Transverse momentum distributions of hadrons in the Tsallis-3 statistics

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Motivation

Experiment:

$$pp,\ HIC \ \rightarrow \ hadrons\ (\pi^{\pm}, \pi^{0}, K^{\pm}, K^{0}, \bar{K}^{0}, p, \bar{p}, n, \bar{n}, \Delta, \Lambda, \Omega \text{ etc.})$$

$$(p_T,\ y) - distributions$$

RHIC, LHC
FAIR, NICA

Theory:

1. Approximate Tsallis p_T -distributions:

Tsallis-like distribution

$$\cancel{E\frac{d^3N_{ch}}{d^3p} = C\frac{dN_{ch}}{dy}\left(1-(1-q)\frac{E_T}{T}\right)^{\frac{1}{1-q}}}$$

Phenomenological Tsallis distribution

$$\boxed{\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1-(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}}$$

2. Exact p_T-distributions of the Tsallis statistics:

Tsallis-1 statistics | ~~**Tsallis-2 statistics**~~ | **Tsallis-3 statistics** | ***q*-dual statistics**

Theory selection:

1. Experiment
2. Consistent derivation of p_T-distribution from the first principles of statistical mechanics
3. Thermodynamic self-consistency of p_T-distribution
4. Approximate results $\approx$ Exact results

Tsallis statistics with escort probabilities (Tsallis-3 statistics)
Grand Canonical Ensemble

Tsallis entropy:

$$\boxed{S = \sum_i \frac{p_i^q - p_i}{1-q} = \frac{1}{\theta}\sum_i p_i^q S_i, \quad S_i = -\theta \frac{p_i^{1-q} - 1}{1-q}}$$

$$\xrightarrow{q \to 1} \quad S_G = -\sum_i p_i \ln p_i$$

- Boltzmann-Gibbs entropy

Norm equation:

$$\sum_i p_i = 1 \qquad \left(\sum_i p_i^q \equiv \theta\right)$$

C. Tsallis, J. Stat. Phys. 52 (1988) 479;
C. Tsallis et al., Physica A 261 (1998) 534

Mean value of the quantum operator A :

$$\boxed{\langle \hat{A} \rangle = \frac{\sum_i p_i^q A_i}{\sum_i p_i^q} = \frac{1}{\theta}\sum_i p_i^q A_i}$$

Thermodynamic potential:

$$\Omega = \langle H \rangle - TS - \mu \langle N \rangle = \frac{1}{\theta}\sum_i p_i^q \Omega_i,$$

- Legendre transform

$$\Omega_i = -TS_i + E_i - \mu N_i = T\theta \frac{p_i^{1-q} - 1}{1-q} + E_i - \mu N_i$$

A.S.P., arXiv:2306.01003 [hep-ph]

Mean energy and mean number of particles:

$$\langle H \rangle = \frac{1}{\theta}\sum_i p_i^q E_i, \qquad \langle N \rangle = \frac{1}{\theta}\sum_i p_i^q N_i$$

Tsallis statistics with escort probabilities (Tsallis-3 statistics)
Grand Canonical Ensemble

The probability is determined from the principle of thermodynamic equilibrium:

$$[d(\Omega-\lambda\varphi)]_{T,V,\mu}=0, \qquad \phi=\sum_i p_i-1=0$$

Probability:

$$\boxed{p_i=\left[1+(1-q)\frac{\Lambda-E_i+\mu N_i}{T\theta^2}\right]^{\frac{1}{1-q}}}$$

A.S.P., arXiv:2306.01003 [hep-ph]

Λ,θ - two norm functions

Two norm equations:

$$\sum_i\left[1+(1-q)\frac{\Lambda-E_i+\mu N_i}{T\theta^2}\right]^{\frac{1}{1-q}}=1$$

$$\sum_i\left[1+(1-q)\frac{\Lambda-E_i+\mu N_i}{T\theta^2}\right]^{\frac{q}{1-q}}=\theta$$

Mean value of the quantum operator:

$$\langle A\rangle=\frac{1}{\theta}\sum_i A_i\left[1+(1-q)\frac{\Lambda-E_i+\mu N_i}{T\theta^2}\right]^{\frac{q}{1-q}}$$

Different variants of nonextensive statistics in grand canonical ensemble

$$S=-\sum_i \frac{p_i-p_i^q}{1-q}$$

Tsallis entropy

$$\xrightarrow{q\to 1/q}$$

$$\sum_i p_i = 1$$

$$S=q\sum_i \frac{p_i^{1/q}-p_i}{q-1}$$

q-dual entropy

Tsallis-1 statistics

$$\langle A\rangle=\sum_i p_i A_i, \qquad \langle 1\rangle = 1$$

$$p_i=\left[1+\frac{q-1}{q}\frac{\Lambda-E_i+\mu N_i}{T}\right]^{\frac{1}{q-1}}$$

$$\sum_i\left[1+\frac{q-1}{q}\frac{\Lambda-E_i+\mu N_i}{T}\right]^{\frac{1}{q-1}}=1$$

Tsallis-2 statistics

$$\langle A\rangle=\sum_i p_i^q A_i, \qquad \langle 1\rangle \neq 1$$

$$p_i=\frac{1}{Z}\left[1-(1-q)\frac{E_i-\mu N_i}{T}\right]^{\frac{1}{1-q}}$$

$$Z=\sum_i\left[1-(1-q)\frac{E_i-\mu N_i}{T}\right]^{\frac{1}{1-q}}$$

q-dual statistics

$$\langle A\rangle=\sum_i p_i A_i, \qquad \langle 1\rangle = 1$$

$$p_i=\left[1+(1-q)\frac{\Lambda-E_i+\mu N_i}{T}\right]^{\frac{q}{1-q}}$$

$$\sum_i\left[1+(1-q)\frac{\Lambda-E_i+\mu N_i}{T}\right]^{\frac{q}{1-q}}=1$$

C. Tsallis, J. Stat. Phys. 52 (1988) 479; C. Tsallis et al., Physica A 261 (1998) 534; A.S.P., Eur. Phys. J. A 51 (2015) 108; A.S.P., Eur. Phys. J. A 53 (2017) 53

A.S.P., Eur. Phys. J. A 56 (2020) 4, 106

- The Tsallis-2 statistics is inconsistent since the mean value of unity is not equal to unity $\langle 1\rangle \neq 1$.
- The Tsallis-1 statistics and q-dual statistics are correctly defined and consistent.

