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The decomposed photon anomalous dimension in QCD and the $\{\beta\}$-expanded representations for the Adler function

A. L. Kataev*[1] and V. S. Molokoedov†[1, 2, 3, 4]

[1]Institute for Nuclear Research of the Russian Academy of Science, 117312, Moscow, Russia
[2]Research Computing Center, Moscow State University, 119991, Moscow, Russia
[3]Moscow Center for Fundamental and Applied Mathematics, 119992, Moscow, Russia
[4]Moscow Institute of Physics and Technology, 141700, Dolgoprudny, Moscow Region, Russia

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*kataev@ms2.inr.ac.ru
†viktor_molokoedov@mail.ru

1 Preliminaries.

Let us start from the consideration of the e^+e^- annihilation Adler function. It is defined in the Euclidean region as

$$D(L, a_s) = -\frac{d\Pi(L, a_s)}{d\ln Q^2} = Q^2 \int\limits_0^\infty ds \frac{R_{e^+e^-}(l, a_s)}{(s+Q^2)^2}, \tag{1}$$

where $a_s(\mu^2) = a_s = \alpha_s/\pi$, α_s is the $\overline{\text{MS}}$-scheme strong coupling constant, μ is the renormalization scale, $L = \ln(\mu^2/Q^2)$ and $l = \ln(\mu^2/s)$ respectively, $Q^2 = -q^2 > 0$ is the Euclidean kinematic variable and $s = q^2 > 0$ is the time-like Minkowskian variable.

The spectral function $R_{e^+e^-}(l, a_s)$ is the theoretical expression for the electron-positron annihilation R-ratio. It is proportional to the experimentally measured total cross section $\sigma(e^+e^- \to \gamma^* \to hadrons)$. $\Pi(L, a_s)$ is the renormalized QCD expression for the photon vacuum polarization function, which enters in the two-point correlator $\Pi_{\mu\nu}(q)$ of the electromagnetic quark vector currents $j^\mu(x)$ as

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|T\{j_\mu(x)j_\nu(0)\}|0\rangle = \frac{1}{12\pi^2}(q_\mu q_\nu - q^2 q_{\mu\nu})\Pi(q^2). \tag{2}$$

Here $j^\mu(x) = \sum_f Q_f \overline{\psi}_f(x)\gamma^\mu\psi_f(x)$, Q_f stands for the electric charge of the quark field $\psi_f(x)$ with flavor f. Note that since the vector current is conserved both in the renormalized and bare cases, the expression for the tensor $\Pi_{\mu\nu}(q)$ is transverse in both cases as well.

The detailed theoretical studies, conducted in the works of Refs.[53, 54] and used later on in Refs.[55, 56], lead to the following renormalization prescription for the photon vacuum polarization function in QCD:

$$\Pi(L, a_s) = Z(a_s) + \Pi_B(L, a_{sB}). \tag{3}$$

Here $\Pi_B(L, a_{sB})$ is the unrenormalized QCD expression for the vacuum polarization function; $a_{sB} = \alpha_{sB}/\pi = \mu^{2\varepsilon}Z_{a_s}(a_s)a_s$, α_{sB} is the bare strong coupling and $Z_{a_s}(a_s)$ is the corresponding renormalization constant, which defines the QCD RG β-function; $Z(a_s) = (Z_{ph}(a_s) - 1)/a$, where $a = \alpha/\pi$ and α is the renormalized QED coupling, defined in the case when the effects of its QED running are not taken into account, $Z_{ph}(a_s)$ is the renormalization constant of the photon wave function, considered in the case when the QCD corrections only are taken into.

Within the class of the MS-like subtraction schemes the expression for $Z(a_s)$ contains the pole terms in ε

$$Z(a_s) = \sum_{p\geq 1} a_s^{p-1}\sum_{k=1}^{p}\frac{Z_{p,-k}}{\varepsilon^k}, \tag{4}$$

whereas the quantity $\Pi_B(L, a_{sB})$ has the following form

$$\Pi_B(L, a_{sB}) = \sum_{p\geq 1}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon p} a_{sB}^{p-1}\sum_{k=-p}^{\infty}\Pi_{p,k}\varepsilon^k \tag{5}$$

with $\varepsilon = (4-d)/2$ and d is the space-time dimension.

The renormalized photon vacuum polarization function $\Pi(L, a_s)$ obeys the following inhomogeneous RG equation:

$$\left(\frac{\partial}{\partial\ln\mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi(L, a_s) = \gamma(a_s), \tag{6}$$

where

$$\gamma(a_s) = \frac{d\Pi(L, a_s)}{d\ln\mu^2} \tag{7}$$

is the QCD anomalous dimension of $\Pi(L, a_s)$ and $\beta(a_s)$ is the defining the scale dependence of the strong coupling RG β-function, namely

$$\beta(a_s) = \frac{\partial a_s}{\partial\ln\mu^2} = -\sum_{n\geq 0}\beta_n a_s^{n+2}. \tag{8}$$

One should note that the RG β-function is included in the renormalized expression for the trace of the energy-momentum tensor in the massless QCD and, therefore, is a measure of violation not only of the symmetry under the dilatation transformations, but under the conformal ones as well.

Application of the renormalization procedure leads to the following perturbative expression for the photon vacuum polarization function:

$$\Pi(L, a_s) = d_R\Big(\sum_f Q_f^2\Big)\Pi^{NS}(L, a_s) + d_R\Big(\sum_f Q_f\Big)^2\Pi^{SI}(L, a_s), \tag{9}$$

where d_R is the dimension of the fundamental representation of the considered generic simple gauge group. In our study we are primarily interested in the case of the $SU(N_c)$ gauge group with $d_R = N_c$. In its particular case of the $SU(3)$ color group, relevant for physical QCD, $N_c = 3$. The quantities $\Pi^{NS}(L, a_s)$ and $\Pi^{SI}(L, a_s)$ are the flavor non-singlet (NS) and singlet (SI) contributions to $\Pi(L, a_s)$ respectively.

Substituting $\Pi(L, a_s)$ from Eq.(9) into (6), one can get the PT expression for the photon anomalous dimension:

$$\gamma(a_s) = d_R\Big(\sum_f Q_f^2\Big)\gamma^{NS}(a_s) + d_R\Big(\sum_f Q_f\Big)^2\gamma^{SI}(a_s), \tag{10}$$

where $\gamma^{NS}(a_s)$ and $\gamma^{SI}(a_s)$ are the PT series in strong coupling :

$$\gamma^{NS}(a_s) = \sum_{n\geq 0}\gamma_n a_s^n, \qquad \gamma^{SI}(a_s) = \sum_{n\geq 3}\gamma_n^{SI} a_s^n. \tag{11}$$

Taking into account Eqs.(6), (8), one arrives to the following RG-improved expressions for $\Pi^{NS}(L, a_s)$ and $\Pi^{SI}(L, a_s)$ at $L = 0$:

$$\Pi^{NS}(0, a_s(Q^2)) = \sum_{n\geq 0}\Pi_n a_s^n(Q^2), \qquad \Pi^{SI}(0, a_s(Q^2)) = \sum_{n\geq 3}\Pi_n^{SI} a_s^n(Q^2). \tag{12}$$

The solution of the RG equation (6) can be found perturbatively. Its explicit form obtained at the $\mathcal{O}(a_s^4)$ level is presented in Appendix A.

Using the expressions, presented above, it is possible to derive the following expression for the Adler function

$$D(L, a_s) = \gamma(a_s) - \beta(a_s)\frac{\partial}{\partial a_s}\Pi(L, a_s). \tag{13}$$

In contrast to the polarization operator, it is the RG-invariant quantity. Therefore, it satisfies the homogeneous RG equation:

$$\frac{dD(L, a_s)}{d\ln\mu^2} = \Big(\frac{\partial}{\partial\ln\mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\Big)D(L, a_s) = 0. \tag{14}$$

Solving the system of the corresponding RG equations, one can get the following PT expression for the Adler function

$$D(a_s(Q^2)) = d_R\Big(\sum_f Q_f^2\Big)D^{NS}(a_s(Q^2)) + d_R\Big(\sum_f Q_f\Big)^2 D^{SI}(a_s(Q^2)), \tag{15}$$

where its NS and SI contributions are defined as

$$D^{NS}(a_s(Q^2)) = \sum_{n\geq 0} d_n a_s^n(Q^2), \tag{16a}$$

$$D^{SI}(a_s(Q^2)) = \sum_{n\geq 3} d_n^{SI} a_s^n(Q^2). \tag{16b}$$

In the massless limit all logarithmic corrections to $D(Q^2)$, controlled by the RG, can be summed up into the running coupling $a_s(Q^2)$.

Using the explicit solution of Eq.(6) for $\Pi(L, a_s)$, one can obtain the solution of the RG equation (14), expressed in terms of the PT coefficients of $\Pi(L, a_s)$, $\beta(a_s)$ and $\gamma(a_s)$. The explicit form of its $\mathcal{O}(a_s^4)$ approximation is given in Appendix A as well.

Comparing solutions the expressions of Eqs.(16a-16b) with the ones following from Eq.(13) and taking into account the dependence $a_s(Q^2)$ on the Euclidean momentum Q^2, we can obtain the following relations:

$$d_0 = \gamma_0, \tag{17a}$$
$$d_1 = \gamma_1, \tag{17b}$$
$$d_2 = \gamma_2 + \beta_0\Pi_1, \tag{17c}$$
$$d_3 = \gamma_3 + 2\beta_0\Pi_2 + \beta_1\Pi_1, \tag{17d}$$
$$d_4 = \gamma_4 + 3\beta_0\Pi_3 + 2\beta_1\Pi_2 + \beta_2\Pi_1, \tag{17e}$$
$$d_3^{SI} = \gamma_3^{SI}, \tag{17f}$$
$$d_4^{SI} = \gamma_4^{SI} + 3\beta_0\Pi_3^{SI}. \tag{17g}$$

One should recall that in the class of the gauge-invariant MS-like schemes, the scheme dependence of the coefficients d_k starts to manifest itself from $k \geq 2$ due to scheme-dependent terms Π_m at $m \geq 1$. The expressions (17a-17g) are derived comparing the RG-based relation directly associating the Adler function to the photon vacuum polarization function and to its anomalous dimension.

The analytical expressions for the coefficients $d_0 \div d_4$ and $\gamma_0 \div \gamma_4$, $\Pi_0 \div \Pi_3$ may be found in Refs.[57] and [58] correspondingly (see also references therein).

In the MS-like scheme the coefficients of the corresponding RG β-function up to β_3-term are known from the results of analytical calculations of Ref. [59], which were effectively confirmed by the foundation of nullification of the three-loop DR- like scheme approximation for the RG β-function of N=4 SUSY YM theory (see e.g. [60]) and by the direct analytical calculations from Ref.[61].

2 The $\{\beta\}$-expansion of $\gamma(a_s)$ and $\Pi(a_s)$.

As was already mentioned in the Introduction, the $\{\beta\}$-expansion formalism implies the representation of the expressions of higher-order PT corrections to the massless RG-invariant quantities, evaluated in the gauge-invariant renormalization schemes, through the sums of the monomials in powers of the RG β-function coefficients with separating the scale-invariant contributions.

For instance, the coefficients $d_1 \div d_4$ and $d_3^{SI} \div d_4^{SI}$ of the e^+e^- annihilation Adler function, defined in Eqs.(15), (16a-16b), have the following $\{\beta\}$-expanded structure:

$$d_1 = d_1[0], \tag{18a}$$
$$d_2 = \beta_0 d_2[1] + d_2[0], \tag{18b}$$
$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0], \tag{18c}$$
$$d_4 = \beta_0^3 d_4[3] + \beta_1\beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0], \tag{18d}$$
$$d_3^{SI} = d_3^{SI}[0], \tag{18e}$$
$$d_4^{SI} = \beta_0 d_4^{SI}[1] + d_4^{SI}[0], \tag{18f}$$

where terms $d_k[\dots]$, $d_k^{SI}[\dots]$ do not contain n_f-dependence (except for the flavor n_f-dependence arising from the contributions to $d_4[0]$ of the light-by-light scattering type diagrams [28, 29], [34]).

Starting from $k \geq 3$, there is an ambiguity associated with fixing the terms $d_k[\dots]$ (apart from the coefficients $d_k[k-1]$ defined by the leading renormalon chains effects, obtained in the case of QED in Ref.[62] and reformulated to the case of QCD in Ref.[7]).

It is related to the difficulty of univocal splitting of the n_f-dependent contributions to higher-order corrections to RG-invariant quantity between flavor-dependent coefficients β_i (with $i \geq 0$) of the RG β-function (see e.g. [4, 32, 39]).

One of the currently existing ways to resolve this problem was proposed in Ref.[28] and wider considered in Refs.[29, 30]. In accordance with these considerations the PT expression for the NS-contribution to the Adler function at the $\mathcal{O}(a_s^4)$ level can be presented through the following double sum:

$$D^{NS}(a_s(Q^2)) = 1 + \sum_{n=0}^{3} \left(\frac{\beta(a_s(Q^2))}{a_s(Q^2)}\right)^n \sum_{k=1}^{4-n} D_{n,k} a_s^k(Q^2) \tag{19}$$
$$= 1 + D_{0,1}a_s(Q^2) + \Big(D_{0,2} - \beta_0 D_{1,1}\Big)a_s^2(Q^2) + \Big(D_{0,3} - \beta_0 D_{1,2} - \beta_1 D_{1,1} + \beta_0^2 D_{2,1}\Big)a_s^3(Q^2)$$
$$+ \Big(D_{0,4} - \beta_0 D_{1,3} - \beta_1 D_{1,2} - \beta_2 D_{1,1} + \beta_0^2 D_{2,2} + 2\beta_0\beta_1 D_{2,1} - \beta_0^3 D_{3,1}\Big)a_s^4(Q^2).$$

We will not touch here the grounds of the presented in Eq.(19)-type representations, including the considered in this paper case of the Adler function. The arguments in its favour are given in Refs.[28–30]. In fact the theoretical ways of fixing analytical expressions of the terms $d_k[\dots]$ are not not unique (see e.g. definite considerations, presented in Refs. [29], [34, 35]). Here we will consider the one, outlined in Ref. [28] and followed in Ref. [30].

