# Weyl group symmetry as an intrinsic color symmetry of QCD

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# Motivation, main problems:

1. Our goal is to find a strict non-perturbative formulation of QCD. A key principal problem is to construct a true vacuum which must satisfy several requirements: (i) quantum stability (ii) color invariance (iii) consistence with color confinement.
2. Construction of one-particle quantum states for gluons and quarks: a principal obstacle is the absence of singlet representations of group $SU(3)$ implying the absence of proper definitions of one-particle singlet quantum states for gluons and quarks. We show that a full space of one-particle states corresponding to non-Abelian gluons represents an infinite but countable space of solutions described by a finite set of integer numbers.
3. results must be consistent with color and quark confinement and hadron phenomenology.

# Main definitions of the Weyl group of $SU(3)$

Generators of $SU(3)$ in fundamental representation are given by eight Gell-Mann matrices $T^a_{\alpha\beta} = \lambda^a_{\alpha\beta}$, $(\alpha = 1, 2, 3)$.
Group $SU(3)$ contains three $SU(2)$ subgroups of I,U,V-types generated by:

I: $\{T^1, T^2, T^3\}$, U: $\{T^4, T^5, \frac{1}{2}(-T^3 + \sqrt{3}T^8)\}$,

V: $\{T^6, T^7, -\frac{1}{2}(T^3 + \sqrt{3}T^8)\}$.

One can choose the Cartan basis, consisting of two Cartan generators $T^{3,8}$, and six off-diagonal generators $T^p_{\pm}$, $p = (1, 2, 3)$ or $(I, U, V)$

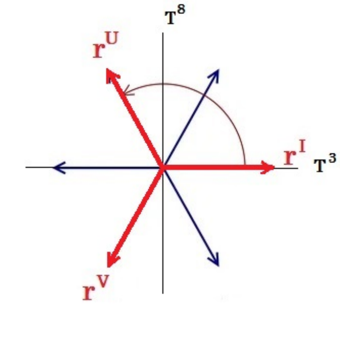
$$T^1_{\pm} = \frac{1}{2}(T^1 \pm iT^2), \quad T^2_{\pm} = \frac{1}{2}(T^4 \pm iT^5), \quad T^3_{\pm} = \frac{1}{2}(T^6 \pm iT^7)$$

Root vectors $r^p_\alpha$ ($\alpha = 3, 8$) are eigenvalues of the 2-component operator $T^\alpha = (T^3, T^8)$ acting on $SU(3)$ Lie algebra as adjoint represn. Roots define color charges of gluons:

$$[T^\alpha, T^p_\pm] = r^p_\alpha T^p_\pm$$

One has six roots in the plane ($X = T^3, Y = T^8$):

$$r^1_\alpha = \pm(1, 0), \quad r^2_\alpha = \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2}), \quad r^3_\alpha = \pm(\frac{1}{2}, \frac{\sqrt{3}}{2})$$




Weyl group $\mathcal{W}$ of $SU(3)$ is a symmetry group of roots of the Lie algebra $\mathfrak{g}(su(3))$, and it is isomorphic to the symmetric group $S_3$, permutation group.

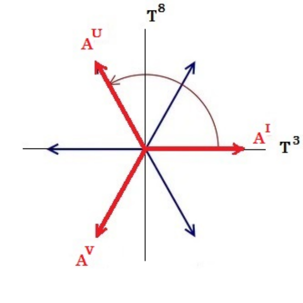
# Representations of the Weyl group

A 3-dim. vector representation $\Gamma_3$ of $S_3$ is realized by all possible permutations of components of a vector $\vec{V} = (V_1, V_2, V_3)$ in $R^3$. Such reprn. is reducible, and it is decomposed into a sum of two irred. represns, $\Gamma_1 \oplus \Gamma_2$. A non-trivial 1-dim. singlet represn $\Gamma_1$ is defined by a constraint $V_1 = V_2 = V_3 \equiv V$. The vector $\vec{V} = (V, V, V)$ is invariant under permutations and represents the only non-trivial irreducible singlet represn of $S_3$. A 2-dim. irred. represn. $\Gamma_2$ is defined by constraint $V_1 + V_2 + V_3 = 0$. The equation defines a 2-dim. plane in $R^3$ which forms invariant vector subspace under permutations of vector components.