Tsallis-3 statistics in the grand canonical ensemble: General formalism $q>1$

Statistical averages of the operators:

A.S.P., arXiv:2306.01003 [hep-ph]

$$\langle A\rangle = \frac{1}{\theta\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'(\Lambda-\Omega_G(\beta'))}\langle A\rangle_G(\beta')\,dt = \frac{1}{\theta}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'\Lambda}(-\beta'\Omega_G(\beta'))^n\langle A\rangle_G(\beta')\,dt$$

Series expansion: $$e^{-\beta'\Omega_G(\beta')} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\beta'\Omega_G(\beta')\right)^n$$

Boltzmann-Gibbs quantities:

$$\Omega_G(\beta') = -\frac{1}{\beta'}\ln Z_G(\beta'),\quad Z_G(\beta')=\sum_i e^{-\beta'(E_i-\mu N_i)},\quad \langle A\rangle_G(\beta') = \frac{1}{Z_G(\beta')}\sum_i A_i e^{-\beta'(E_i-\mu N_i)},\quad \beta' = \frac{-t(1-q)}{T\theta^2}$$

Norm functions Λ, θ :

$$1 = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)}\int_0^\infty t^{\frac{2-q}{q-1}} e^{-t+\beta'(\Lambda-\Omega_G(\beta'))}dt = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{q-1}\right)}\int_0^\infty t^{\frac{2-q}{q-1}} e^{-t+\beta'\Lambda}(-\beta'\Omega_G(\beta'))^n\,dt$$

$$\theta = \frac{1}{\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'(\Lambda-\Omega_G(\beta'))}dt = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{q}{q-1}\right)}\int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'\Lambda}(-\beta'\Omega_G(\beta'))^n\,dt$$

p_T - distribution in Tsallis-3 statistics: Relativistic ideal gas $q>1$

Relativistic transverse momentum distribution in grand canonical ensemble ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \frac{1}{\theta\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'(\Lambda-\Omega_G(\beta'))} \frac{1}{e^{\beta'(m_T\cosh y-\mu)}+\eta} dt$$

$$= \frac{gV}{(2\pi)^2} p_T m_T \cosh y \frac{1}{\theta} \sum_{n=0}^\infty \frac{1}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'\Lambda} \frac{(-\beta'\Omega_G(\beta'))^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta} dt$$

Norm functions Λ, θ :

A.S.P., arXiv:2306.01003 [hep-ph]

$$1 = \sum_{n=0}^\infty \frac{1}{n!\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty t^{\frac{2-q}{q-1}} e^{-t+\beta'\Lambda} (-\beta'\Omega_G(\beta'))^n dt$$

$$\theta = \sum_{n=0}^\infty \frac{1}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'\Lambda} (-\beta'\Omega_G(\beta'))^n dt$$

$$-\beta'\Omega_G(\beta') = \sum_{\mathbf{p},\sigma} \ln\left[1+\eta e^{-\beta'(\varepsilon_{\mathbf{p}}-\mu)}\right]^{\frac{1}{\eta}}$$

$$\varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2} = m_T \cosh y$$

$$m_T = \sqrt{p_T^2+m^2}$$

$$\beta' = \frac{-t(1-q)}{T\theta^2}$$

p_T - distribution in Tsallis-3 statistics: Maxwell-Boltzmann statistics of particles $\eta=0$

Relativistic transverse momentum distribution in grand canonical ensemble for Maxwell-Boltzmann statistics of particles: $q>1$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \frac{1}{\theta\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}} e^{-t+\beta'(\Lambda - m_T \cosh y + \mu) + \omega t^{-1} e^{\beta'\mu} K_2(\beta' m)} dt$$

$$= \frac{gV}{(2\pi)^2} p_T m_T \cosh y \frac{1}{\theta} \sum_{n=0}^{\infty} \frac{\omega^n}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}-n} e^{-t+\beta'(\Lambda - m_T \cosh y + \mu(n+1))} \left(K_2\left(\beta' m\right)\right)^n dt$$

$$\Omega_G\left(\beta'\right) = -\frac{gV}{2\pi^2}\frac{m^2}{\beta'^2} e^{\beta'\mu} K_2\left(\beta' m\right), \qquad \omega = \frac{gV}{2\pi^2}\frac{m^2 T\theta^2}{q-1}$$

A.S.P., arXiv:2306.01003 [hep-ph]

Norm functions Λ, θ :

$$1 = \sum_{n=0}^{\infty} \frac{\omega^n}{n!\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty t^{\frac{2-q}{q-1}-n} e^{-t+\beta'(\Lambda+\mu n)} \left(K_2\left(\beta' m\right)\right)^n dt$$

$$\theta = \sum_{n=0}^{\infty} \frac{\omega^n}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}-n} e^{-t+\beta'(\Lambda+\mu n)} \left(K_2\left(\beta' m\right)\right)^n dt$$

$$\beta' = \frac{-t(1-q)}{T\theta^2}$$

Ultrarelativistic p_T- distribution in Tsallis-3 statistics: Maxwell-Boltzmann statistics

$\eta = 0$

Ultrarelativistic p_T -distribution in grand canonical ensemble for Maxwell-Boltzmann statistics of particles: $(m=0)$

$q > 1$

$$\boxed{\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T^2 \cosh y \frac{1}{\theta} \sum_{n=0}^{\infty} \frac{\tilde{\omega}^n}{n!} \frac{\Gamma\left(\frac{q}{q-1} - 3n\right)}{(q-1)^{3n} \Gamma\left(\frac{q}{q-1}\right)} \left[1 + (1-q) \frac{\Lambda - p_T \cosh y + \mu(n+1)}{T\theta^2}\right]^{\frac{q}{1-q}+3n}}$$

$$\Omega_G(\beta') = -\frac{gV}{\pi^2} \frac{1}{\beta'^4} e^{\beta'\mu}, \qquad \tilde{\omega} = \frac{gV}{\pi^2} T^3 \theta^6$$

Norm functions Λ, θ :