As observed by us there, it is applicable to the wider class of the functions and quantities in the corresponding RG equations. One may ask us the following question: what are the theoretical and phenomenological reasons for separating the scale-invariant contributions $d_k[0]$ from the total expressions for the coefficients d_k?

The problem of careful consideration of the status of the PMC-related expressions and of the unraveling in them of the effects related to the scale-invariant limit and to its violation by the CSB effects amongst the answers to this question.

The PMC-related considerations enable to eliminate β-dependent terms in the coefficients d_k of Eqs,(18b-18d) and Eq. (18f) by redefining the scale parameter in every order of PT and to leave in the coefficients of the PT expressions for the related to the Green functions quantities the scale-invariant parts $d_k[0]$ only.

As a result, the scale parameter becomes the coupling-dependent function (for the concrete realization of this feature within large n_f-expansion see Refs. [14, 15] and Ref.[16] while its PMC-type realizations are given in Refs.[21, 32]). It is also important that the PT coefficients of higher-order corrections to the corresponding physical quantities, studied in the gauge-invariant schemes are becoming independent on the choice of scale.

The representation of Eq.(19) allows not only to separate the scale-invariant contributions $d_k[0]$, but to reproduce the structure of the $\{\beta\}$-expansion in Eqs, (18b-18d) as well. It also imposes essential restrictions on the terms of this decomposition, namely:

$$d_2[1] = d_3[0,1] = d_4[0,0,1] = -D_{1,1}, \tag{20a}$$
$$d_3[1] = d_4[0,1] = -D_{1,2}, \tag{20b}$$
$$d_3[2] = d_4[1,1]/2 = D_{2,1}. \tag{20c}$$

This property is in correspondence with the feature of *special degeneracy* observed in Ref.[23] while applying considered there R_δ-procedure.

Application of these relations allowed the authors of Ref.[28] to get the analytical expressions for the terms $d_k[\dots]$ with $k \leq 4$ in the $\{\beta\}$-expansions (18b-18d). Their explicit form is given in Appendix B.

The representation (19) enabled also to fix several terms of the $\{\beta\}$-expansion of the totally unknown at present coefficient d_5 [29].

In the approach described above, for finding terms $d_k[\dots]$ of the $\{\beta\}$-decomposed corrections to the e^+e^- annihilation Adler function, it is not necessary to use any information about the possible $\{\beta\}$-structure of the RG-related quantities such as the photon anomalous dimension or the vacuum polarization function. In this case we deal directly with the RG invariant quantity $D(Q^2)$. However, when we pass to consideration of the relation (13) between $D(L, a_s)$, $\gamma(a_s)$ and $\Pi(L, a_s)$ and to the expressions (17a-17g) following from it, the important issue whether or not to decompose the coefficients of $\gamma(a_s)$ and $\Pi(L, a_s)$ is vividly arising.

We adhere here to the statement made previously in Refs.[29, 32, 39] that it is really necessary to decompose them in combinations of the β-function coefficients.

In accordance with this opinion, in order to extract the scale-invariant contributions from the PT expressions for the photon anomalous dimension, we should apply the $\{\beta\}$-expansion procedure to the coefficients γ_k and γ_k^{SI} of the photon anomalous dimension function $\gamma(a_s)$ in Eq.(11) as well.

The additional arguments in favor of this assertion will be given below. Following the proposal of Ref.[39], we will write down:

$$\gamma_1 = \gamma_1[0], \tag{21a}$$
$$\gamma_2 = \beta_0\gamma_2[1] + \gamma_2[0], \tag{21b}$$
$$\gamma_3 = \beta_0^2\gamma_3[2] + \beta_1\gamma_3[0,1] + \beta_0\gamma_3[1] + \gamma_3[0], \tag{21c}$$
$$\gamma_4 = \beta_0^3\gamma_4[3] + \beta_1\beta_0\gamma_4[1,1] + \beta_2\gamma_4[0,0,1] + \beta_0^2\gamma_4[2] + \beta_1\gamma_4[0,1] + \beta_0\gamma_4[1] + \gamma_4[0], \tag{21d}$$
$$\gamma_3^{SI} = \gamma_3^{SI}[0], \tag{21e}$$
$$\gamma_4^{SI} = \beta_0\gamma_4^{SI}[1] + \gamma_4^{SI}[0]. \tag{21f}$$

The equation (13) leads to the relations $d_k[0] = \gamma_k[0]$ and $d_k^{SI}[0] = \gamma_k^{SI}[0]$. This fact in conjunction with the $\{\beta\}$-expansion (18a-18d) and equalities (20a-20c) entails the following relationships for terms $\gamma_k[\ldots]$ of the photon anomalous dimension:

$$\gamma_2[1] = \gamma_3[0,1] = \gamma_4[0,0,1], \qquad \gamma_3[1] = \gamma_4[0,1], \qquad \gamma_3[2] = \gamma_4[1,1]/2. \tag{22}$$

Accommodating the explicit analytical expressions for the coefficients $\gamma_0 \div \gamma_4$ [58], $\beta_0 \div \beta_2$ [59] and using the relations (22), we obtain the analytical expressions for terms $\gamma_k[\ldots]$ and $\gamma_k^{SI}[\ldots]$:

$$\gamma_1[0] = \frac{3}{4}C_F, \quad \gamma_2[0] = -\frac{3}{32}C_F^2 + \frac{1}{16}C_FC_A, \quad \gamma_2[1] = \gamma_3[0,1] = \gamma_4[0,0,1] = \frac{11}{16}C_F, \tag{23a}$$
$$\gamma_3[1] = \gamma_4[0,1] = \left(\frac{239}{192} - \frac{11}{4}\zeta_3\right)C_F^2 + \left(\frac{163}{288} + \frac{11}{4}\zeta_3\right)C_FC_A, \; \gamma_3[2] = \frac{1}{2}\gamma_4[1,1] = -\frac{77}{144}C_F, \tag{23b}$$
$$\gamma_3[0] = -\frac{69}{128}C_F^3 + \left(-\frac{101}{256} + \frac{33}{16}\zeta_3\right)C_F^2C_A + \left(-\frac{53}{192} - \frac{33}{16}\zeta_3\right)C_FC_A^2, \tag{23c}$$
$$\gamma_4[2] = \left(\frac{5467}{1536} - \frac{119}{16}\zeta_3 + \frac{99}{32}\zeta_4\right)C_F^2 + \left(-\frac{123}{512} + \frac{629}{64}\zeta_3 - \frac{99}{32}\zeta_4\right)C_FC_A, \tag{23d}$$

$$\begin{aligned}\gamma_4[1] = &\left(-\frac{1477}{256} - \frac{135}{32}\zeta_3 + \frac{435}{32}\zeta_5\right)C_F^3 + \left(\frac{4733}{2048} + \frac{1167}{128}\zeta_3 - \frac{297}{128}\zeta_4 - \frac{765}{64}\zeta_5\right)C_F^2C_A \\ &+ \left(-\frac{16453}{18432} - \frac{2109}{256}\zeta_3 + \frac{297}{128}\zeta_4 - \frac{135}{128}\zeta_5\right)C_FC_A^2, \quad \gamma_4[3] = \left(3 - \frac{107}{384} - \frac{3}{8}\zeta_3\right)C_F,\end{aligned} \tag{23e}$$

$$\begin{aligned}\gamma_4[0] = &\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right)C_F^4 + \left(-\frac{3509}{1536} - \frac{73}{128}\zeta_3 - \frac{165}{32}\zeta_5\right)C_F^3C_A \\ &+ \left(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\right)C_F^2C_A^2 + \left(-\frac{30863}{36864} - \frac{147}{128}\zeta_3 + \frac{165}{64}\zeta_5\right)C_FC_A^3 \\ &+ \left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} + \left(-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f,\end{aligned} \tag{23f}$$

$$\gamma_3^{SI}[0] = \left(\frac{11}{192} - \frac{1}{8}\zeta_3\right)\frac{d^{abc}d^{abc}}{d_R}, \qquad \gamma_4^{SI}[1] = \left(\frac{55}{256} - \frac{123}{256}\zeta_3 + \frac{9}{64}\zeta_4 + \frac{15}{64}\zeta_5\right)\frac{d^{abc}d^{abc}}{d_R}, \tag{23g}$$
$$\gamma_4^{SI}[0] = \left(\left(-\frac{13}{64} - \frac{1}{4}\zeta_3 + \frac{5}{8}\zeta_5\right)C_F + \left(\frac{205}{1536} - \frac{13}{64}\zeta_3 - \frac{5}{32}\zeta_5\right)C_A\right)\frac{d^{abc}d^{abc}}{d_R}, \tag{23h}$$

where $\zeta_n = \sum_{k\geq 1}^{\infty} k^{-n}$ is the Riemann zeta-function; C_F and C_A are the quadratic Casimir operator in the fundamental and adjoint representation of the gauge group correspondingly, $d^{abc} = 2\,\mathrm{Tr}(t^a t^{\{b} t^{c\}})$, $d_F^{abcd} = \mathrm{Tr}(t^a t^{\{b} t^c t^{d\}})/6$ and $d_A^{abcd} = \mathrm{Tr}(C^a C^{\{b} C^c C^{d\}})/6$, $(C^a)_{bc} = -if^{abc}$ are the generators of the adjoint representation with the anti-symmetric structure constants f^{abc}: $[t^a, t^b] = if^{abc}t^c$. The terms proportional to $d_F^{abcd}d_F^{abcd}n_f/d_R$ and $d_F^{abcd}d_A^{abcd}/d_R$ group structures are the light-by-light scattering effects and they have to be included in the scale-invariant "n_f-independent" coefficient $\gamma_4[0]$ [28, 29].

Using now the expressions (17a-17g), (21a-21f) and taking into account the following $\{\beta\}$-expansion structure of the vacuum polarization function (9), (12)

$$\Pi_0 = \Pi_0[0], \quad \Pi_1 = \Pi_1[0], \quad \Pi_2 = \Pi_2[0] + \beta_0\Pi_2[1], \quad \Pi_3 = \Pi_3[0] + \beta_0\Pi_3[1] + \beta_1\Pi_3[0,1] + \beta_0^2\Pi_3[2], \tag{24}$$

we arrive to the substantial relationships between terms $d_k[\dots]$, $\gamma_k[\dots]$ and $\Pi_k[\dots]$:

$$d_1[0]=\gamma_1[0],\quad d_2[0]=\gamma_2[0],\qquad d_2[1]=\gamma_2[1]+\Pi_1[0], \tag{25a}$$
$$d_3[0]=\gamma_3[0],\quad d_3[1]=\gamma_3[1]+2\Pi_2[0],\quad d_3[0,1]=\gamma_3[0,1]+\Pi_1[0],\quad d_3[2]=\gamma_3[2]+2\Pi_2[1], \tag{25b}$$
$$d_4[0]=\gamma_4[0],\quad d_4[1]=\gamma_4[1]+3\Pi_3[0],\quad d_4[0,1]=\gamma_4[0,1]+2\Pi_2[0],\quad d_4[2]=\gamma_4[2]+3\Pi_3[1], \tag{25c}$$
$$d_4[3]=\gamma_4[3]+3\Pi_3[2],\qquad d_4[1,1]=\gamma_4[1,1]+3\Pi_3[0,1]+2\Pi_2[1], \tag{25d}$$
$$d_4[0,0,1]=\gamma_4[0,0,1]+\Pi_1[0],\quad d_3^{SI}[0]=\gamma_3^{SI}[0],\quad d_4^{SI}[0]=\gamma_4^{SI}[0],\quad d_4^{SI}[1]=\gamma_4^{SI}[1]+3\Pi_3^{SI}[0], \tag{25e}$$

where $\Pi_3^{SI}[0]=\Pi_3^{SI}$. Applying these relations and using the explicit expressions for $d_k[\dots]$ from Refs.[28, 29] and for $\gamma_k[\dots]$ from Eqs.(23a-23h), we get the values of terms $\Pi_k[\dots]$:

$$\Pi_0[0]=\frac{5}{3},\qquad \Pi_1[0]=\left(\frac{55}{16}-3\zeta_3\right)C_F,\qquad \Pi_2[1]=\left(\frac{3701}{288}-\frac{19}{2}\zeta_3\right)C_F, \tag{26a}$$
$$\Pi_2[0]=\left(-\frac{143}{96}-\frac{37}{8}\zeta_3+\frac{15}{2}\zeta_5\right)C_F^2+\left(\frac{73}{72}-\frac{3}{4}\zeta_3-\frac{5}{4}\zeta_5\right)C_FC_A, \tag{26b}$$

$$\Pi_3[2]=\left(\frac{196513}{3456}-\frac{809}{24}\zeta_3-15\zeta_5\right)C_F,\qquad \Pi_3[0,1]=\left(\frac{3701}{432}-\frac{19}{3}\zeta_3\right)C_F, \tag{26c}$$
$$\begin{aligned}\Pi_3[1]=&\left(-\frac{22103}{4608}-\frac{1439}{24}\zeta_3+9\zeta_3^2-\frac{33}{32}\zeta_4+\frac{125}{2}\zeta_5\right)C_F^2\\&+\left(\frac{29353}{1536}-\frac{473}{192}\zeta_3-\frac{3}{2}\zeta_3^2+\frac{33}{32}\zeta_4-\frac{185}{12}\zeta_5\right)C_FC_A,\end{aligned} \tag{26d}$$
$$\begin{aligned}\Pi_3[0]=&\left(-\frac{31}{256}+\frac{39}{32}\zeta_3+\frac{735}{32}\zeta_5-\frac{105}{4}\zeta_7\right)C_F^3+\left(-\frac{520933}{55296}+\frac{5699}{384}\zeta_3-\frac{33}{4}\zeta_3^2+\frac{99}{128}\zeta_4\right.\\&\left.-\frac{565}{64}\zeta_5+\frac{105}{8}\zeta_7\right)C_F^2C_A-\left(\frac{112907}{55296}+\frac{5839}{768}\zeta_3-\frac{33}{4}\zeta_3^2+\frac{99}{128}\zeta_4-\frac{835}{384}\zeta_5+\frac{35}{16}\zeta_7\right)C_FC_A^2,\end{aligned} \tag{26e}$$
$$\Pi_3^{SI}=\Pi_3^{SI}[0]=\left(\frac{431}{2304}-\frac{63}{256}\zeta_3-\frac{1}{8}\zeta_3^2-\frac{3}{64}\zeta_4+\frac{15}{64}\zeta_5\right)\frac{d^{abc}d^{abc}}{d_R}. \tag{26f}$$

