression takes into account more adequately both the relativistic character of interacting particles and their spin and masses.


Thanks for your attention !