A.S.P., arXiv:2306.01003 [hep-ph]

$$\boxed{1 = \sum_{n=0}^{\infty} \frac{\tilde{\omega}^n}{n!} \frac{\Gamma\left(\frac{1}{q-1} - 3n\right)}{(q-1)^{3n} \Gamma\left(\frac{1}{q-1}\right)} \left[1 + (1-q) \frac{\Lambda + \mu n}{T\theta^2}\right]^{\frac{1}{1-q}+3n}, \qquad \theta = \sum_{n=0}^{\infty} \frac{\tilde{\omega}^n}{n!} \frac{\Gamma\left(\frac{q}{q-1} - 3n\right)}{(q-1)^{3n} \Gamma\left(\frac{q}{q-1}\right)} \left[1 + (1-q) \frac{\Lambda + \mu n}{T\theta^2}\right]^{\frac{q}{1-q}+3n}}$$

p_T - distribution in Tsallis-3 statistics: Zeroth term approximation $(n=0)$

Relativistic p_T-distribution in zeroth term approximation ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics, $\eta=0$ - M-B statistics)

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \left[1-(k+1)(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \qquad \eta=-1,0,1$$

-quantum

A.S.P., arXiv:2306.01003 [hep-ph]

$$\Lambda = 0, \qquad \theta = 1, \qquad S = \frac{\theta - 1}{1-q} = 0$$

- Entropy in the zeroth term approximation is zero for all values of temperature T and chemical potential μ

Relativistic p_T-distribution in zeroth term approximation for Maxwell-Boltzmann statistics of particles:

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1-(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \qquad \eta=0$$

-classical

Tsallis-2
Tsallis-3
***q*-dual statistics**

- p_T -distribution of the Tsallis-3 statistics in the zeroth term approximation for Maxwell-Boltzmann statistics of particles exactly coincides with the phenomenological Tsallis distribution defined in [**J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160**].
- In the Tsallis-3 statistics, the phenomenological Tsallis distribution for the Maxwell-Boltzmann statistics of particles is inconsistent as it corresponds to the unphysical condition of zero entropy of system.
- In the Tsallis-2 statistics, the phenomenological Tsallis distribution for the Maxwell-Boltzmann statistics of particles is inconsistent as the Tsallis-2 statistics is incorrectly defined: $\langle 1 \rangle \neq 1$.
- The phenomenological Tsallis distribution for the Maxwell-Boltzmann statistics of particles is consistent in *q*-dual statistics as the *q*-dual statistics is correctly defined: $\langle 1 \rangle = 1$.

Zeroth term approximation for Tsallis statistics was introduced in: **A.S.P., Eur. Phys. J. A 53 (2017) 53; A.S.P., Eur. Phys. J. A 52 (2016) 355**

Tsallis-3 statistics: Quantum spectra in factorization approximation of the zeroth term approximation

Let us consider the factorization approximation, which implies the following mathematically unsanctioned replacement:

$$\left[1-(k+1)(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\frac{q}{1-q}} \approx \left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{\frac{q}{1-q}(k+1)}$$

H. Hasegawa, Phys. Rev. E 80, 011126 (2009)

Relativistic p_T-distribution in the factorization approximation of the zeroth term approximation ($\eta=-1$ -B-E statistics, $\eta=1$ -F-D statistics) for Tsallis-3 statistics

A.S.P., arXiv:2306.01003 [hep-ph]

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \frac{1}{\left[1-(1-q)\frac{\varepsilon_{\mathbf{p}}-\mu}{T}\right]^{-\frac{q}{1-q}}+\eta} \qquad \eta=-1,1$$

Tsallis-2
Tsallis-3
***q*-dual statistics**

- The form of this quantum distribution is similar to the quantum Tsallis-like distributions used in the high-energy physics.
- Thus, the quantum Tsallis-like distributions of this form are mathematically inconsistent because of equation of factorization given above.

Numerical results
Comparison with experimental data

Cut-off of infinite sum of p_T -distribution in the Tsallis-3 statistics

In the series expansion, a few terms are only convergent. The integrals are not convergent due to the divergence of the integrants as $t \to 0$.

The single upper cut-off limit of summation:

$$n_0 = \left[\frac{\nu}{3} \left(\frac{2-q}{q-1} + \delta \right) \right] \qquad 0 \leqslant \nu \leqslant 1, \quad \delta = 0.98$$

ν - a fixed parameter

The Maxwell-Boltzmann transverse momentum distribution of the Tsallis-3 statistics for $q > 1$

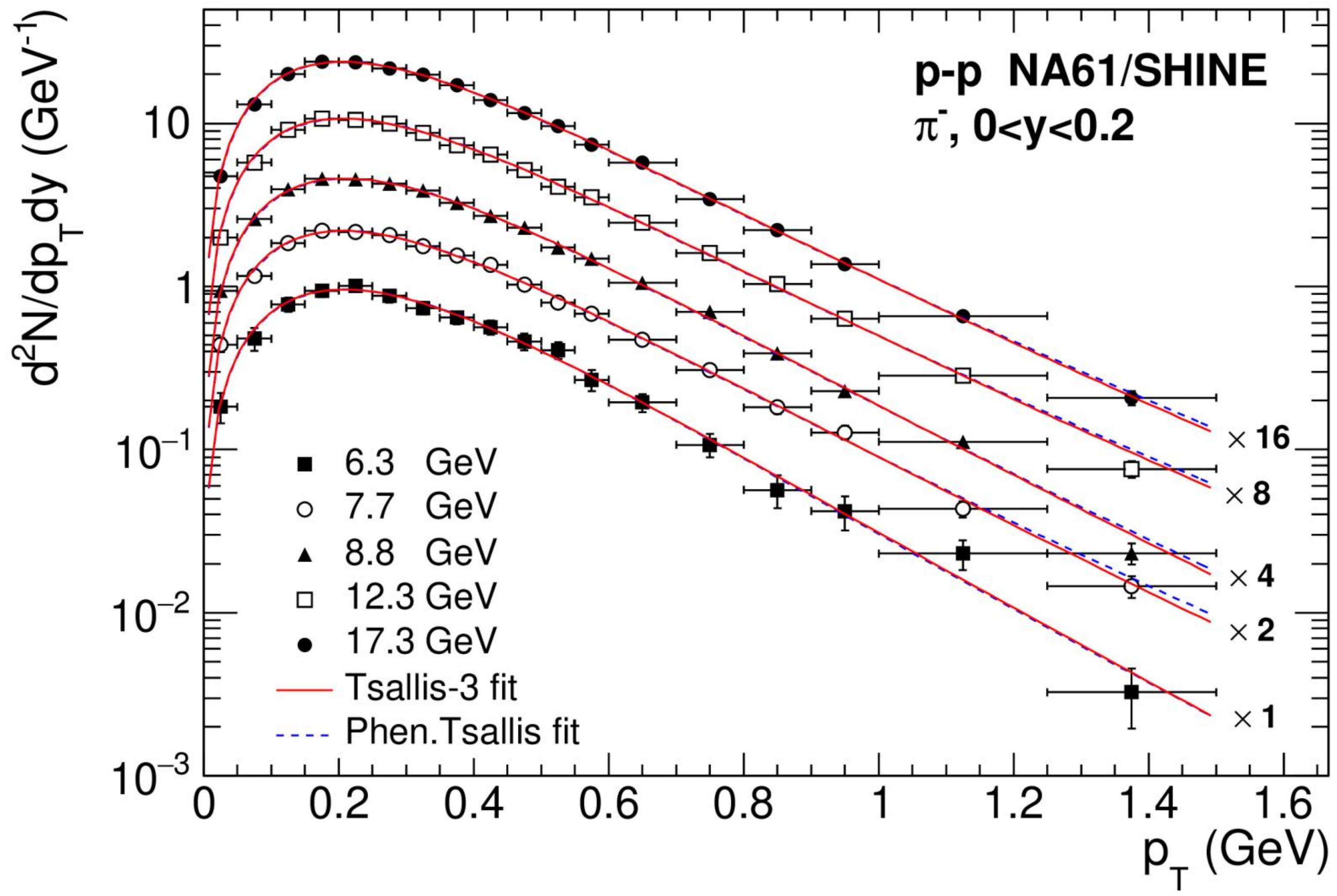
$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \frac{1}{\theta} \sum_{n=0}^{n_0} \frac{\omega^n}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}-n} e^{-t+\beta'(\Lambda - m_T \cosh y + \mu(n+1))} (K_2(\beta' m))^n dt$$