One should note that in contrast to Eqs.(20a) and (22), for the analogous $\{\beta\}$-expanded terms of the photon vacuum polarization function one has $\Pi_3[0,1]\neq\Pi_2[1]$. However, they turn out to be proportional to each other, namely $\Pi_3[0,1]=2/3\cdot\Pi_2[1]$. This follows from the fact that the derivative $\partial/\partial a_s$ is included in Eq.(13). Thus, the analogue of the double sum representation (19) for the two-point correlator $\Pi(a_s)$ is not fulfilled, but is held for the term $\beta(a_s)\partial\Pi(a_s)/\partial a_s$ in Eq.(13) as the whole.

After receiving the concrete results (23a-23h) and (26a-26f) for the $\{\beta\}$-expanded coefficients of $\gamma(a_s)$ and $\Pi(a_s)$ respectively, we are moving now on to presenting extra arguments in favour of the necessity of their $\{\beta\}$-decomposition.

3 Arguments in favour of the $\{\beta\}$-expansion of $\gamma(a_s)$.

While realizing the expressed in Refs.[16, 21] PMC ideas to the $\overline{MS}$-scheme expression for the quantities, related to the Adler function, the authors of the initial manuscripts of Refs.[23, 31] adhere the point of view that the $\{\beta\}$-expansion procedure should not be applied to entering in the related to above presented Eq.(13) photon anomalous dimension $\gamma(a_s)$.

In the work of Ref.[32] within the effective QCD model with multiplet of massless gluinos the attempt to clarify that the non-application of the $\{\beta\}$-expansion approach to $\gamma(a_s)$ contradicts the renormalizability principles was made.

However, the arguments, given within this effective QCD related model in the related to Ref.[32] works, nor the arguments expressed within QCD itself in the work of Ref.[29] were not heard and the ideas of the applications of the PMC approximants without using $\{\beta\}$-expansion of the photon anomalous dimension are continuing to be applied to the related to Adler function quantities (see e.g. the works of Refs.[40–43] and the citations in the related discussions in even experimentally related works).

To our understanding, the opinion of the authors of the papers of e.g. of Refs.[23, 31, 40–43] may be summarized in the form of the following statements: *since the anomalous dimension $\gamma(a_s)$ is a scheme-invariant in the class of the MS-like schemes it is associated with the renormalization of the QED coupling by the QCD corrections only. These corrections are not related to the running of the strong coupling constant and thus the photon anomalous dimension RG function should be considered as a conformal contribution during the PMC scale setting analysis.*

Let us try to clarify once more not heard arguments given previously in Refs.[32, 39] and Ref. [29] rephrasing them by other way as

- we disagree that $\gamma(a_s)$ is not associated with the renormalization of the strong coupling constant. On the contrary, the QCD photon anomalous dimension is inseparably related with the renormalization of the QCD charge. Indeed, the coefficients of $\gamma(a_s)$ are expressed through the first-order pole terms $Z_{n+1,-1}$ of $Z(a_s)$ (4):

$$\gamma(a_s) = -\varepsilon Z(a_s) + \frac{dZ(a_s)}{d\ln\mu^2} = -\varepsilon Z(a_s) + (-\varepsilon a_s + \beta(a_s))\frac{\partial Z(a_s)}{\partial a_s} = -\sum_{n\geq 0}(n+1)Z_{n+1,-1}\, a_s^n. \tag{27}$$

  This fact directly follows from the condition of cancellation of the divergent terms in Eq.(3) and from the relation between the bare coupling a_{sB} and the renormalized one a_s.

  However, the renormalization prescription (3) allows to present the PT coefficients of the anomalous dimension $\gamma(a_s)$ through the terms $\Pi_{p,k}$ of the ε-expansion of the bare polarization operator $\Pi_B(L, a_{sB})$ (5) as well:

$$\gamma_0 = \Pi_{1,-1}, \tag{28a}$$
$$\gamma_1 = 2\Pi_{2,-1}, \tag{28b}$$
$$\gamma_2 = 3\Pi_{3,-1} - 3\beta_0\Pi_{2,0}, \tag{28c}$$
$$\gamma_3 = 4\Pi_{4,-1} - 8\beta_0\Pi_{3,0} - 2\beta_1\Pi_{2,0} + 4\beta_0^2\Pi_{2,1}, \tag{28d}$$
$$\gamma_4 = 5\Pi_{5,-1} - 15\beta_0\Pi_{4,0} - 5\beta_1\Pi_{3,0} - \frac{5}{3}\beta_2\Pi_{2,0} + \frac{35}{6}\beta_0\beta_1\Pi_{2,1} + 15\beta_0^2\Pi_{3,1} - 5\beta_0^3\Pi_{2,2}. \tag{28e}$$

  Here the total prefactor $d_R\sum_f Q_f^2$ in the r.h.s. of Eqs.(28a-28e) is omitted.

  The expressions (28a-28e) reveal explicitly the structure of the $\{\beta\}$-expansion in powers of the coefficients of the QCD RG β-function. Moreover, the β-dependent terms in Eqs.(28c-28e) appear after the QCD charge renormalization only.

  Therefore, the total expression for the photon anomalous dimension should not be considered as the conformal part of $D(Q^2)$-function. the related to this understanding statement is presented below.

- Since the strong coupling is running then the QCD photon anomalous dimension is not a scale-invariant object. Indeed, $d\gamma(a_s)/d\ln(Q^2) = \beta(a_s)\partial\gamma(a_s)/\partial a_s = -\beta_0\gamma_1 a_s^2 - (2\beta_0\gamma_2 + \beta_1\gamma_1)a_s^3 + \ldots \neq 0$ where $a_s{=}a_s(Q^2)$. Therefore, the scheme-invariance of *its coefficients* in the class of the gauge-invariant renormalization MS-like schemes (due to the relations $\gamma_k = -(k+1)Z_{k+1,-1}$ (27) and the scheme-invariance of the first-order pole terms in $Z(a_s)$) is not the argument against $\{\beta\}$-decomposition of $\gamma(a_s)$.

Let us repeat now another given in Ref. [29] serious extra argument for applying β-expansion to $\gamma(a_s)$.

- In the QED limit the term $\widetilde{d}_2[0]$ (52b) becomes equal to $\widetilde{d}_2^{\rm QED}[0] = -3/32 - 11/48N$, where N is the number of the charged leptons.

  This expression is N-dependent and does not correspond to the Rosner's result [63] of calculating of the divergent part of the photon field renormalization constant Z_{ph} in the quenched QED, formulated in the diagrammatic level in Ref.[64]. In this finite approximation the constant Z_{ph} does not contain the internal subgraphs renormalizing electromagnetic charge. The result of this work is $(Z_{ph}^{-1})_{div} = \dfrac{a_B}{3}\left(1 + \dfrac{3}{4}a_B \boxed{-\dfrac{3}{32}}\, a_B^2\right)\ln\dfrac{M^2}{m^2}$, where $a_B = \alpha_B/\pi$, α_B is the bare fine-structure constant, m is the lepton mass and M is the large scale cutoff mass.

The boxed term does not match the expression for $\widetilde{d}_2^{\rm QED}[0]$, obtained when the photon anomalous dimension is not $\{\beta\}$-decomposed, but it is in full agreement with the result for $d_2^{\rm QED}[0]$, following from the $U(1)$-limit of the given in Eq.(23a) $SU(N_c)$ expression for $\gamma_2[0]$ after fixing $C_F = 1$ and $C_A = 0$.

There is also the $\mathcal{N} = 1$ SUSY QCD argument in favour of the title of this section

- The NSVZ-like relation for the Adler function in the $\mathcal{N} = 1$ SUSY QCD derived in Ref.[65] and its detailed consideration at the three-loop level made in Ref.[66] serve as the extra arguments in favor of the $\{\beta\}$-expansion of the SUSY analog of the photon anomalous dimension, namely the anomalous dimension of the matter superfields. Indeed, the NSVZ relation will be violated at the three-loop level if one does not decompose this anomalous dimension in the first coefficient of the corresponding β-function[1].

Note that if we consider the Adler function defined in Eq.(15) directly, without involving Eq.(13) linking the functions $D(a_s)$, $\gamma(a_s)$ and $\Pi(a_s)$, then $\{\beta\}$-expansion for its PT expression should not depend on $\gamma(a_s)$ and $\Pi(a_s)$.

Moreover, from a formal point of view, for the $\{\beta\}$-decomposition there is no principal difference, for example, between the PT series for $D(Q^2)$-function, the Bjorken polarized sum rule or for the static interaction potential of the heavy quark-antiquark pair.

However, the results of the $\{\beta\}$-expansion for the Adler function, presented in Ref.[31], depend on $\gamma(a_s)$ in any case.

At the same time in the same paper the $\{\beta\}$-decomposition for the static QCD Coulomb-like potential, calculated analytically at the three-loop level in Ref.[67], is implemented on a general grounds as in Ref.[30] and for an arbitrary RG-invariant quantity.

Therefore, the agreement of the results of $\{\beta\}$-expansion for the static potential derived in Ref.[31] with those obtained in Ref.[30] in the framework of our formalism, is rather natural.

Thus, the photon anomalous dimension is the convenient ingredient for analytical calculations, but it should not affect the structure of the $\{\beta\}$-expansion of the Adler function.

If one follows the logic of the works [23, 31, 40–43] and does not decompose the quantity $\gamma(a_s)$ in powers of β_k-coefficients, then analytical expressions for the analogous to $d_k[\dots]$-coefficients will be different.

We denote these terms by $\widetilde{d}_k[\dots]$ to distinguish them from ours $d_k[\dots]$. For comparison of their analytical structure see expressions for $d_k[\dots]$ and $\widetilde{d}_k[\dots]$ in Appendix B.

Note, that the explicit analytical expressions for γ_4 and Π_3 contain the Riemann ζ_4-contributions [58], which, however, are mutually cancelled out in d_4 [57], i.e.

$$d_4^{(\zeta_4)} = \gamma_4^{(\zeta_4)} + 3\beta_0\Pi_3^{(\zeta_4)} = 0. \tag{29}$$

If we properly expand γ_4 and Π_3 (in accordance with Eqs.(21d) and (24)), we will naturally arrive to the absence of the ζ_4-contributions in expression for $d_4[0]$ [28, 29]. However, if one assumes that γ_4 is a scale-invariant term, then the coefficient $\widetilde{d}_4[0]$ (see Eq.(52g)) will definitely contain ζ_4-contributions.

This fact contradicts the consequences of the no-π theorem [68], explaining why ζ_4-contribution should appear in the expressions for higher-order PT corrections to the Adler function starting from the coefficient d_5 only.

Let us now discuss the consequences stemming from the results of Refs.[23, 31, 40–42] for the terms $\widetilde{d}_k[\dots]$ obtained when $\gamma(a_s)$ is not $\{\beta\}$-expanded.

- In this case the $\{\beta\}$-decomposition of the coefficients d_k of the Adler function has the following form:

$$d_2 = \beta_0\widetilde{d}_2[1] + \widetilde{d}_2[0], \tag{30a}$$

$$d_3 = \underbrace{\beta_0^2\widetilde{d}_3[2]}_{=0} + \beta_1\widetilde{d}_2[1] + 2\beta_0\widetilde{d}_3[1] + \widetilde{d}_3[0], \tag{30b}$$

$$d_4 = \underbrace{\beta_0^3\widetilde{d}_4[3]}_{=0} + \underbrace{3\beta_0^2\widetilde{d}_4[2]}_{=0} + 3\beta_0\widetilde{d}_4[1] + \underbrace{\frac{5}{2}\beta_1\beta_0\widetilde{d}_3[2]}_{=0} + 2\beta_1\widetilde{d}_3[1] + \beta_2\widetilde{d}_2[1] + \widetilde{d}_4[0], \tag{30c}$$

[1]It may be interesting to get the arguments in favor of this statement in the $\mathcal{N} = 1$ SUSY QCD at the four-loop level.

where curly brackets indicate terms with identically zero coefficients. It should be emphasized that this representation does not correspond to the well-known renormalon asymptotics $d_{k+1} \sim \beta_0^k k!$ at $k \gg 1$ for higher-order PT coefficients of the Adler function in the large-β_0 approximation (see e.g. [1, 2, 52]).