Example: I,U,V-type Abelian potentials form 2-dim. reprn. $\Gamma_2$:

$$A^I_\mu = (A^3_\mu, 0); \quad A^U_\mu = (-1/2A^3_\mu, \sqrt{3}/2A^8_\mu),$$
$$A^V_\mu = (-1/2A^3_\mu, -\sqrt{3}/2A^8_\mu), \quad A^I_\mu + A^U_\mu + A^V_\mu = 0,$$

Corresponding $I, U, V$-type field strengths $F^{I,U,V}_{\mu\nu} \in \Gamma_2$ If invariants $(H^{I,U,V}_{\mu\nu})^2$ are equalled then they form singlet reprn. $\Gamma_1$.




** Continious subgroups of $SU(3)$ have no non-trivial 1-dim. singlet reprns. This causes a problem of existence and construction of a non-degenerate color invariant vacuum and single particles.

# Ansatz for singlet Weyl symmetric solutions

The Weyl group is a finite color subgroup of $SU(3)$ and it is the only color symmetry remaining after removing all pure gauge degrees of freedom! Magnetic ansatz contains $I, U, V$ sectors:

$$I: \quad A_t^2 = K_0,\ A_r^2 = K_1,\ A_\theta^2 = K_2, \quad A_\varphi^1 = K_4, \quad A_\varphi^I = K_3$$

$$U: \quad A_t^5 = Q_0,\ A_r^2 = Q_1,\ A_\theta^5 = Q_2, \quad A_\varphi^4 = Q_4,\ A_\varphi^U = -\frac{1}{2}K_3 + \frac{\sqrt{3}}{2}K_8$$

$$V: \quad A_t^7 = S_0,\ A_r^7 = S_1,\ A_\theta^7 = S_2, \quad A_\varphi^6 = S_4,\ A_\varphi^V = -\frac{1}{2}K_3 - \frac{\sqrt{3}}{2}K_8$$

$$A_\varphi^3 = K_3;\ A_\varphi^8 = K_8, \qquad \{A_\varphi^p : A_\varphi^I + A_\varphi^U + A_\varphi^V = 0\} \in \Gamma_2,$$

Singlet structure is provided by constraints $(i = 0, 1, 2)$:
$Q_i = S_i = K_i \Rightarrow (K_i, K_i, K_i) \in \Gamma_1$.
If $K_3 = K_8$ then $A_\alpha^p = K_3 r_\alpha^p$ match the root diagram $r^p$ with one field $K_3$. So, $I, U, V$ components of the Abelian field $A_\alpha^p$ can realize singlet Weyl representation. Vector field $K_{\mu=0,1,2,3}$ realizes four singlet Weyl reprns.Numeric solutions confirm the singlet structure.

The lowest energy solutions with lowest $\theta$ modes in the leading order are given by $K_\mu(r,\theta,t) = \tilde{K}_\mu(r,\theta)(b_1\cos(Mt) + b_2\sin(Mt))$