Norm functions Λ, θ :

A.S.P., arXiv:2306.01003 [hep-ph]

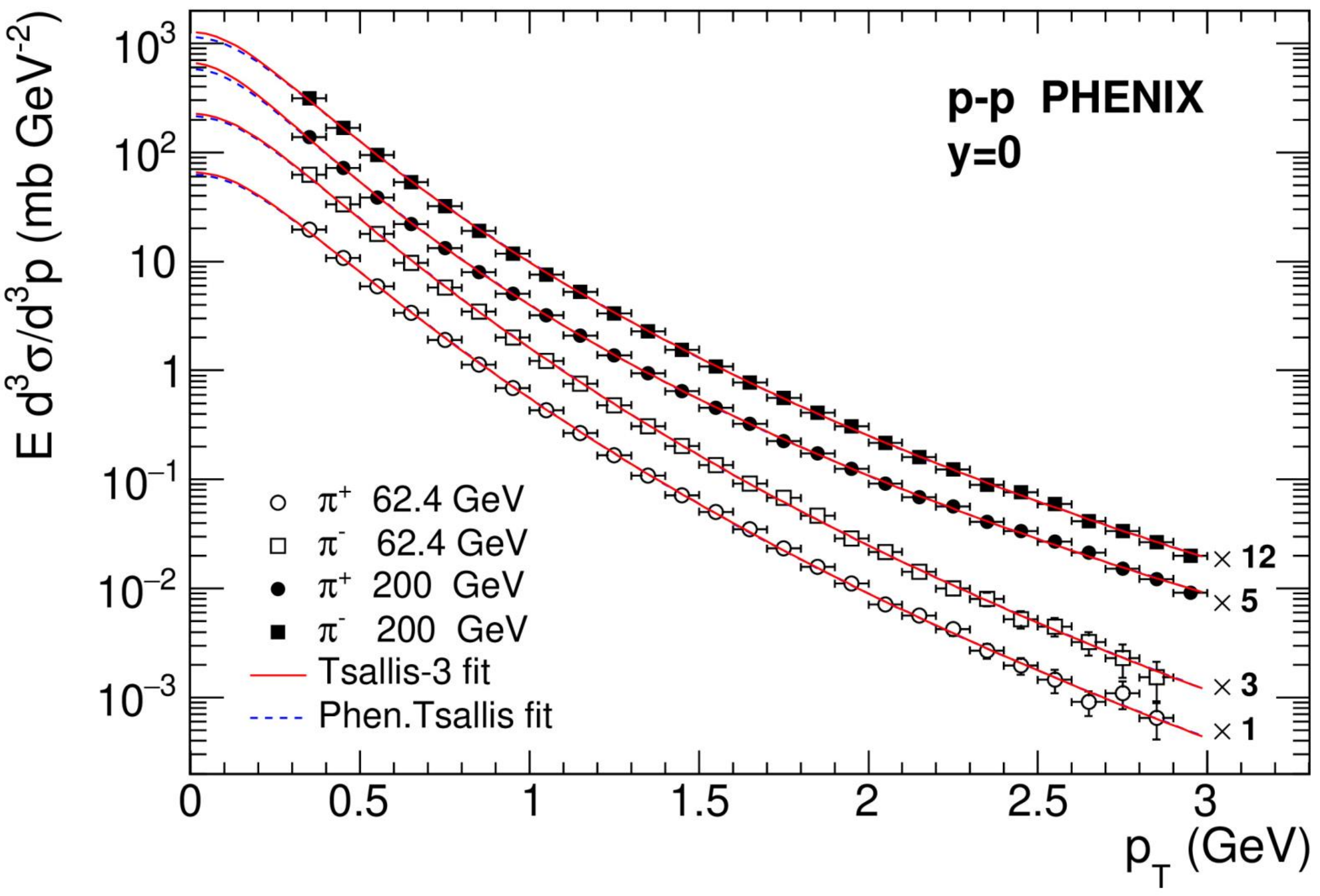
$$1 = \sum_{n=0}^{n_0} \frac{\omega^n}{n!\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty t^{\frac{2-q}{q-1}-n} e^{-t+\beta'(\Lambda+\mu n)} (K_2(\beta' m))^n dt, \qquad \theta = \sum_{n=0}^{n_0} \frac{\omega^n}{n!\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}-n} e^{-t+\beta'(\Lambda+\mu n)} (K_2(\beta' m))^n dt$$

$$\omega = \frac{gV}{2\pi^2} \frac{m^2 T \theta^2}{q-1}, \qquad \beta' = \frac{-t(1-q)}{T\theta^2}$$