Indeed, all terms $\widetilde{d}_{k+1}[k]$ in Eqs.(30b-30c) at $k \geq 2$ *are identically equal to zero.* Thus, if we do not decompose the coefficients γ_k and Π_k in powers of the RG β-function coefficients, then we will not reproduce the large-β_0 asymptotics for d_k in any order of PT starting from $k = 3$.

The leading renormalon chain contribution, whose explicit general formula for arbitrary order k follows from analytical results, given in Refs.[7, 62], is fixed correctly when the photon anomalous dimension undergoes the $\{\beta\}$-expansion procedure only. In its turn, in Refs.[23, 31, 40–42] the missing n_f-dependent contributions are hidden in the expressions for non-zero terms $\widetilde{d}_2[0], \widetilde{d}_2[1]; \widetilde{d}_3[0], \widetilde{d}_3[1]; \widetilde{d}_4[0], \widetilde{d}_4[1]$ in Eqs.(30a-30c).

One more important point, which follows from toatally decomposed β-expanded represntation for the Adler function is the recovery of its original BLM NLO expression.

Thus the worries of Ref.[64] on non-recovery of the BLM results within the considered in Ref.[23] application of the R_δ procedure to the PT expression for the Adler function without proper expansion of the PT QCD series for the photon anomalous dimension $\gamma(s_s)$, critically commented in the more detailed PMC-related work of Ref.[31], turned out to have rather solid background.

Since after applying the formulated in Eq.(19) multiple β-function representation to the Adler function we reproduce its NLO BLM expression, we will call it as PMC/BLM approach.

4 Application of the PMC/BLM to the Adler function.

4.1 Modified PMC expressions.

At the first stage of the BLM-related applications one should consider the scale transformations $\mu \to \mu'$ and introduce the shift parameter $\Delta = L - L' = \ln(\mu^2/\mu'^2)$, where $L' = \ln(\mu'^2/Q^2)$.

Using now the scaling operator (which may also be called the dilatation operator), one can obtain the relation between $a_s(\mu^2)$ and $a_s(\mu'^2)$ in the following form considered previously in Ref.[4] and [32]:

$$\begin{aligned} a_s(\mu^2) &= a_s(\exp(\Delta)\cdot\mu'^2) = \exp\left(\Delta\frac{d}{d\ln\mu'^2}\right)a'_s = \exp\left(\Delta\beta(a'_s)\frac{\partial}{\partial a'_s}\right)a'_s \\ &= a'_s + \frac{\Delta}{1!}\beta(a'_s) + \frac{\Delta^2}{2!}\beta(a'_s)\frac{\partial}{\partial a'_s}\beta(a'_s) + \frac{\Delta^3}{3!}\beta(a'_s)\frac{\partial}{\partial a'_s}\left(\beta(a'_s)\frac{\partial}{\partial a'_s}\beta(a'_s)\right) + \dots\ , \end{aligned} \tag{31}$$

where $a'_s = a_s(\mu'^2)$.

At the next step, we choose the PMC/BLM scale shift Δ as a PT series in powers of $\beta_0 a'_s$:

$$\Delta = \ln\left(\frac{\mu^2}{\mu'^2}\right) = \Delta_0 + \sum_{k\geq 1}\Delta_k(\beta_0 a'_s)^k. \tag{32}$$

Taking into account this representation, one can rewrite the relation (31) in the fourth order of approximation in the following form:

$$\begin{aligned} a_s &= a'_s - \beta_0\Delta_0 {a'_s}^2 + (\beta_0^2\Delta_0^2 - \beta_1\Delta_0 - \beta_0^2\Delta_1){a'_s}^3 \\ &+ \left(\frac{5}{2}\beta_0\beta_1\Delta_0^2 - \beta_0\beta_1\Delta_1 + 2\beta_0^3\Delta_0\Delta_1 - \beta_0^3\Delta_0^3 - \beta_2\Delta_0 - \beta_0^3\Delta_2\right){a'_s}^4. \end{aligned} \tag{33}$$

Using now Eq.(33) at $\mu'^2 = Q^2$, bearing in mind the RG invariance of the Adler function and its $\{\beta\}$-expansion pattern (18a-18f), it is possible to get the expressions for the coefficients d'_k of $D(a'_s)$-function, normalized at the new scale, in the form given in Refs.[4, 32]:

$$\begin{aligned}
d'_1 &= d_1[0], && (34a)\\
d'_2 &= \beta_0(d_2[1]-\Delta_0 d_1[0]) + d_2[0], && (34b)\\
d'_3 + \delta_f (d_3^{SI})' &= \beta_0^2(d_3[2] - 2\Delta_0 d_2[1] + \Delta_0^2 d_1[0] - \Delta_1 d_1[0]) + \beta_1(d_3[0,1] - \Delta_0 d_1[0]) && (34c)\\
&+ \beta_0(d_3[1] - 2\Delta_0 d_2[0]) + d_3[0] + \delta_f d_3^{SI}[0],\\
d'_4 + \delta_f (d_4^{SI})' &= \beta_0^3(d_4[3] - 3\Delta_0 d_3[2] + 3\Delta_0^2 d_2[1] - 2\Delta_1 d_2[1] + 2\Delta_0\Delta_1 d_1[0] - \Delta_0^3 d_1[0] && (34d)\\
&- \Delta_2 d_1[0]) + \beta_0\beta_1(d_4[1,1] - 3\Delta_0 d_3[0,1] - 2\Delta_0 d_2[1] + 5\Delta_0^2 d_1[0]/2\\
&- \Delta_1 d_1[0]) + \beta_2(d_4[0,0,1] - \Delta_0 d_1[0]) + \beta_0^2(d_4[2] - 3\Delta_0 d_3[1] + 3\Delta_0^2 d_2[0]\\
&- 2\Delta_1 d_2[0]) + \beta_1(d_4[0,1] - 2\Delta_0 d_2[0]) + \beta_0(d_4[1] - 3\Delta_0 d_3[0])\\
&+ \beta_0\delta_f(d_4^{SI}[1] - 3\Delta_0 d_3^{SI}[0]) + d_4[0] + \delta_f d_4^{SI}[0],
\end{aligned}$$

where $\delta_f = (\sum Q_f)^2/\sum Q_f^2$. Recall that the coefficients $d_k[\dots]$ are presented in Appendix B.

Setting initially

$$\Delta_0 = \Delta_{BLM} = \frac{d_2[1]}{d_1[0]} = \left(\frac{33}{8} - 3\zeta_3\right)C_F, \tag{35}$$

one can introduce the energy scale Q_0^2:

$$Q_0^2 = Q^2 \exp(-\Delta_0). \tag{36}$$

At this new scale the expression for the Adler function reads

$$D(Q^2) = 3\sum_f Q_f^2\left(1 + d_1[0]a_s(Q_0^2) + d_2[0]a_s^2(Q_0^2) + \mathcal{O}(a_s^3(Q_0^2))\right). \tag{37}$$

Further on, taking into account the relation (35) and absorbing the remaining n_f and β_i-dependent contributions in Eq.(34c) into parameter Δ_1, we arrive to the following expression:

$$\beta_0\Delta_1(n_f) = \beta_0\left(\frac{d_3[2]}{d_1[0]} - \frac{d_2^2[1]}{d_1^2[0]}\right) + \frac{d_3[1]}{d_1[0]} - \frac{2d_2[0]d_2[1]}{d_1^2[0]} + \frac{\beta_1}{\beta_0}\frac{d_3[0,1]-d_2[1]}{d_1[0]}. \tag{38}$$

Application of the PMC/BLM approach at the $\mathcal{O}(a_s^3)$ level eventually yields:

$$D(Q^2) = 3\sum_f Q_f^2\left(1 + d_1[0]a_s(Q_1^2) + d_2[0]a_s^2(Q_1^2) + (d_3[0] + \delta_f d_3^{SI}[0])a_s^3(Q_1^2) + \mathcal{O}(a_s^4(Q_1^2))\right), \tag{39}$$

where Q_1^2 is defined in accordance with Eqs.(32), (35-36), (38) as

$$Q_1^2 = Q^2 \exp\left(-\Delta_0 - \beta_0\Delta_1(n_f)a_s(Q_0^2)\right). \tag{40}$$

Following this logic and using Eq.(34d), one can fix the parameter $\beta_0^2\Delta_2$ as

$$\begin{aligned}
\beta_0^2\Delta_2(n_f) &= \beta_0^2\left(\frac{d_4[3]}{d_1[0]} - 3\frac{d_2[1]d_3[2]}{d_1^2[0]} + 2\frac{d_2^3[1]}{d_1^3[0]}\right) + \beta_1\left(\frac{d_4[1,1]}{d_1[0]} - 3\frac{d_2[1]d_3[0,1]}{d_1^2[0]}\right. && (41)\\
&+ \left.\frac{3}{2}\frac{d_2^2[1]}{d_1^2[0]} - \frac{d_3[2]}{d_1[0]}\right) + \beta_0\left(\frac{d_4[2]}{d_1[0]} - 3\frac{d_3[1]d_2[1]}{d_1^2[0]} + 5\frac{d_2[0]d_2^2[1]}{d_1^3[0]} - 2\frac{d_2[0]d_3[2]}{d_1^2[0]}\right)\\
&+ \frac{d_4[1]}{d_1[0]} - 3\frac{d_3[0]d_2[1]}{d_1^2[0]} + \delta_f\left(\frac{d_4^{SI}[1]}{d_1[0]} - 3\frac{d_3^{SI}[0]d_2[1]}{d_1^2[0]}\right) - 2\frac{d_2[0]d_3[1]}{d_1^2[0]} + 4\frac{d_2^2[0]d_2[1]}{d_1^3[0]}\\
&+ \frac{\beta_1^2}{\beta_0^2}\frac{d_2[1]-d_3[0,1]}{d_1[0]} + \frac{\beta_1}{\beta_0}\left(\frac{d_4[0,1]-d_3[1]}{d_1[0]} - \frac{2d_2[0]}{d_1^2[0]}(d_3[0,1]-d_2[1])\right)\\
&+ \frac{\beta_2}{\beta_0}\frac{d_4[0,0,1]-d_2[1]}{d_1[0]}.
\end{aligned}$$

In this case, instead of the expressions (39) and (40) we obtain their higher order counterparts:

$$D(Q^2) = 3\sum_f Q_f^2\Big(1 + d_1[0]a_s(Q_2^2) + d_2[0]a_s^2(Q_2^2) + (d_3[0] + \delta_f d_3^{SI}[0])a_s^3(Q_2^2) \quad (42)$$
$$+ (d_4[0] + \delta_f d_4^{SI}[0])a_s^4(Q_2^2) + \mathcal{O}(a_s^5(Q_2^2))\Big),$$
$$Q_2^2 = Q^2 \exp\Big(- \Delta_0 - \beta_0\Delta_1(n_f)a_s(Q_1^2) - \beta_0^2\Delta_2(n_f)a_s^2(Q_1^2)\Big). \quad (43)$$

In a particular case of the $SU(3)$ color gauge group the numerical forms of the parameters Δ_0, $\beta_0\Delta_1$ and $\beta_0^2\Delta_2$ are defined in correspondingly and are included into the determination of the scale Q_2^2 in Eq.(43), read:

$$\begin{cases} \Delta_0 = \dfrac{11}{2} - 4\zeta_3 \approx 0.6918, & \text{(44a)}\\ \beta_0\Delta_1(n_f) = \beta_0\Big(\dfrac{119}{36} + \dfrac{56}{3}\zeta_3 - 16\zeta_3^2\Big) + \dfrac{51}{8} - \dfrac{47}{3}\zeta_3 + \dfrac{50}{3}\zeta_5 \approx 2.6249\beta_0 + 4.8249, & \text{(44b)}\\ \beta_0^2\Delta_2(n_f) = -3.599\beta_0^2 + 2.386\beta_1 + 7.128\beta_0 - 54.535 - 0.292\delta_f. & \text{(44c)}\end{cases}$$

Note that in the reality the expressions (44b) and (44c) do not contain the terms proportional to the factor β_1/β_0 and β_1^2/β_0^2, β_1/β_0, β_2/β_0, which are contained in their corresponding analytical forms, presented in Eq.(38)and Eq.(41).

This pleasant fact is the consequence of the relationships (20a-20b) stemming from the multiple β-function expansion (19), proposed in Ref.[28].

Using the values of the coefficients $d_k[\dots]$, given in Appendix B and originally obtained in Ref.[28] within the same decomposition (19) of the Adler function in powers of $\beta(a_s)/a_s$, advocated in our work, we get its numerical expression in the case of the $SU(3)$ color group relevant for physical QCD:

$$D(Q^2) = 3\sum_f Q_f^2\Big(1 + a_s(Q_2^2) + \frac{1}{12}a_s^2(Q_2^2) + (-23.2227 - 0.4132\delta_f)a_s^3(Q_2^2) \quad (45)$$
$$+ (81.1571 + 0.0802n_f - 2.7804\delta_f)a_s^4(Q_2^2) + \mathcal{O}(a_s^5(Q_2^2))\Big).$$

It is worth mentioning that the numerical results of Eq.(45) were previously presented in Ref.[29].