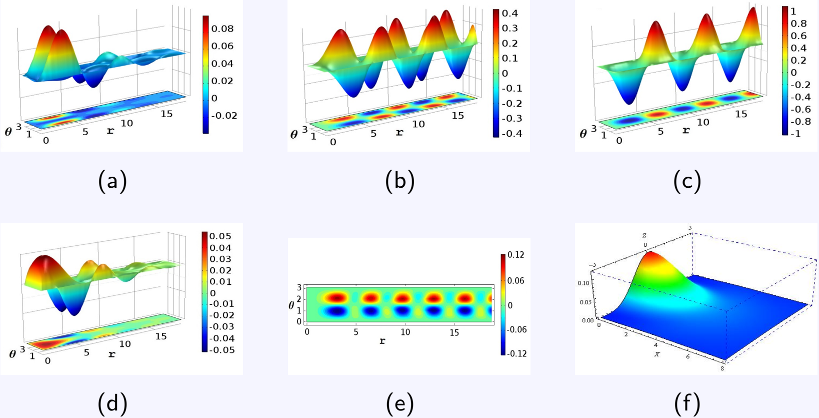



Figure: Solution profile functions in the leading order: (a) $\tilde{K}_1$; (b) $\tilde{K}_2$; (c) $\tilde{K}_3$; (d) $\tilde{K}_0$; (e) the time averaged radial color magnetic field $\overline{B_r^3}$; (f) the time averaged energy density $\bar{\mathcal{E}}(\rho,z)$ in the plane $(\rho,z)$ in cylindrical coordinates $(g=1, M=1)$.

# Color confinement and color symmetry of the vacuum

1974-1975, Wilson, Kogut and Susskind: the vacuum should be color invariant and non-degenerate to provide confinement of color.
1981, 't Hooft's conjecture: confinement phase is described by Abelian projected QCD
*a problem: After gauge fixing and removing all pure gauge degrees of freedom a residual global color symmetry should survives to define color invariant vacuum solutions. However, none of continious subgroups of $SU(3)$ ($SU(2), U(1)$ or their products) admit a non-trivial singlet representation. This implies degeneracy of the vacuum space and spontaneous color symmetry breaking, which excludes the color confinement, since color symmetry must be preserved. Fortunately, this puzzle is resolved due to existence of a discrete color subgroup, the Weyl group! which admits non-trivial singlet irreducible representations providing a non-degenerate color invariant vacuum and color confinement.

# Weyl symmetry and non-degenerate vacuum: example

One-loop effective potential for constant background chromo-magnetic field is known ( //Flyvbjerg 1980). It is Weyl invariant and has non-trivial minimums which correspond to possible vacuum energy. Weyl symmetric solutions have higher symmetry compare to other solutions, and the singlet solution has the highest intrinsic color symmetry which provides the deepest non-degenerate color singlet vacuum.

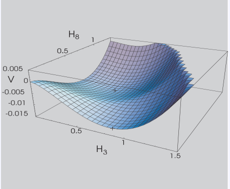



Figure: Degenerate vacuums form Weyl reprn. $\Gamma_2$ when $(\vec{H}_3 \vec{H}_8) \neq 0$. Not all invariants $H_{I,U,V}$ are equal.

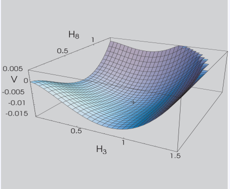



Figure: The deepest vacuum represents Weyl singlet when $(\vec{H}_3 \vec{H}_8) = 0$ implying $H_I = H_U = H_V$.

One has 3 parameters describing constant magnetic field: $(H_{3\mu\nu})^2$, $(H_{8\mu\nu})^2$, and angle $\theta$ between magnetic field vectors $\vec{H}_3, \vec{H}_8$. Or one can use gauge invariant $I, U, V$ variables: $H^p = \sqrt{H^2_{\mu\nu}(A^p)}$, $(p = I, U, V)$. Weyl symmetric effective potential has an absolute minimum at the symmetric point $(H^I = H^U = H^V \equiv H_0)$ describing a Weyl singlet vacuum. Weyl symmetry describes intrinsic color properties of gluon field. Weyl symmetric gluon solutions have higher symmetry to compare other non-Weyl symmetric fields and form degenerate unstable vacuums. This color symmetry has relative character, it provide relationship between members of Wel multiplet. Non-degenerate vacuum has intrinsic color symmetry which provides a deepest color singlet vacuum. Note, the Weyl singlet representation is realized on field strengths $H^{I,U,V}$, not on $I, U, V$-components of the gauge potential. We have constructed Weyl singlet stationary gluon solutions which provide a color singlet vacuum which is stable against quantum fluctuations /Pak, P.M. Zhang,Y. Kim, T.Tsukioka, PLB-2018; Pak, R.-G. Cai, Y.-F. Zhou, P.M. Zhang, T.Tsukioka, PLB-2023.