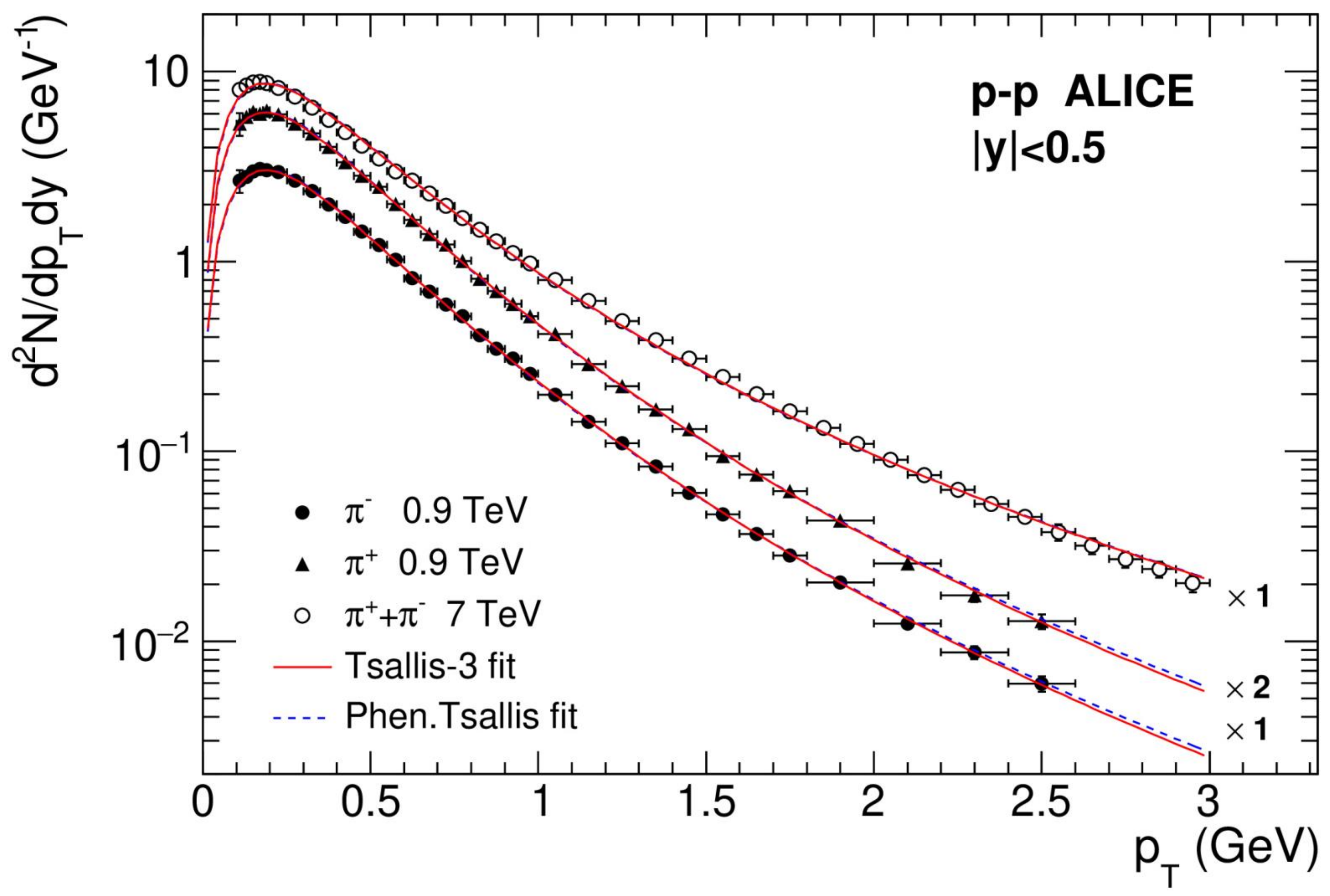
Comparison of exact p_T -distribution of the Tsallis-3 statistics with experimental data of NA61/SHINE Collaboration

Experimental Data: **NA61/SHINE, EPJC 74 (2014) 2794**



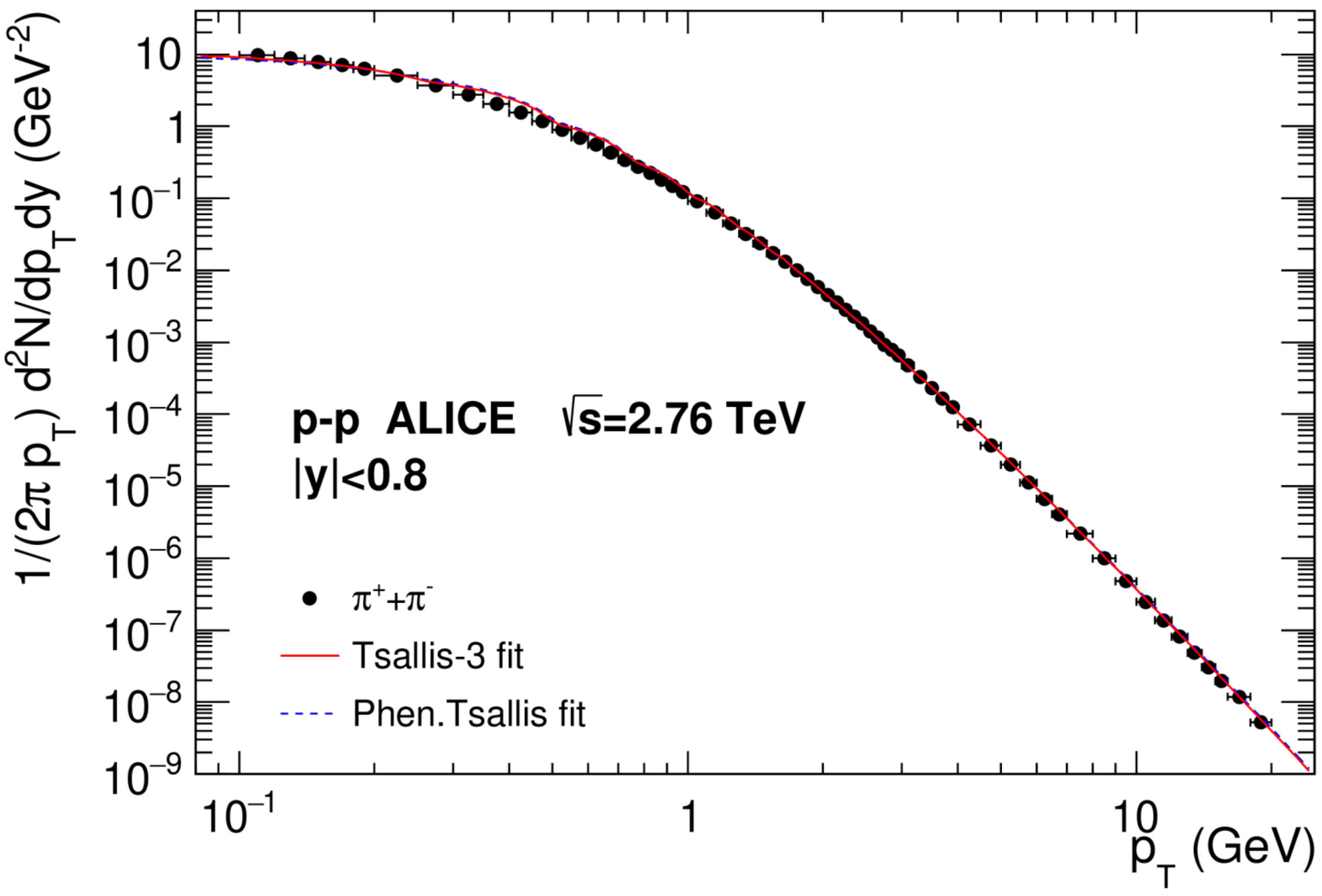
Comparison of exact p_T -distribution of the Tsallis-3 statistics with experimental data of PHENIX Collaboration