The magnitudes of their $\mathcal{O}(a_s^2)$ and $\mathcal{O}(a_s^3)$ coefficients are in agreement with the ones, received in Ref.[14] with help of the generalized BLM and the large-n_f expansion (see the related work of Ref. [16] where the numerical expression for the related $\mathcal{O}(a_s^4)$ coefficient in Eq.(45) was found.

This expression should be compared with analogous one, which follows from the analytical form of the $\{\beta\}$-decomposed representation of the Adler function, obtained and discussed in Refs.[34, 35]

Our expression (**??**) should be also compared with its counterpart following from the PMC-type considerations of Refs.[23, 31, 40–43] with the $\{\beta\}$-non-expanded photon anomalous dimension :

$$D(Q^2) = 3\sum_f Q_f^2\Big(1 + a_s(\widetilde{Q}_2^2) + (2.6042 - 0.1528n_f)a_s^2(\widetilde{Q}_2^2) \quad (46)$$
$$+ (9.7418 - 2.0426n_f - 0.0198n_f^2 - 0.4132\delta_f)a_s^3(\widetilde{Q}_2^2)$$
$$+ (41.0141 - 12.9110n_f + 0.4887n_f^2 + 0.0045n_f^3 + (-2.3829 - 0.0241n_f)\delta_f)a_s^4(\widetilde{Q}_2^2) + \mathcal{O}(a_s^5(\widetilde{Q}_2^2))\Big),$$

where as before we do not specify the explicit form of the corresponding scale $\widetilde{Q}_2^2$, which does not coincide with Q_2^2, but has the related to Eq.(43) order $a_s^2(\widetilde{Q}_2^2)$ representation.

As we have already discussed above, the coefficients in the expression (46) are n_f-dependent ones.

This essential difference of Eq.(46) from Eq. (45) is the consequence of the not applied $\{\beta\}$-expansion procedure to the photon anomalous dimension $\gamma(a_s)$ in the the papers of the members of the group of authors of Refs [23, 31, 40–43].

This fact was critically commented by in Sec. 3 of this work.

For the sake of completeness, we also present here the numerical $\overline{\rm MS}$-scheme result for the Adler function, which follows from obtained in [57, 58] and confirmed in [69] analytical $O(a_s^4)$-expression, written down in the numerical form as

$$D(Q^2) = 3\sum_f Q_f^2\Big(1 + a_s(Q^2) + (1.9857 - 0.1153n_f)a_s^2(Q^2) \\ + (18.2427 - 4.2158n_f + 0.0862n_f^2 - 0.4132\delta_f)a_s^3(Q^2) \\ + (135.7916 - 34.4402n_f + 1.8753n_f^2 - 0.0101n_f^3 + (-5.9422 + 0.1916n_f)\delta_f)a_s^4(Q^2) + \mathcal{O}(a_s^5(Q_2^2)\Big). \tag{47}$$

It is worth clarifying that the leading large n_f-contributions to Eq.(47) do agree with the numerical form of analytical QED result, obtained previously in Ref.[62], but disagree with the analogous numbers, given in Eq.(46) above. This is the consequence of the PMC-related feature, that the non-properly non-expanded expression for the photon anomalous dimension $\gamma(a_s)$ is containing the bulk of the renormalon-related contributions, while the remaining ones are absorbed into the $O(a_s^2)$-representation into not perfectly, to our minds, fixed scale $\widetilde{Q}_2^2$ of the non-perfectly , to our minds, realizations of in general theoretically interesting, to our minds, idea of applying the PMC-type expansion to the expression for the Adler function.

4.2 Energy dependence of the PMC/BLM and $\overline{\rm MS}$-scheme Adler function approximants.

4.2.1 PMC/BLM inputs.

Let us now specify what do we mean under the expression for the expansion parameters $a_s(Q_0^2)$, $a_s(Q_1^2)$ and $a_s(Q_2^2)$ in the next-to-leading order (NLO), next-to-next-to-leading order (N^{2}LO) and next-to-next-to-next-to-leading order (N^{2}LO) PMC/BLM Adler function approximants which are presented in Eqs.(37),(39) and Eq.(42) above.

In fact they are related to the given in Eq.(9.5) of the QCD PDG review of Ref.[81] of the $\overline{\rm MS}$-scheme QCD coupling constant represntation through the inverse powers of $ln(Q^2/\Lambda^{(n_f)2})$-terms with $\Lambda^{(n_f)}$ being the $\overline{\rm MS}$-scheme scale parameter, while the related NLO, N^2LO and N^3LO energy scales Q_0^2, Q_1^2 and Q_2^2 being fixed at the related orders of these representations through the relatively applied PMC/BLM ways of their fixations, given in Eqs.(36), (40), (43 with their numerical values of the parameters fixed at By Eqs.(44a-(44a) above.

In the concrete applications these ways of fixation can be re-written through the unique $\overline{\rm MS}$-scheme representation of the QCD coupling constant related to the arbitrary energy scale Q^2, but with the appropriately re-defined expressions of the $\overline{\rm MS}$-scheme scale parameters.

The related flavor-dependent expressions for the scale parameter of the PMC/BLM procedure, defined at the NLO, N^{2}LO and N^{3}LO forms of the the corresponding scale-transformation expressions read:

$$\Lambda^{\rm (BLM)}_{\rm NLO}(n_f) = \Lambda^{(n_f)}_{\rm NLO}\cdot \exp\Big[-\frac{1}{2}\Delta_0\Big], \tag{48a}$$

$$\Lambda^{\rm (PMC)}_{\rm N^2LO}(n_f) = \Lambda^{(n_f)}_{\rm N^2LO}\cdot \exp\Big[-\frac{1}{2}\Big(\Delta_0 + \beta_0\Delta_1(n_f)a_s^{\rm NLO}(Q^2/(\Lambda^{\rm (BLM)}_{\rm NLO})^2)\Big)\Big], \tag{48b}$$

$$\Lambda^{\rm (PMC)}_{\rm N^3LO}(n_f) = \Lambda^{(n_f)}_{\rm N^3LO}\cdot \exp\Big[-\frac{1}{2}\Big(\Delta_0 + \beta_0\Delta_1(n_f)a_s^{\rm N^2LO}(Q^2/(\Lambda^{\rm (PMC)}_{\rm N^2LO})^2) \\ + \beta_0^2\Delta_2(n_f)(a_s^{\rm N^2LO}(Q^2/(\Lambda^{\rm (PMC)}_{\rm N^2LO})^2))^2\Big)\Big], \tag{48c}$$

where $\Lambda^{(n_f)}_{\rm NLO}$, $\Lambda^{(n_f)}_{\rm N^2LO}$, $\Lambda^{(n_f)}_{\rm N^3LO}$ are the expressions for the QCD scale parameter defined in the $\overline{\rm MS}$-scheme in the corresponding order of PT, while the parameters Δ_0, $\beta_0\Delta_1(n_f)$, $\beta_0^2\Delta_2(n_f)$ are defined by Eqs.(44a)- (44c) presented above.

We will use the correspondingly truncated solutions of the RG-improved expressions for the running QCD coupling $a_s(Q^2)$ through the coefficients of the QCD β-function up to N^{3}LO four-loop corrections, analytically evaluated in Ref. [70] and confirmed in Ref.[71] and the inverse powers of $ln(Q^2/\Lambda^{(n_f)2})$-terms.

In the $\overline{MS}$-scheme $\Lambda^{(n_f)}$ is the n_f-dependent QCD scale parameter. Note, that its values are also sensitive to the the order of the truncation of the corresponding approximations for the the QCD RG β-function with taking into account its N^{2}LO $O(a_s^3)$ three-loop β_2 coefficient and including the known at present its $O(a_s^5)$ β_5-term , analytically evaluated in Ref.[72] and confirmed in Refs.[73, 74] . However, for the sake of consistency the order of truncation of the a Adler function PT approximation we will not use this term below.

4.2.2 The $\overline{\rm MS}$-scheme benchmarks.

We will fix as the initial normalization point the τ-lepton pole mass $M_\tau = 1776.8$ MeV, will consider $n_f = 3$ number of active flavours and will use the rounded strong coupling constant value $\alpha_s(M_\tau^2) = 0.312$, extracted in Ref.[75] from the QCD sum rules analysis of the ALEPH Collaboration τ-lepton decay data. Despite the fact in view of the qualitative aims of our studies to be presented below we will neglect the effects of the corresponding theoretical and experimental uncertainties in not only this input number, we note, that this result of Ref.[75] falls into the bands of the related results, independently obtained in Ref.[76] from the more detailed re-analysis of the same ALEPH data.

Considering now the properly truncated at the NLO, N^{2}LO and N^{3}LO representations $\alpha_s(M_\tau^2)$ through the inverse powers of logarithms from $M_\tau^2/\Lambda^{(3)2}$-ratio, we arrive to the following, of course rather rough, values for the $\overline{\rm MS}$ scale QCD parameter $\Lambda^{(3)}$ at $n_f = 3$

$$\Lambda^{(3)}_{\rm NLO} = 361 \text{ MeV} \quad , \quad \Lambda^{(3)}_{\rm N^2LO} = 330 \text{ MeV} \quad , \quad \Lambda^{(3)}_{\rm N^3LO} = 325 \text{ MeV}.$$

To transform these them to the cases of $n_f = 4$ and $n_f = 5$ effective number of quarks flavours we will use the expressions for the threshold transformation formulas from the works of Refs.[**?**],[79],[80] with the corresponding matching scales fixed at $\sqrt{Q^2} = 2\overline{m}_c(\overline{m}_c^2) = 2.54$ GeV and $\sqrt{Q^2} = 2\overline{m}_b(\overline{m}_b^2) = 8.36$ GeV. They are related to the following values of the $\overline{\rm MS}$-scheme running c- and b-quark masses $\overline{m}_c(\overline{m}_c^2) = 1.27$ GeV and $\overline{m}_b(\overline{m}_b^2) = 4.18$ GeV, which are taken from the PDG(2022) volume of Ref.[81].

Following these steps we obtain the related to the cases of $n_f = 4$ and $n_f = 5$ numbers of active flavours sets of the numerical values of the $\overline{\rm MS}$-scheme scale parameter

$$\Lambda^{(4)}_{\rm NLO} = 315 \text{ MeV} \quad , \quad \Lambda^{(4)}_{\rm N^2LO} = 286 \text{ MeV} \quad , \quad \Lambda^{(4)}_{\rm N^3LO} = 282 \text{ MeV}$$

and

$$\Lambda^{(5)}_{\rm NLO} = 223 \text{ MeV} \quad , \quad \Lambda^{(5)}_{\rm N^2LO} = 205 \text{ MeV} \quad , \quad \Lambda^{(5)}_{\rm N^3LO} = 203 \text{ MeV}$$

The choice of the concrete threshold energies is also ambiguous and will introduce additional inaccuracies [82]. However, these effects are also not substantial for our aims and we will neglect them as well.

Using these values, the relations (48a-48c), the inverse logarithmic representation of strong coupling in the NLO, N^{2}LO, N^{3}LO approximations and accommodating the explicit expressions for $D(Q^2)$ in the $\overline{\rm MS}$-scheme (47) and within the PMC/BLM procedure we can get the corresponding energy dependence of the Adler function for the $\overline{\rm MS}$ and PMC/BLM approximants and to compare them with each other.

It was also checked that the evolution of the taken value $\alpha_s(M_\tau^2) = 0.312$ up to the mass $M_Z = 91.188$ GeV [81] of Z^0-boson in QCD at the $\mathcal{O}(\alpha_s^4)$ level yields $\alpha_s(M_Z^2)$=0.1175. It is consistent with the results of the work [75] and with the average value of PDG(2022) [81] within not taken into account uncertainties.

4.2.3 The phenomenologically relevant $n_f = 3$ case.

To illustrate the characteristic behavior of the discussed approximants of the Adler function in the $\overline{\rm MS}$-scheme and in the PMC/BLM approach in the case of $n_f = 3$, we consider the chosen by us region of the Euclidean transferred momentum 1.5 GeV $\leq \sqrt{Q^2} \leq$ 2.4 GeV, where the lower limit is slightly smaller than the τ-lepton mass and the upper limit is a bit smaller than the twice charm-quark mass.

Note that in the Minkowskian time-like domain in the similar energy region 1.84 GeV $\leq \sqrt{s} \leq$ 3.88 GeV the subprocess of the production of the light quark-antiquark u, d, s pairs dominates in the e^+e^- annihilation into hadrons process. In this domain the experimental data for the total cross section of the discussed subprocess was extracted from measurements provided by KEDR [83, 84] and BES III [85] Collaborations.

Taking into account the results of studies and benchmarks presented above, we obtain Figure 1a, demonstrating the energy behavior of the NLO, N^{2}LO and N^{3}LO massless approximants for the Adler function in the $\overline{\rm MS}$-scheme

(47) and in the PMC/BLM approach . For comparison the Born quark-parton result $3\sum Q_f^2$ is presented there as well.