# Weyl symmetric Abelian projection

Weyl symmetric Abelian projection is described by Weyl symmetric ansatz reduced to a special case with only one non-zero Abelian field $A^3_\mu = A^8_\mu \equiv K_3$. A complete set of Abelian fields is given by spherical vector harmonics forming a basis in the Hilbert space

$$\vec{A}^{\mathfrak{m}}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L} j_l(kr) Y_{lm}(\theta, \varphi) e^{i\omega t},$$
$$\vec{A}^{\mathfrak{e}}_{lm} = \frac{-i}{\sqrt{l(l+1)}} \vec{\nabla} \times (\vec{L} j_l(kr) Y_{lm}(\theta, \varphi)) e^{i\omega t}, \qquad (1)$$

$\{\vec{A}^{\mathfrak{m},\mathfrak{e}}_{lm}\}$ are eigenfunctions of the total angular momentum operator with $J = l \geq 1$, $J_z = m$, $\vec{L}$ - orbital momentum operator. We use dimensionless units $\tilde{M} = Ma_0, x = r/a_0, \tau = t/a_0$ with an effective size of hadron $a_0$. Gauge fields are localized inside a sphere of unit radius $x = 1$ (a node or antinode).

# Localization of a single gluon

We start with known one-loop effective Lagrangian of $SU(3)$ QCD

$$\mathcal{L}^{\text{1-l}} = -\frac{1}{4}\tilde{F}^2 - k_0 g^2 \tilde{F}^2 \left( \log\left(\frac{g^2\tilde{F}^2}{\Lambda^4_{\rm QCD}}\right) - c_0 \right),$$

where $\tilde{F}^2 \equiv \tilde{F}^2_{\mu\nu}$ is a squared Abelian field strength of magnetic type, and we treat $k_0, c_0$ as free parameters. Effective potential $V^{\text{1-l}} = -\mathcal{L}^{\text{1-l}}$ has an absolute minimum at a positive vacuum gluon condensate value

$$g^2 B^2_{0\,\mu\nu} = \Lambda_{\rm QCD} \exp\left(c_0 - 1 - \frac{1}{2k_0 g^2}\right).$$

Splitting the full gauge field strength $\tilde{F}_{\mu\nu} = B_{\mu\nu} + F_{\mu\nu}$ into gluon condensate part $B_{\mu\nu}$ and single gluon part $F_{\mu\nu}$ one can obtain an effective Lagrangian for a single gluon

$$\mathcal{L}^{(2)}_{\rm eff}[A] = -2k_0 g^2 \frac{(B^{\mu\nu}F_{\mu\nu})^2}{B^2} \equiv -\kappa (B^{\mu\nu}F_{\mu\nu})^2$$

Substituting expression for vacuum gluon condensate function $B_\varphi(r,\theta,t) = r\,j_1(r)\sin^2\theta\,\sin(t)$, one can solve Euler equation for the single gluon potential $A_\varphi(r,\theta,t) = a(r,\theta)\sin(t+\phi_0)$:

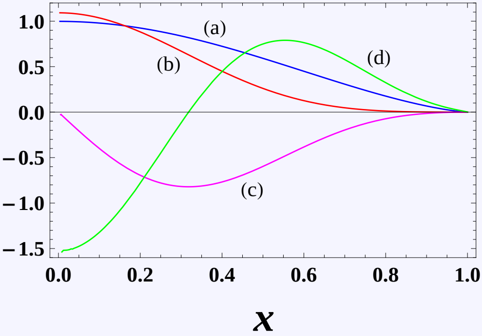



Figure: (a) Solution $f(x)$; (b) a radial energy density $\bar{\mathcal{E}}/4\pi\kappa$(in red); (c) the first derivative of the energy density; (d) the second derivative of the energy density; The bound state energy is minimal at phase shift $\phi_0 = \pm\pi/2$, and describes a lightest glueball with $J = 0$.