Experimental Data: **PHENIX, PRC 83 (2011) 064903**



Comparison of exact p_T -distribution of the Tsallis-3 statistics with experimental data of ALICE Collaboration

Experimental Data: **ALICE, EPJC 71 (2011) 1655; ALICE, EPJC 75 (2015) 226**



Comparison of exact p_T -distribution of the Tsallis-3 statistics with experimental data of ALICE Collaboration

Experimental Data: **ALICE, PLB 736 (2014) 196**

The exact p_T-distribution of the Tsallis-3 statistics

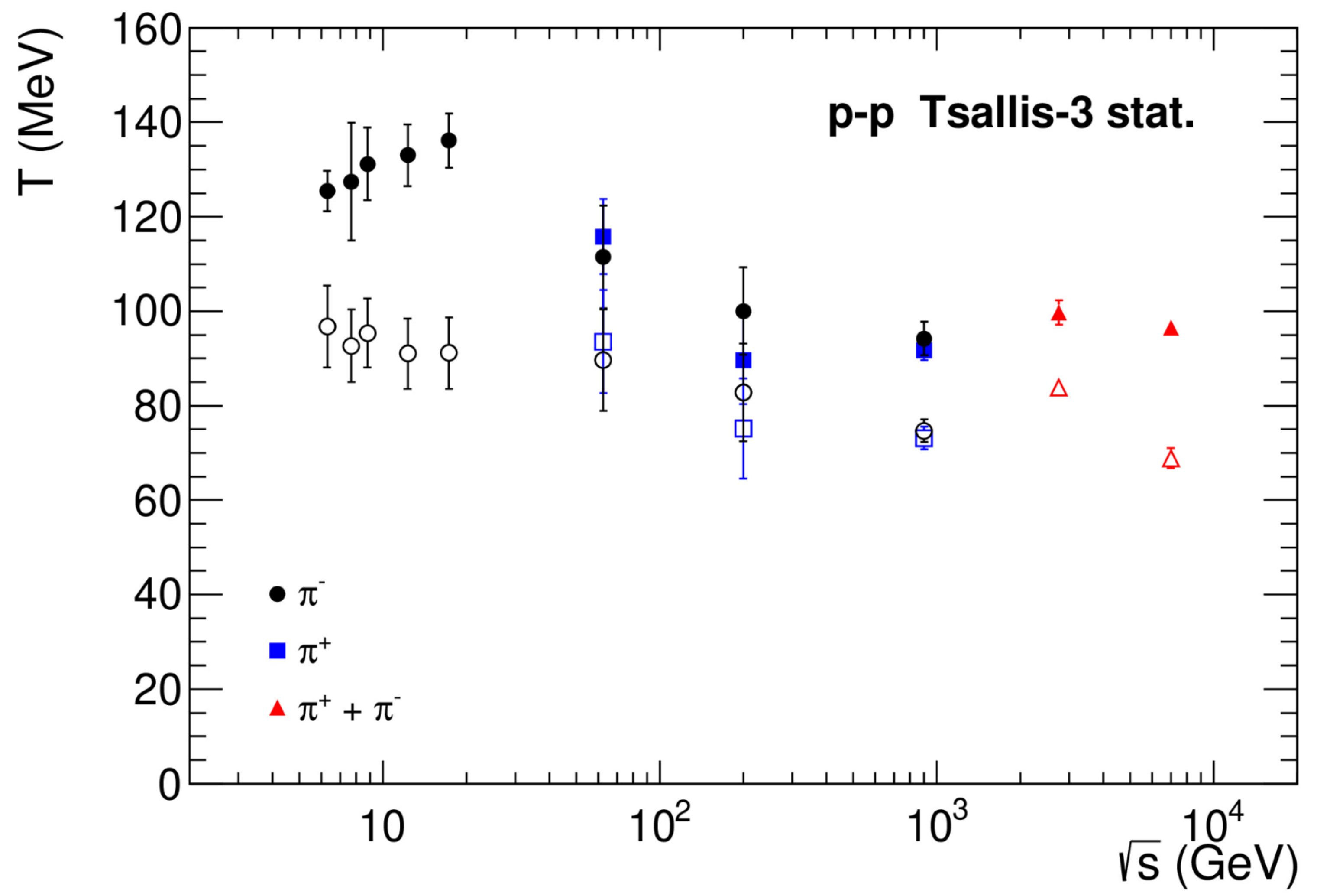
Table 1. Parameters of the Tsallis-3 statistics fit for the pions produced in pp collisions at different energies. The chemical potential $\mu = 0$.