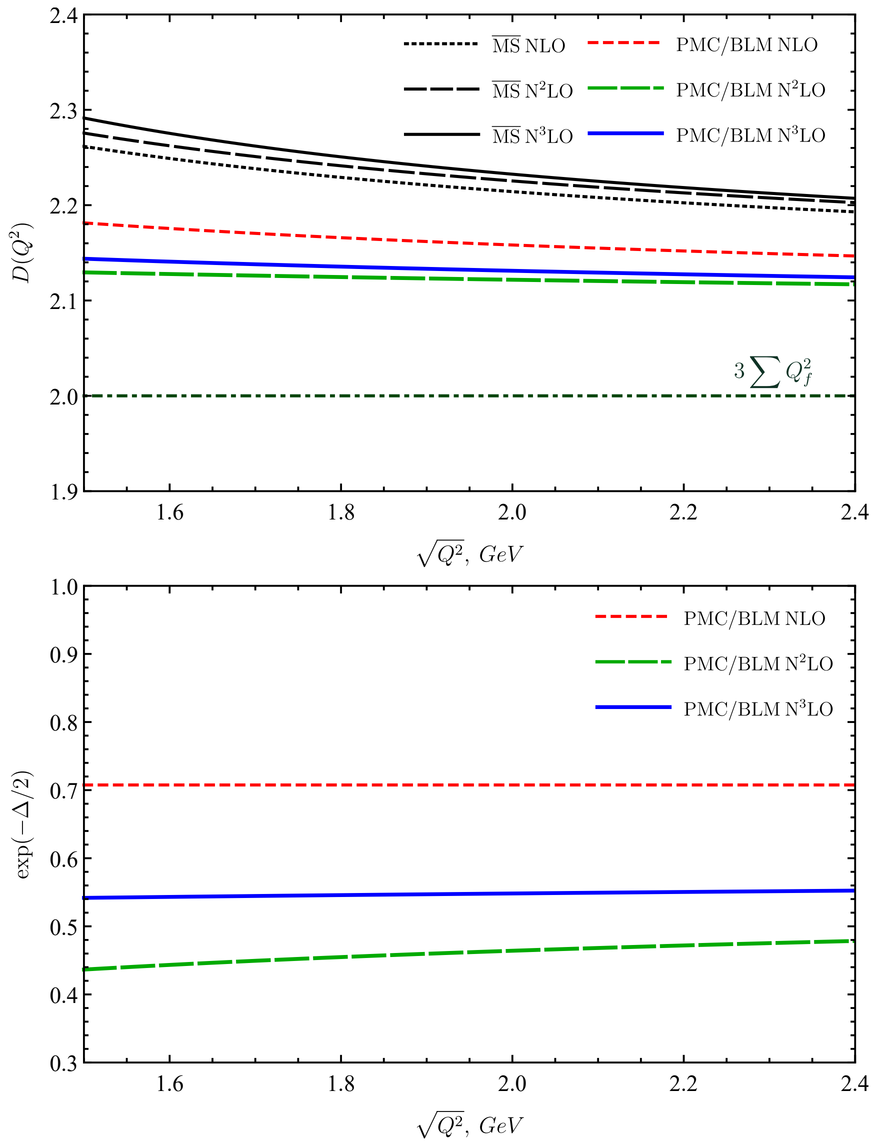

Figure 1: (1a) The dependence of the PT Adler function $D(Q^2)$ on $\sqrt{Q^2}$ at $n_f = 3$ in the massless limit. (1b) The dependence of the factor $\exp(-\Delta/2)$ on $\sqrt{Q^2}$ at $n_f = 3$.

The Figure 1b shows the energy dependence of the factor $\exp(-\Delta/2)$, defining the rescaling effect of the $\overline{\text{MS}}$ parameter $\Lambda^{(n_f)}$ after application of the PMC/BLM procedure in the considered orders of PT.

Let us comment the definite consequences following from the comparison of behaviour of various curves presented in Figure 1a.

1. One can see that the NLO PT corrections to the Adler function are leading to the corrections, which are quantitatively defining the main contribution to the Adler function in both $\overline{\text{MS}}$ and PMC/BLM cases.

2. While taking into account higher order PT corrections we observe the characteristic difference in the fine structure of sets of $\overline{\text{MS}}$-scheme and PMC/BLM approximants. Indeed, the $\overline{\text{MS}}$ results satisfy the inequalities $D_{Born} < D_{\text{NLO}}(Q^2) < D_{\text{N}^2\text{LO}}(Q^2) < D_{\text{N}^2\text{LO}}(Q^2)$, whereas for PMC/BLM we have $D_{Born} < D_{\text{NLO}}(Q^2)$, but $D_{NLO}(Q^2) > D_{\text{N}^2\text{LO}}(Q^2)$ and $D_{\text{N}^2\text{LO}}(Q^2) < D_{(\text{N}^3\text{LO}}(Q^2)$.

3. It is interesting that the sign-structure of the related PT QCD expressions are changing from the pattern $++++$ in the $\overline{\rm MS}$-scheme case to the pattern ++-+ N^{2}LO in the PMC/BLM case.

4. The PMC/BLM approximants are located below the $\overline{\rm MS}$-ones. Together with the recent detailed phenomenologically based analysis of the work of Ref. [86] and the less detailed described analysis of Refs.[87], [88] of the previous e^+e^- to hadrons experimental data demonstrates that *in the case of* $n_f = 3$ *numbers of active flavours taking into account massless PT QCD corrections within PMC/BLM procedure considerably increases the deviations from the experimentally based results for the Adler function in the considered kinematical region.*

Indeed, as it was shown in the works [86–88], in the considered low energy region

Meanwhile within $\overline{\rm MS}$ the massive and ...other effects not taken

the incorporation of the quark mass effects (especially the charm-quark contributions) and the nonperturbative corrections to the massless expression for PT Adler function increases noticeably its $\overline{\rm MS}$-value (say at $\sqrt{Q^2} = 1.5$ GeV its magnitude rises on average by 15% [86].)

Note, however, that it also has positive features. Indeed, in contrast to the case of the pure $\overline{\rm MS}$-scheme, in the PMC/BLM approach we observe almost full independence of the Adler function on the energy scale in a concrete order of perturbation theory.

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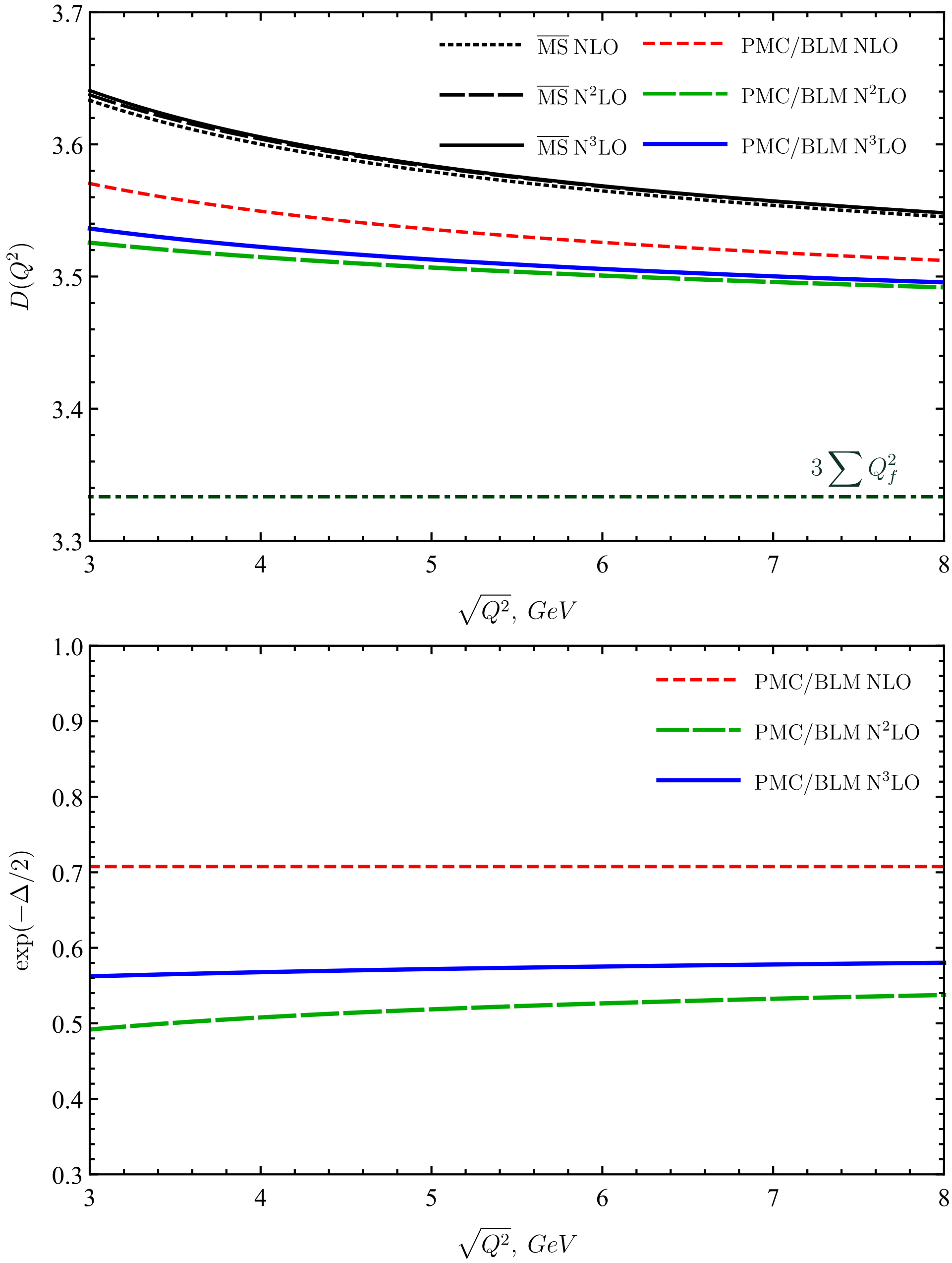

Figure 2: (Upper) The dependence of the PT Adler function $D(Q^2)$ on $\sqrt{Q^2}$ at $n_f = 4$ in the massless limit. The Born factor $3\sum Q_f^2$ is presented as well. (Bottom) The dependence of the factor $\exp(-\Delta/2)$ on $\sqrt{Q^2}$ at $n_f = 4$.

Appendix A.

The solution of the RG equation (6) for the photon vacuum polarization function can be found perturbatively

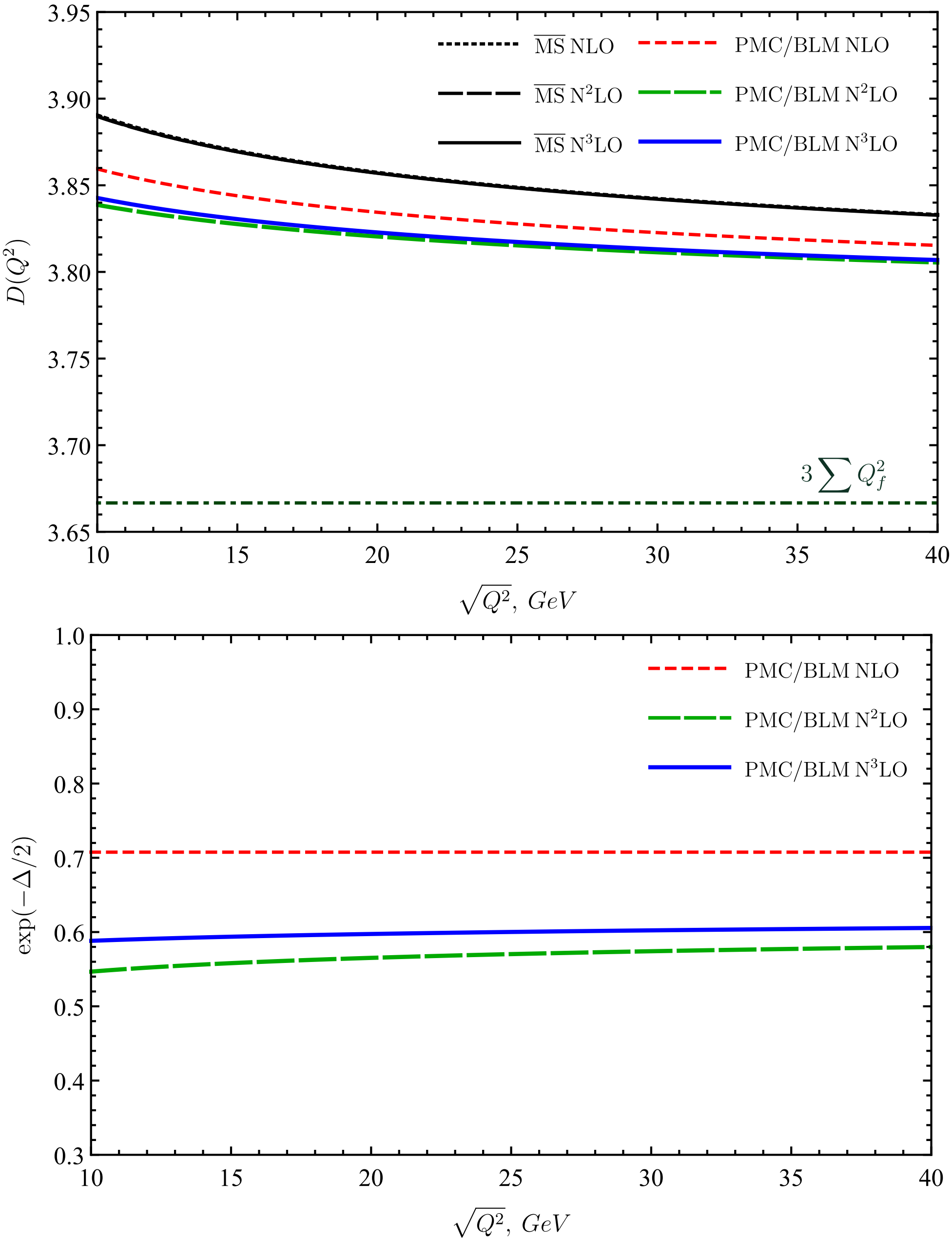

Figure 3: (Upper) The dependence of the PT Adler function $D(Q^2)$ on $\sqrt{Q^2}$ at $n_f = 5$ in the massless limit. The Born factor $3\sum Q_f^2$ is presented as well. (Bottom) The dependence of the factor $\exp(-\Delta/2)$ on $\sqrt{Q^2}$ at $n_f = 5$.