# Weyl symmetric quarks

Consider a simple Abelian projected QCD with Abelian gluon fields $A_\mu^{3,8}$ corresponding to Cartan algebra generators $T_{3,8} = \lambda^{3,8}$. $T_{3,8}$ have three common eigenvectors $\hat{u}^p$

$\hat{u}^1 = (1,0,0); \quad \hat{u}^2 = (0,1,0); \quad \hat{u}^3 = (0,0,1)$ with corresponding eigenvalues represented by 2-dim. weight vectors $w_\alpha^p$ $(\alpha = 3,8)$

$T_\alpha \hat{u}_\alpha^p = w_\alpha^p \hat{u}_\alpha^p,$

$w^1 = (1, 1/\sqrt{3}), \quad w^2 = (-1, 1/\sqrt{3}), \quad w^3 = (0, -2/\sqrt{3}).$

$SU(3)$ quark triplet can be decomposed as: $\Psi = \psi(x)^p u^p$, where $\psi^p(x)$ are Dirac spinor functions describing three color quarks.

Substitution of the decomposition for $\Psi$ into the quark Lagrangian leads to the Lagrangian in explicit Weyl symmetric form:

$\mathcal{L}_q = \sum_p \bar{\psi}^p (i\gamma^\mu \partial_\mu - m + \frac{g}{2}\gamma^\mu A_\mu^p)\psi^p,$

where $(p = I, U, V)$ $A_\mu^p = w_3^p A_{3\mu} + w_8^p A_{8\mu}$ – $I, U, V$-components of the gauge potentials $A_{3,8\mu}$ defined with using weights.

QCD equations take the form:
$\partial^\mu F_{\mu\nu}(A^3) = -\frac{g}{2}\bar{\psi}^1\gamma^\mu\psi^1 + \frac{g}{2}\bar{\psi}^2\gamma^\mu\psi^2,$
$\partial^\mu F_{\mu\nu}(A^8) = -\frac{g}{2\sqrt{3}}\bar{\psi}^1\gamma^\mu\psi^1 - \frac{g}{2\sqrt{3}}\bar{\psi}^2\gamma^\mu\psi^2 + \frac{g}{\sqrt{3}}\bar{\psi}^3\gamma^\mu\psi^3$
$[i\gamma^\mu\partial_\mu - m + \frac{g}{2}\gamma^\mu(A^p_\mu)_w]\psi^p = 0;$
From these eqs one can derive three separated systems of eqs for p-components for quark $\psi^p$ and for two gluon fields $A^{3,8}_\mu$ in explicit Weyl symmetric form:
$\partial^\mu F_{\mu\nu}(A^p_w) = -\frac{2g}{3}\bar{\psi}^p\gamma^\mu\psi^p,$
$\partial^\mu F_{\mu\nu}(A^p_r) = 0;$
$[i\gamma^\mu\partial_\mu - m + \frac{g}{2}\gamma^\mu(A^p_\mu)_w]\psi^p = 0;$
Three solutions form 3-dim Weyl representation $\Gamma_3$.

Surprisingly, there is a Weyl singlet solution for one single quark dressed in gluon field. Consider a system of eqs. corresponding to I-component quark and gluon fields. The system contains two Weyl symmetric combinations of gluon fields $A^3, A^8$:

$(A^I)_w = A^3 + \frac{1}{\sqrt{3}} A^8$

$(A^I)_r = -\frac{1}{2} A^3 + \frac{\sqrt{3}}{2} A^8$

Impose constraints reducing number of gluons:

$A^3 = \mathcal{A}; \quad A^8 = \frac{1}{\sqrt{3}} \mathcal{A},$

With this one has: $(A^I)_w = \frac{4}{3} \mathcal{A}, \quad (A^I)_r \equiv 0.$

By using a similar reduction for $U, V$ quarks and gluons one obtains the same system of two eqs for each quark $\psi^p$ and dressing gluon field $\mathcal{A}$:

$\partial^\mu F_{\mu\nu}(\mathcal{A}) = -\frac{g}{2} \bar{\psi}^p \gamma^\mu \psi^p,$

$[i\gamma^\mu \partial_\mu - m + \frac{2g}{3} \gamma^\mu \mathcal{A}] \psi^p = 0;$

By choosing the same solution for gluon field $\mathcal{A}$ and for quark one obtains Weyl singlet solution for constituent quark dressed in singlet gluon field.