Collaboration	Type	$\sqrt{s}$, GeV	T, MeV	R, fm	q	χ^2/ndf	ν
NA61/SHINE	π^-	6.3	125.45±4.23	4.502±0.171	$1.0217\pm\ 2\cdot 10^{-6}$	2.70/15	0.4
NA61/SHINE	π^-	7.7	127.45±12.49	4.893±0.589	1.0258±0.0053	1.15/15	0.4
NA61/SHINE	π^-	8.8	131.20±7.66	4.864±0.371	1.0249±0.0039	0.82/15	0.4
NA61/SHINE	π^-	12.3	133.02±6.47	5.330±0.342	1.0396±0.0050	0.77/15	0.4
NA61/SHINE	π^-	17.3	136.13±5.74	5.515±0.310	1.0398±0.0044	0.44/15	0.4
PHENIX	π^+	62.4	115.80±7.97	4.260±0.566	1.0694±0.0045	1.96/23	0.4
PHENIX	π^-	62.4	111.44±10.86	4.563±0.813	1.0705±0.0062	1.11/23	0.4
PHENIX	π^+	200	89.65±9.37	5.349±0.874	1.0913±0.0038	1.40/24	0.4
PHENIX	π^-	200	100.05±9.28	4.610±0.700	1.0859±0.0040	0.92/24	0.4
ALICE	π^+	900	91.74±2.14	5.093±0.103	1.0995±0.0016	3.19/30	0.4
ALICE	π^-	900	94.22±3.57	4.960±0.165	1.0976±0.0024	1.38/30	0.4
ALICE	$\pi^+ + \pi^-$	2760	99.71±2.59	4.965±0.111	1.0782±0.0004	10.52/60	0.6
ALICE	$\pi^+ + \pi^-$	7000	96.44±1.38	6.219±0.074	1.1163±0.0006	11.45/38	0.6

The approximate phenomenological Tsallis distribution

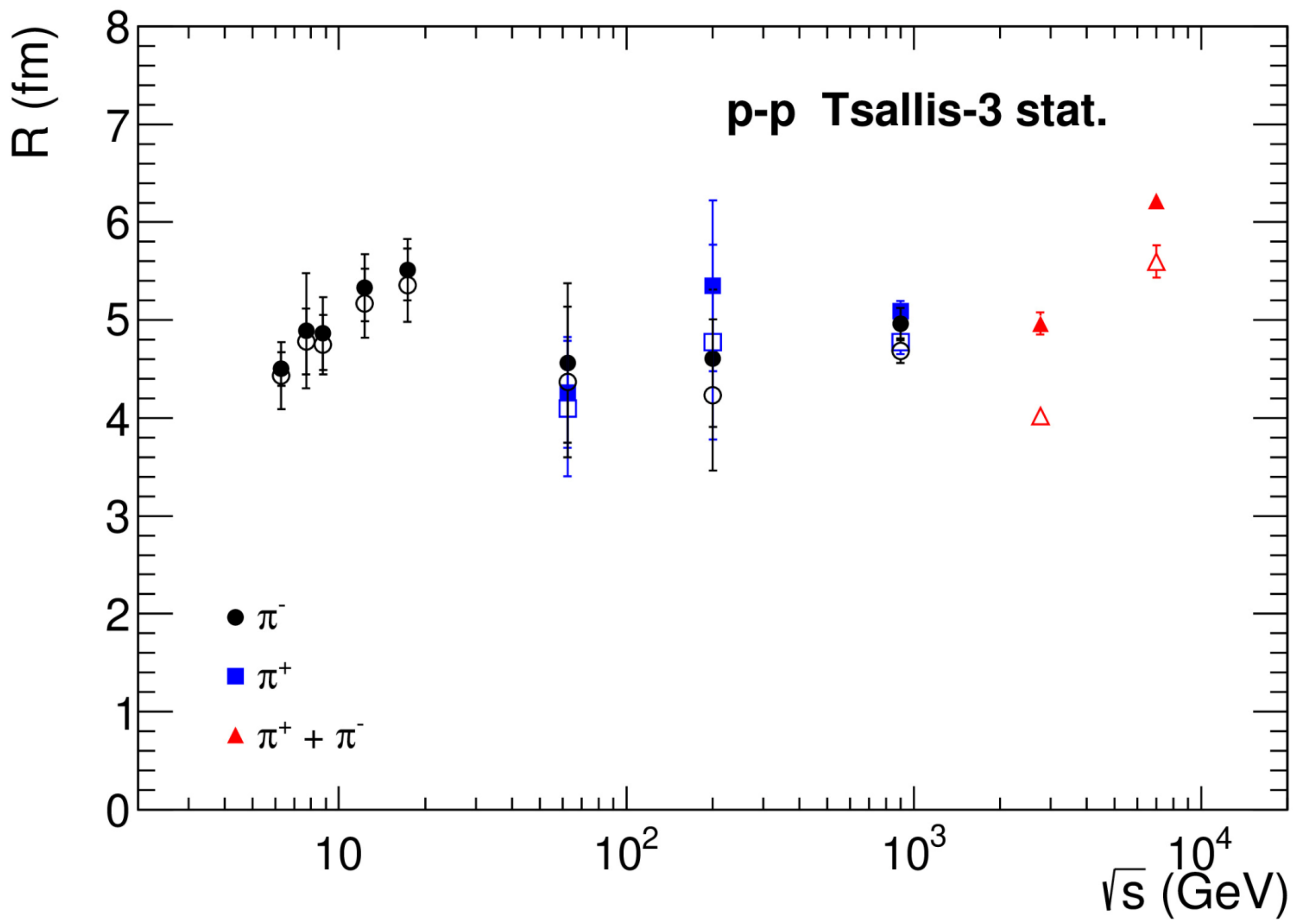
Table 2. Parameters of the approximate phenomenological Tsallis distribution (the Tsallis-3 distribution in the zeroth term approximation) fit for the pions produced in pp collisions at different energies (the parameters for π^- are practically the same as in Ref. [24]). The chemical potential $\mu = 0$.

Collaboration	Type	$\sqrt{s}$, GeV	T, MeV	R, fm	q	χ^2/ndf
NA61/SHINE	π^-	6.3	96.76±8.69	4.431±0.344	1.0449±0.0223	2.70/15
NA61/SHINE	π^-	7.7	92.68±7.67	4.782±0.334	1.0647±0.0208	1.14/15
NA61/SHINE	π^-	8.8	95.39±7.33	4.749±0.301	1.0580±0.0204	0.99/15
NA61/SHINE	π^-	12.3	91.04±7.44	5.172±0.350	1.0741±0.0209	0.89/15
NA61/SHINE	π^-	17.3	91.18±7.57	5.357±0.376	1.0736±0.0205	0.46/15
PHENIX	π^+	62.4	93.57±10.91	4.098±0.692	1.0893±0.0101	2.02/23
PHENIX	π^-	62.4	89.67±10.69	4.370±0.769	1.0908±0.0098	1.17/23
PHENIX	π^+	200	75.17±10.62	4.773±0.994	1.1266±0.0092	1.45/24
PHENIX	π^-	200	82.81±10.37	4.234±0.772	1.1174±0.0089	1.01/24
ALICE	π^+	900	73.14±2.36	4.778±0.129	1.1482±0.0051	4.08/30
ALICE	π^-	900	74.68±2.37	4.686±0.125	1.1446±0.0051	2.18/30
ALICE	$\pi^+ + \pi^-$	2760	83.84±1.61	4.021±0.084	1.1480±0.0017	18.74/60
ALICE	$\pi^+ + \pi^-$	7000	68.86±2.14	5.597±0.165	1.1784±0.0035	14.05/38