and at the $\mathcal{O}(a_s^4)$ level it has the following form for both the NS and SI contributions:

$$\Pi^{NS}(L, a_s) = \Pi_0 + \gamma_0 L + \Big(\Pi_1 + \gamma_1 L\Big) a_s(\mu^2) + \Big(\Pi_2 + (\gamma_2 + \beta_0\Pi_1)L + \frac{1}{2}\beta_0\gamma_1 L^2\Big) a_s^2(\mu^2) \tag{49a}$$
$$+ \Big(\Pi_3 + (\gamma_3 + \beta_1\Pi_1 + 2\beta_0\Pi_2)L + (\beta_0\gamma_2 + \frac{1}{2}\beta_1\gamma_1 + \beta_0^2\Pi_1)L^2 + \frac{1}{3}\beta_0^2\gamma_1 L^3\Big) a_s^3(\mu^2)$$
$$+ \Big(\Pi_4 + (\gamma_4 + \beta_2\Pi_1 + 2\beta_1\Pi_2 + 3\beta_0\Pi_3)L + (\beta_1\gamma_2 + \frac{1}{2}\beta_2\gamma_1 + \frac{3}{2}\beta_0\gamma_3 + \frac{5}{2}\beta_0\beta_1\Pi_1 + 3\beta_0^2\Pi_2)L^2$$
$$+ (\frac{5}{6}\beta_0\beta_1\gamma_1 + \beta_0^2\gamma_2 + \beta_0^3\Pi_1)L^3 + \frac{1}{4}\beta_0^3\gamma_1 L^4\Big) a_s^4(\mu^2) + \dots,$$
$$\Pi^{SI}(L, a_s) = \Big(\Pi_3^{SI} + \gamma_3^{SI} L\Big) a_s^3(\mu^2) + \Big(\Pi_4^{SI} + (\gamma_4^{SI} + 3\beta_0\Pi_3^{SI})L + \frac{3}{2}\beta_0\gamma_3^{SI} L^2\Big) a_s^4(\mu^2) + \dots \tag{49b}$$

The explicit solution of the RG equation (14), expressed in terms of the PT coefficients of the photon vacuum polarization function, its anomalous dimension and the RG β-function, reads:

$$D^{NS}(L, a_s) = \gamma_0 + \gamma_1 a_s(\mu^2) + \Big(\gamma_2 + \beta_0\Pi_1 + \beta_0\gamma_1 L\Big) a_s^2(\mu^2) \tag{50a}$$

$$+ \Big(\gamma_3 + \beta_1\Pi_1 + 2\beta_0\Pi_2 + (\beta_1\gamma_1 + 2\beta_0\gamma_2 + 2\beta_0^2\Pi_1)L + \beta_0^2\gamma_1 L^2\Big) a_s^3(\mu^2)$$

$$+ \Big(\gamma_4 + \beta_2\Pi_1 + 2\beta_1\Pi_2 + 3\beta_0\Pi_3 + (2\beta_1\gamma_2 + \beta_2\gamma_1 + 3\beta_0\gamma_3 + 5\beta_0\beta_1\Pi_1 + 6\beta_0^2\Pi_2)L$$

$$+ (3\beta_0^2\gamma_2 + \frac{5}{2}\beta_0\beta_1\gamma_1 + 3\beta_0^3\Pi_1)L^2 + \beta_0^3\gamma_1 L^3\Big) a_s^4(\mu^2) + \ldots,$$

$$D^{SI}(L, a_s) = \gamma_3^{SI} a_s^3(\mu^2) + \Big(\gamma_4^{SI} + 3\beta_0\Pi_3^{SI} + 3\beta_0\gamma_3^{SI} L\Big) a_s^4(\mu^2) + \ldots . \tag{50b}$$

Appendix B.

Coefficients $d_k[\dots]$.

Application of the $\{\beta\}$-decomposition procedure (**??**) to the PT series for the Adler function enables to obtain the expressions for terms $d_k[\dots]$ and $d_k^{SI}[\dots]$ in relations (18a-18f).

Within this procedure these terms were defined previously in Refs.[28, 29]. The scale-invariant contributions $d_k[0]$ and $d_k^{SI}[0]$ satisfy the relations

$$d_k[0] = \gamma_k[0], \qquad d_k^{SI}[0] = \gamma_k^{SI}[0], \tag{51a}$$

and terms $\gamma_k[0]$, $\gamma_k^{SI}[0]$ were fixed in Eqs.(23a), (23c), (23f), (23g), (23h).

The remaining scale non-invariant contributions to $D(Q^2)$-function have the following analytical form:

$$d_2[1] = d_3[0,1] = d_4[0,0,1] = \boxed{\left(\frac{33}{8} - 3\zeta_3\right)C_F}, \tag{51b}$$

$$d_3[1] = d_4[0,1] = \left(-\frac{111}{64} - 12\zeta_3 + 15\zeta_5\right)C_F^2 + \left(\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F C_A, \tag{51c}$$

$$d_3[2] = \frac{1}{2}d_4[1,1] = \boxed{\left(\frac{151}{6} - 19\zeta_3\right)C_F}, \tag{51d}$$

$$\begin{aligned} d_4[1] = {} & \left(-\frac{785}{128} - \frac{9}{16}\zeta_3 + \frac{165}{2}\zeta_5 - \frac{315}{4}\zeta_7\right)C_F^3 + \left(-\frac{3737}{144} + \frac{3433}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{615}{16}\zeta_5\right. \\ & \left. + \frac{315}{8}\zeta_7\right)C_F^2 C_A + \left(-\frac{2695}{384} - \frac{1987}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{175}{32}\zeta_5 - \frac{105}{16}\zeta_7\right)C_F C_A^2, \end{aligned} \tag{51e}$$

$$d_4[2] = \left(-\frac{4159}{384} - \frac{2997}{16}\zeta_3 + 27\zeta_3^2 + \frac{375}{2}\zeta_5\right)C_F^2 + \left(\frac{14615}{256} + \frac{39}{16}\zeta_3 - \frac{9}{2}\zeta_3^2 - \frac{185}{4}\zeta_5\right)C_F C_A, \tag{51f}$$

$$d_4[3] = \boxed{\left(\frac{6131}{36} - \frac{203}{2}\zeta_3 - 45\zeta_5\right)C_F}, \tag{51g}$$

$$33 d_4^{SI}[1] = \left(\frac{149}{192} - \frac{39}{32}\zeta_3 + \frac{15}{16}\zeta_5 - \frac{3}{8}\zeta_3^2\right)\frac{d^{abc}d^{abc}}{d_R}. \tag{51h}$$

The boxed analytical expressions are the obtained in Ref.[62] and presented in Ref.[7] $\overline{\mathrm{MS}}$ chain diagram.

Coefficients $\widetilde{d}_k[\dots]$.

In the case when the photon vacuum polarization function and its anomalous dimension are not $\{\beta\}$-decomposed, the counterparts of the expressions (23a), (23c), (23f), (23g), (23h), (51b-51h) were obtained in Refs.[23, 31, 40–42] and read:

$$\widetilde{d}_1[0] = \gamma_1 = \frac{3}{4}C_F, \tag{52a}$$

$$\widetilde{d}_2[0] = \gamma_2 = -\frac{3}{32}C_F^2 + \frac{133}{192}C_F C_A \boxed{-\frac{11}{48}C_F T_F n_f}, \tag{52b}$$

$$\widetilde{d}_2[1] = \widetilde{d}_3[0,1] = \widetilde{d}_4[0,0,1] = \Pi_1 = \boxed{\left(\frac{55}{16} - 3\zeta_3\right)C_F}, \tag{52c}$$

$$\begin{aligned}\widetilde{d}_3[0] = \gamma_3 = &-\frac{69}{128}C_F^3 + \left(\frac{215}{288} - \frac{11}{24}\zeta_3\right)C_F^2 C_A + \left(\frac{5815}{20736} + \frac{11}{24}\zeta_3\right)C_F C_A^2 \\ &- \left(\frac{169}{288} - \frac{11}{12}\zeta_3\right)C_F^2 T_F n_f - \left(\frac{769}{5184} + \frac{11}{12}\zeta_3\right)C_F C_A T_F n_f \boxed{\boxed{-\frac{77}{1296}C_F T_F^2 n_f^2}},\end{aligned} \tag{52d}$$

$$\begin{aligned}\widetilde{d}_3[1] = \widetilde{d}_4[0,1] = \Pi_2 = &\left(-\frac{143}{96} - \frac{37}{8}\zeta_3 + \frac{15}{2}\zeta_5\right)C_F^2 + \left(\frac{44215}{3456} - \frac{227}{24}\zeta_3 - \frac{5}{4}\zeta_5\right)C_F C_A \\ &\boxed{\boxed{-\left(\frac{3701}{864} - \frac{19}{6}\zeta_3\right)C_F T_F n_f}},\end{aligned} \tag{52e}$$

$$\widetilde{d}_3[2] = \widetilde{d}_4[2] = \widetilde{d}_4[3] = 0, \tag{52f}$$

$$\begin{aligned}\widetilde{d}_4[0] = \gamma_4 = &\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right)C_F^4 - \left(\frac{7755}{1024} + \frac{71}{16}\zeta_3 - \frac{935}{128}\zeta_5\right)C_F^3 C_A + \left(\frac{882893}{110592}\right. \\ &\left.+ \frac{11501}{4608}\zeta_3 + \frac{121}{256}\zeta_4 - \frac{2145}{256}\zeta_5\right)C_F^2 C_A^2 - \left(\frac{1192475}{663552} - \frac{5609}{4608}\zeta_3 + \frac{121}{256}\zeta_4\right. \\ &\left.- \frac{825}{512}\zeta_5\right)C_F C_A^3 + \left(\frac{2509}{1536} + \frac{67}{32}\zeta_3 - \frac{145}{32}\zeta_5\right)C_F^3 T_F n_f - \left(\frac{66451}{18432} - \frac{2263}{1152}\zeta_3\right. \\ &\left.+ \frac{143}{128}\zeta_4 - \frac{255}{64}\zeta_5\right)C_F^2 C_A T_F n_f + \left(\frac{22423}{41472} - \frac{9425}{2304}\zeta_3 + \frac{143}{128}\zeta_4 + \frac{45}{128}\zeta_5\right)C_F C_A^2 T_F n_f \\ &+ \left(\frac{4961}{13824} - \frac{119}{144}\zeta_3 + \frac{11}{32}\zeta_4\right)C_F^2 T_F^2 n_f^2 - \left(\frac{8191}{41472} - \frac{563}{576}\zeta_3 + \frac{11}{32}\zeta_4\right)C_F C_A T_F^2 n_f^2 \\ &\boxed{\boxed{\boxed{+\left(\frac{107}{10368} + \frac{1}{72}\zeta_3\right)C_F T_F^3 n_f^3}}} + \left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} \\ &- \left(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f,\end{aligned} \tag{52g}$$

$$\begin{aligned}\widetilde{d}_4[1] = \Pi_3 = &\left(-\frac{31}{256} + \frac{39}{32}\zeta_3 + \frac{735}{32}\zeta_5 - \frac{105}{4}\zeta_7\right)C_F^3 - \left(\frac{382033}{27648} + \frac{46219}{1152}\zeta_3 + \frac{11}{64}\zeta_4\right. \\ &\left.- \frac{9305}{192}\zeta_5 - \frac{105}{8}\zeta_7\right)C_F^2 C_A + \left(\frac{34499767}{497664} - \frac{147473}{3456}\zeta_3 + \frac{55}{8}\zeta_3^2 + \frac{11}{64}\zeta_4 - \frac{28295}{1152}\zeta_5\right. \\ &\left.- \frac{35}{16}\zeta_7\right)C_F C_A^2 - \left(\frac{7505}{13824} - \frac{1553}{72}\zeta_3 + 3\zeta_3^2 - \frac{11}{32}\zeta_4 + \frac{125}{6}\zeta_5\right)C_F^2 T_F n_f \\ &- \left(\frac{5559937}{124416} - \frac{41575}{1728}\zeta_3 - \frac{1}{2}\zeta_3^2 + \frac{11}{32}\zeta_4 - \frac{515}{36}\zeta_5\right)C_F C_A T_F n_f \\ &\boxed{\boxed{\boxed{+\left(\frac{196513}{31104} - \frac{809}{216}\zeta_3 - \frac{5}{3}\zeta_5\right)C_F T_F^2 n_f^2}}}\end{aligned} \tag{52h}$$

(52i)

$$\widetilde{d}_3^{SI}[0] = \gamma_3^{SI} = \left(\frac{11}{192} - \frac{1}{8}\zeta_3\right)\frac{d^{abc}d^{abc}}{d_R}, \tag{52j}$$

$$\widetilde{d}_4^{SI}[0] = \gamma_4^{SI} = \left(\left(-\frac{13}{64} - \frac{1}{4}\zeta_3 + \frac{5}{8}\zeta_5\right)C_F + \left(\frac{1015}{3072} - \frac{659}{1024}\zeta_3 + \frac{33}{256}\zeta_4 + \frac{15}{256}\zeta_5\right)C_A \right. \tag{52k}$$

$$\left. + \left(-\frac{55}{768} + \frac{41}{256}\zeta_3 - \frac{3}{64}\zeta_4 - \frac{5}{64}\zeta_5\right)T_F n_f\right)\frac{d^{abc}d^{abc}}{d_R}, \tag{52l}$$

$$\widetilde{d}_4^{SI}[1] = \Pi_3^{SI} = \left(\frac{431}{2304} - \frac{63}{256}\zeta_3 - \frac{1}{8}\zeta_3^2 - \frac{3}{64}\zeta_4 + \frac{15}{64}\zeta_5\right)\frac{d^{abc}d^{abc}}{d_R}, \tag{52m}$$

$$\tag{52n}$$

where $d_3^{SI} = \widetilde{d}_3^{SI}[0]$ and $d_4^{SI} = \widetilde{d}_4^{SI}[0] + 3\beta_0\widetilde{d}_4^{SI}[1]$.