Note that existence of quark solution from singlet Weyl representation $\Gamma_1$implies decomposition of 3-dim. quark representation: $\Gamma_3 = \Gamma_1 + \Gamma_2$. Indeed, quark solutions for two quarks $\psi^1, \psi^2$ form $\Gamma_2$. This can be seen by using dual Abelian projection which produces equations for quark dressed in gluon fields of $I, U, V$ type with respect to root vectors. Such Abelian projection is realized if we choose a different basis for Cartan subalgebra. Namely, define two generators as follows:

$$L = \lambda^1 + \lambda^4 + \lambda^6 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, Q = \lambda^2 + \lambda^5 + \lambda^7 = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix},$$

The matrices commute to each other and form Cartan subalgebra: $[L, Q] = 0$.

The matrices $L, Q$ have three common eigenvectors $u^0, u^\pm$ with corresponding eigenvalues

$$u^0 = (1,1,1) \in \Gamma_1$$
$$Lu^0 = 2u^0, \qquad Qu^0 = 0,$$
$$u^\pm = (-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, -\frac{1}{2} \mp \frac{i\sqrt{3}}{2}, 1) \in \Gamma_2,$$
$$Lu^\pm = -u^\pm, \qquad Qu^\pm = \pm\sqrt{3}u^\pm,$$
$$u^+u^+ = 0, \qquad u^-u^- = 0, \qquad u^+u^- = u^-u^+ = 3$$

The color vector $u^0$ belongs to the standard irreducible singlet representation $\Gamma_1$ of the Weyl group, and the color vectors $u^+, u^-$ form a two-dimensional complex irreducible vector representation $\Gamma_2$ due to the property $u_1^\pm + u_2^\pm + u_3^\pm = 0.$ One can verify that decomposition of $SU(3)$ quark triplet in the basis of color vectors $u^{0,\pm}$ with corresponding quarks $\psi^{0,\pm}$ leads to similar Weyl structure of quark solutions.

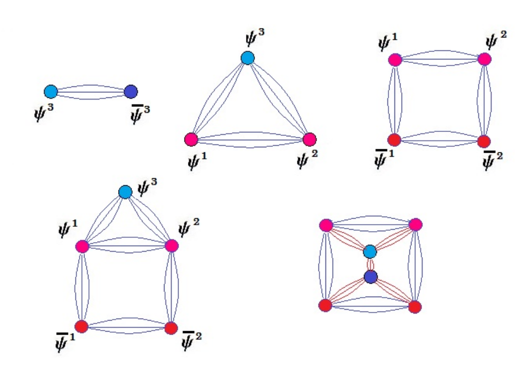
# Quark model with Weyl group symmetry

Consider interaction of quarks and gluons from the eqs of motion

$\partial^\mu F_{\mu\nu}(A^3) = -\frac{g}{2}\bar{\psi}^1\gamma^\mu\psi^1 + \frac{g}{2}\bar{\psi}^2\gamma^\mu\psi^2,$

$\partial^\mu F_{\mu\nu}(A^8) = -\frac{g}{2\sqrt{3}}\bar{\psi}^1\gamma^\mu\psi^1 - \frac{g}{2\sqrt{3}}\bar{\psi}^2\gamma^\mu\psi^2 + \frac{g}{\sqrt{3}}\bar{\psi}^3\gamma^\mu\psi^3$

$[i\gamma^\mu\partial_\mu - m + \frac{g}{2}\gamma^\mu(A^p_\mu)_w]\psi^p = 0;$



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