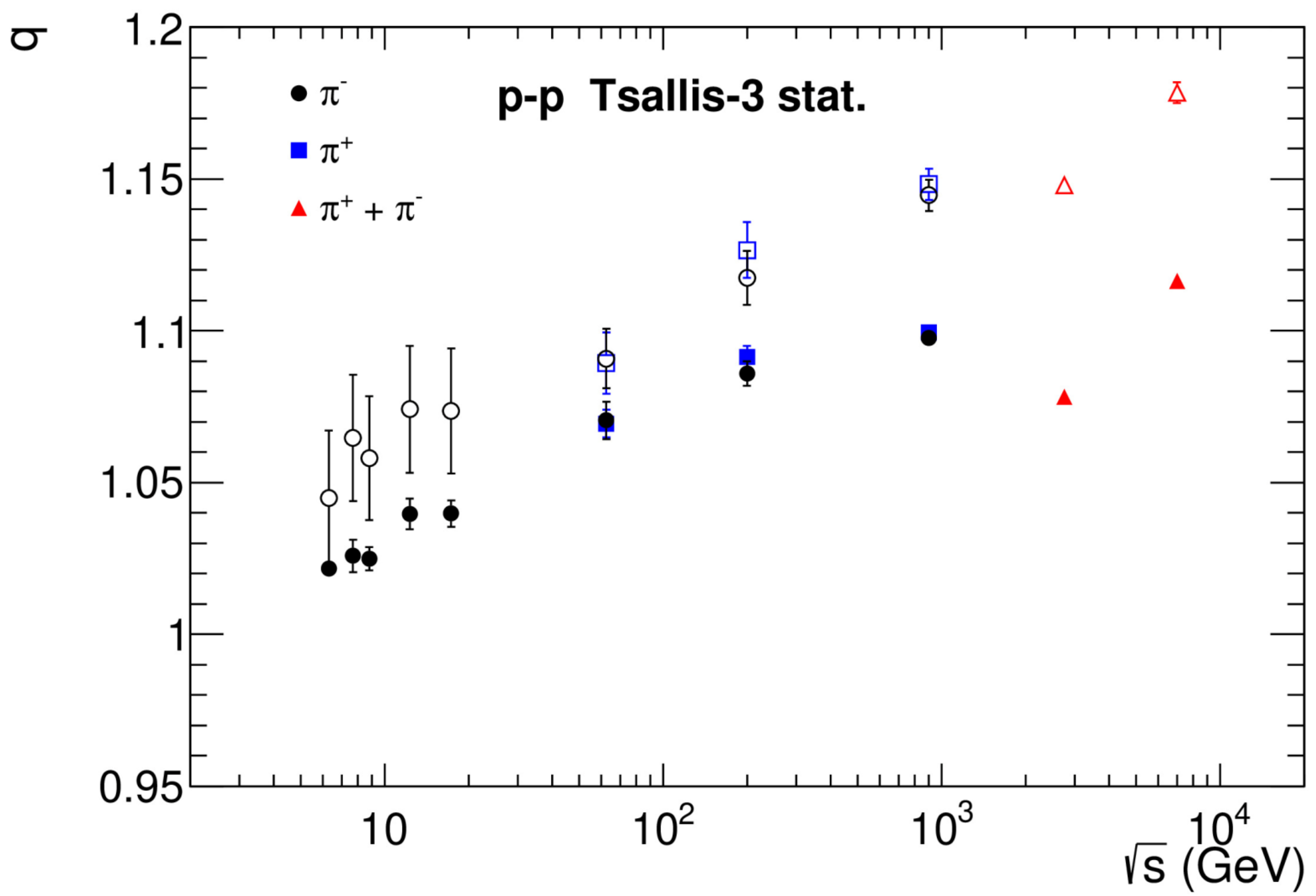
Temperature T of the Tsallis-3 statistics and the approximate phenomenological Tsallis distribution

- **Solid symbols** – the exact Tsallis-3 statistics distribution
- **Open symbols** – the approximate phenomenological Tsallis distribution



Radius *R* of system for the Tsallis-3 statistics and the approximate phenomenological Tsallis distribution

- **Solid symbols** – the exact Tsallis-3 statistics distribution
- **Open symbols** – the approximate phenomenological Tsallis distribution



Parameter *q* for the Tsallis-3 statistics and the approximate phenomenological Tsallis distribution

- **Solid symbols** – the exact Tsallis-3 statistics distribution
- **Open symbols** – the approximate phenomenological Tsallis distribution

Conclusions

1. The exact transverse momentum distributions for the Tsallis-3 statistics in the grand canonical ensemble for the Fermi-Dirac, Bose-Einstein and Maxwell-Boltzmann statistics of particles have been found.

2. The exact Tsallis-3 classical p_T-distribution and the classical approximate phenomenological Tsallis distribution have been applied to describe the experimental spectra of the charged pions produced in the pp collisions at LHC and RHIC energies.

3. We have revealed that the well-known classical approximate phenomenological Tsallis distribution in the framework of the Tsallis-3 statistics corresponds to the zeroth term approximation and to the unphysical condition of zero entropy of the system in the whole range of state variables. Moreover, we have found that it approximates the exact Tsallis-3 classical distribution unsatisfactory in the whole energy range of pp collision.

4. We have found that the quantum approximate phenomenological Tsallis distribution and the quantum Tsallis-like distribution used in high-energy physics are similar to the quantum transverse momentum distribution obtained in the Tsallis-3 statistics by introducing a mathematically inconsistent factorization approximation in the zeroth term approximation.

5. We have revealed that the transverse momentum distributions in the zeroth term approximation and in the factorization approximation of the zeroth term approximation are the same in the Tsallis-3, Tsallis-2 and q-dual statistics.

Thank you for your attention!