The sum of the single, double and triple and analytical expressions coincie with the boxed analytical expressions presented in Abstract In fact the second terms from these pairs of boxed terms were absorbed in non-properly fefined in eqs PMC scales while the remaing boxed analtical values are still remaining in the non-beta expanded anonalous deimension Therefore whiloe not considering we are parts of renormalon effects while in the expanded case all of them are really containedv in the PMC scales

References

[1] M. Beneke, Phys. Rept. **317**, 1-142 (1999) [arXiv:hep-ph/9807443 [hep-ph]].

[2] M. Beneke and V. M. Braun, [arXiv:hep-ph/0010208 [hep-ph]].

[3] A. G. Grozin, [arXiv:hep-ph/0311050 [hep-ph]].

[4] S. V. Mikhailov, JHEP **06**, 009 (2007) [arXiv:hep-ph/0411397 [hep-ph]].

[5] E. Laenen, C. Marinissen and M. Vonk, [arXiv:2302.13715 [hep-ph]].

[6] R. J. Crewther, Phys. Rev. Lett. **28**, 1421 (1972)

[7] D. J. Broadhurst and A. L. Kataev, Phys. Lett. B **315**, 179-187 (1993) [arXiv:hep-ph/9308274 [hep-ph]].

[8] K. G. Chetyrkin, Nucl. Phys. B **985**, 115988 (2022) [arXiv:2206.12948 [hep-ph]].

[9] A. L. Kataev and V. S. Molokoedov, [arXiv:2302.03443 [hep-ph]], tp by published in Theor. Math. Phys. (2023)

[10] A. L. Kataev and S. V. Mikhailov, Theor. Math. Phys. **170**, 139-150 (2012) [arXiv:1011.5248 [hep-ph]].

[11] A. V. Garkusha, A. L. Kataev and V. S. Molokoedov, JHEP **02**, 161 (2018) [arXiv:1801.06231 [hep-ph]].

[12] J. A. Gracey and R. H. Mason, [arXiv:2306.11416 [hep-ph]].

[13] S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983)

[14] G. Grunberg and A. L. Kataev, Phys. Lett. B **279**, 352-358 (1992)

[15] G. Grunberg, Phys. Rev. D **46**, 2228-2239 (1992)

[16] S. J. Brodsky and X. G. Wu, Phys. Rev. D **85**, 034038 (2012) [erratum: Phys. Rev. D **86**, 079903 (2012)] [arXiv:1111.6175 [hep-ph]].

[17] M. Khellat and A. Mirjalili, EPJ Web Conf. **138**, 02004 (2017) [arXiv:1611.03817 [hep-ph]].

[18] M. Beneke and V. M. Braun, Phys. Lett. B **348**, 513-520 (1995) [arXiv:hep-ph/9411229 [hep-ph]].

[19] M. Neubert, Phys. Rev. D **51**, 5924-5941 (1995) [arXiv:hep-ph/9412265 [hep-ph]].

[20] G. Mishima, Y. Sumino and H. Takaura, Phys. Lett. B **759**, 550-554 (2016) [arXiv:1602.02790 [hep-ph]].

[21] S. J. Brodsky and L. Di Giustino, Phys. Rev. D **86**, 085026 (2012) [arXiv:1107.0338 [hep-ph]].

[22] V. A. Matveev, R. M. Muradyan and A. N. Tavkhelidze, Lett. Nuovo Cim. **5S2**, 907-912 (1972)

[23] M. Mojaza, S. J. Brodsky and X. G. Wu, Phys. Rev. Lett. **110**, 192001 (2013) [arXiv:1212.0049 [hep-ph]].

[24] G. Cvetič and C. Valenzuela, Phys. Rev. D **74**, 114030 (2006) [erratum: Phys. Rev. D **84**, 019902 (2011)] [arXiv:hep-ph/0608256 [hep-ph]].

[25] D. V. Shirkov and I. L. Solovtsov, Phys. Rev. Lett. **79**, 1209-1212 (1997) [arXiv:hep-ph/9704333 [hep-ph]].

[26] A. P. Bakulev, S. V. Mikhailov and N. G. Stefanis, Phys. Rev. D **72**, 074014 (2005) [arXiv:hep-ph/0506311 [hep-ph]].

[27] A. V. Kotikov and I. A. Zemlyakov, Phys. Rev. D **107**, no.9, 094034 (2023) [arXiv:2302.12171 [hep-ph]].

[28] G. Cvetič and A. L. Kataev, Phys. Rev. D **94**, no. 1, 014006 (2016) [arXiv:1604.00509 [hep-ph]].

[29] I. O. Goriachuk, A. L. Kataev and V. S. Molokoedov, JHEP **05**, 028 (2022) [arXiv:2111.12060 [hep-ph]].

[30] A. L. Kataev and V. S. Molokoedov, Phys. Part. Nucl. **54** (2023), 931-941 (to appear) [arXiv:2211.10242 [hep-ph]].

[31] S. J. Brodsky, M. Mojaza and X. G. Wu, Phys. Rev. D **89**, 014027 (2014) [arXiv:1304.4631 [hep-ph]].

[32] A. L. Kataev and S. V. Mikhailov, Phys. Rev. D **91**, no.1, 014007 (2015) [arXiv:1408.0122 [hep-ph]].

[33] S. V. Mikhailov, JHEP **04**, 169 (2017) [arXiv:1610.01305 [hep-ph]].

[34] P. A. Baikov and S. V. Mikhailov, JHEP **09**, 185 (2022) [arXiv:2206.14063 [hep-ph]].

[35] P. A. Baikov and S. V. Mikhailov, JHEP **03**, 53 (2023). Addendum to JHEP **09**, 185 (2022).

[36] M. F. Zoller, JHEP **10**, 118 (2016) [arXiv:1608.08982 [hep-ph]].

[37] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **259**, 345-352 (1991)

[38] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn and J. Rittinger, Phys. Lett. B **714**, 62-65 (2012) [arXiv:1206.1288 [hep-ph]].

[39] A. L. Kataev and S. V. Mikhailov, JHEP **11**, 079 (2016) [arXiv:1607.08698 [hep-th]].

[40] X. G. Wu, J. M. Shen, B. L. Du, X. D. Huang, S. Q. Wang and S. J. Brodsky, Prog. Part. Nucl. Phys. **108**, 103706 (2019) [arXiv:1903.12177 [hep-ph]].

[41] X. D. Huang, X. G. Wu, X. C. Zheng, Q. Yu, S. Q. Wang and J. M. Shen, Eur. Phys. J. C **81**, no.4, 291 (2021) [arXiv:2008.07362 [hep-ph]].

[42] J. M. Shen, B. H. Qin, J. Yan, S. Q. Wang and X. G. Wu, JHEP **07**, 109 (2023) [arXiv:2303.11782 [hep-ph]].

[43] L. Di Giustino, S. J. Brodsky, P. G. Ratcliffe, X. G. Wu and S. Q. Wang, [arXiv:2307.03951 [hep-ph]].

[44] E. C. G. Stueckelberg de Breidenbach and A. Petermann, Helv. Phys. Acta **26** (1953), 499-520

[45] M. Gell-Mann and F. E. Low, Phys. Rev. **95** (1954), 1300-1312

[46] N. N. Bogolyubov and D. V. Shirkov, Nuovo Cim. **3** (1956), 845-863

[47] P. M. Stevenson, "Renormalized Perturbation Theory and its Optimization by the Principle of Minimal Sensitivity," World Scientific, 2022, ISBN 978-981-12-5568-7

[48] P. M. Stevenson, Phys. Rev. D **23** (1981), 2916

[49] S. J. Brodsky and X. G. Wu, Phys. Rev. D **86** (2012), 054018 [arXiv:1208.0700 [hep-ph]].

[50] S. J. Brodsky and H. J. Lu, [arXiv:hep-ph/9211308 [hep-ph]].

[51] P. M. Stevenson, [arXiv:hep-ph/9211327 [hep-ph]].

[52] V. I. Zakharov, Nucl. Phys. B **385**, 452-480 (1992)

[53] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Preprint IYaI-P-0170 (1980).

[54] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Nucl. Phys. B **174**, 345-377 (1980)

[55] S. G. Gorishnii, A. L. Kataev and S. A. Larin, Phys. Lett. B **259**, 144-150 (1991)

[56] L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66**, 560-563 (1991) [erratum: Phys. Rev. Lett. **66**, 2416 (1991)]

[57] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. **104**, 132004 (2010) [arXiv:1001.3606 [hep-ph]].

[58] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn and J. Rittinger, JHEP **07**, 017 (2012) [arXiv:1206.1284 [hep-ph]].

[59] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B **93**, 429-432 (1980)

[60] O. V. Tarasov and A. A. Vladimirov, Phys. Part. Nucl. **44** (2013), 791-802 [arXiv:1301.5645 [hep-ph]].

[61] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **303** (1993), 334-336 [arXiv:hep-ph/9302208 [hep-ph]].

[62] D. J. Broadhurst, Z. Phys. C **58** (1993), 339-346

[63] J. L. Rosner, Phys. Rev. Lett. **17**, 1190-1192 (1966)

[64] A. L. Kataev, JHEP **02**, 092 (2014) [arXiv:1305.4605 [hep-th]].

[65] M. A. Shifman and K. .V. Stepanyantz, Phys. Rev. Lett. **114**, no.5, 051601 (2015) [arXiv:1412.3382 [hep-th]].

[66] S. S. Aleshin, A. L. Kataev and K. V. Stepanyantz, JHEP **03**, 196 (2019) [arXiv:1902.08602 [hep-th]].

[67] R. N. Lee, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. D **94**, no.5, 054029 (2016) [arXiv:1608.02603 [hep-ph]].

[68] P. A. Baikov and K. G. Chetyrkin, JHEP **06**, 141 (2018) [arXiv:1804.10088 [hep-ph]].

[69] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP **08** (2017), 113 [arXiv:1707.01044 [hep-ph]].

[70] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B **400** (1997), 379-384 [arXiv:hep-ph/9701390 [hep-ph]].

[71] M. Czakon, Nucl. Phys. B **710** (2005), 485-498 [arXiv:hep-ph/0411261 [hep-ph]].

[72] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. **118** (2017) no.8, 082002 [arXiv:1606.08659 [hep-ph]].

[73] T. Luthe, A. Maier, P. Marquard and Y. Schroder, JHEP **03** (2017), 020 [arXiv:1701.07068 [hep-ph]].

[74] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP **02** (2017), 090

[75] C. Ayala, G. Cvetič and D. Teca, Eur. Phys. J. C **81**, no.10, 930 (2021) [arXiv:2105.00356 [hep-ph]].

[76] A. Pich and A. Rodríguez-Sánchez, JHEP **07**, 145 (2022) [arXiv:2205.07587 [hep-ph]].

[77] W. Bernreuther and W. Wetzel, Nucl. Phys. B **197** (1982), 228-236 [erratum: Nucl. Phys. B **513** (1998), 758-758]

[78] S. A. Larin, T. van Ritbergen and J. A. M. Vermaseren, Nucl. Phys. B **438** (1995), 278-306 [arXiv:hep-ph/9411260 [hep-ph]].

[79] K. G. Chetyrkin, B. A. Kniehl and M. Steinhauser, Phys. Rev. Lett. **79** (1997), 2184-2187 [arXiv:hep-ph/9706430 [hep-ph]].

[80] B. A. Kniehl, A. V. Kotikov, A. I. Onishchenko and O. L. Veretin, Phys. Rev. Lett. **97**, 042001 (2006) [arXiv:hep-ph/0607202 [hep-ph]].

[81] R. L. Workman *et al.* [Particle Data Group], PTEP **2022**, 083C01 (2022)

[82] J. Blumlein and W. L. van Neerven, Phys. Lett. B **450** (1999), 417-426 doi:10.1016/S0370-2693(99)00152-5 [arXiv:hep-ph/9811351 [hep-ph]].

[83] V. V. Anashin *et al.* [KEDR], Phys. Lett. B **788**, 42-51 (2019) [arXiv:1805.06235 [hep-ex]].

[84] V. V. Anashin *et al.* [KEDR], Phys. Part. Nucl. **54**, no.1, 185-226 (2023)

[85] M. Ablikim *et al.* [BESIII], Phys. Rev. Lett. **128**, no.6, 062004 (2022) [arXiv:2112.11728 [hep-ex]].

[86] M. Davier, D. Díaz-Calderón, B. Malaescu, A. Pich, A. Rodríguez-Sánchez and Z. Zhang, JHEP **04**, 067 (2023) [arXiv:2302.01359 [hep-ph]].

[87] S. Eidelman, F. Jegerlehner, A. L. Kataev and O. Veretin, Phys. Lett. B **454**, 369-380 (1999) [arXiv:hep-ph/9812521 [hep-ph]].

[88] F. Jegerlehner, Nucl. Phys. B Proc. Suppl. **181-182**, 135-140 (2008) [arXiv:0807.4206 [hep-ph]].