BEC Stars in Rastall and Rainbow Gravities

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- *Bose-Einstein Condensate Stars in Rastall and Rainbow Gravities*, *under construction!* - arXiv:2311:xxxx
- *Rotating Bose-Einstein Condensate Stars at finite temperature* *under review in* Phys. Rev. D.

Outline

1. Introduction
   - General Relativity (GR)
   - Stellar Structure Equations (TOV)
   - GR - limitations
   - Modified gravities
2. Modified TOV in Rastall & Rainbow theories
3. BEC stars
4. BEC: Equation of States
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Introduction

- Einstein's General Relativity (GR):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu},$$

  $R_{\mu\nu}$: Ricci tensor, R : Ricci scalar, $T_{\mu\nu}$: Energy-momentum tensor
- Conservation law: ${T^{\mu\nu}}_{;\mu} = 0$, valid in both Special and General Relativity
- General form of spherically symmetric static metric is given by

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

  $A(r), B(r)$: metric functions
- For a perfect fluid with mass density ρ and pressure p,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu},$$

  u^μ : 4-velocity of fluid

Tolman-Oppenheimer-Volkoff Equations

- Tolman-Oppenheimer-Volkoff (TOV) Equations - set of coupled differential equations for $\rho(r)$ and $p(r)$

$$\frac{dp}{dr} = -\frac{[\rho + p]\left[m + 4\pi r^3 p\right]}{r(r-2m)},$$
$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

- TOV equations are integrated from center to the surface of the star $(r = R)$ for a given equation of state: $p = p(\rho)$
- Boundary conditions: $p(r = 0) = p_c$ (central pressure) and $m(r = 0) = 0$
- The radius R of the star is defined as the distance at which the pressure drops to zero, i.e. $p(r = R) = 0$; mass of the star is then given by $M = m(R)$

Einstein's General Relativity

- GR has successfully explained many aspects of the known universe
  - Observation of gravitational waves by binary black hole [LIGO & Virgo (2016)]
  - Neutron star mergers [LIGO & Virgo (2017)]
  - First photograph of black hole by Event Horizon Telescope
- Limitations:
  - Dark matter and dark energy problem, early inflation
- Several modified gravity theories have been proposed

Rastall Gravity

- Generalization of Einstein's GR, by relaxing the constraint ${T^{\mu\nu}}_{;\mu} = 0$
- Conservation law proposed by Rastall:

$$ {T^{\mu\nu}}_{;\mu} = \frac{1-\kappa}{16\pi G} R^{,\nu}, $$

  where κ is the Rastall parameter
- With $R \longrightarrow 0$, conservation law in flat space-time can be recovered
- Modified Einstein field equations consistent with the conservation law:

$$ {R^{\nu}}_{\mu} - \frac{\kappa}{2} {\delta^{\nu}}_{\mu} R = 8\pi G {T^{\nu}}_{\mu} $$

- With $\kappa = 1$, field equations of GR can be obtained

[P. Rastall, PRD **6**, 3357 (1972)]

Rainbow formalism

- Energy-momentum relation of test particle of mass m is modified as

$$E^2\Xi(x)^2 - p^2\Sigma(x)^2 = m^2,$$

where $x = E/E_p$ is the ratio of energy of the test particle to the Planck energy $E_p = \sqrt{\hbar/G}$

- $\Xi(x)$ and $\Sigma(x)$ are arbitrary energy dependent Rainbow functions
- In the Infrared regime,

$$\lim_{x\to 0}\Xi(x) = 1, \quad \lim_{x\to 0}\Sigma(x) = 1$$

which gives back the standard E-p relation

- Geometry of the space-time is energy dependent: Families of metric parametrised by x

[J. Magueijo & L. Smolin, Class. Quant. Grav. **21**, 1725–1736 (2004)]

Combined Rastall-Rainbow theory

- Effects of Rainbow gravity can be incorporated to the Rastall theory by considering an energy dependent metric with spherical symmetry

$$ds^2 = -\frac{B(r)}{\Xi^2}dt^2 + \frac{A(r)}{\Sigma^2}dr^2 + \frac{r^2}{\Sigma^2}(d\theta^2 + \sin\theta^2 d\phi^2)$$

- Rastall-Rainbow field equations are given by

$$R^{\nu}{}_{\mu}(x) - \frac{\kappa}{2}\delta^{\nu}{}_{\mu}R(x) = 8\pi G(x)T^{\nu}{}_{\mu}(x)$$

- Assuming the stellar matter to be made of perfect fluid, above equations can be solved to obtain new set of equations describing the stellar equilibrium - modified TOV equations

[C. E. Mota, L. C. N. Santos et al., PRD **100**, 024043 (2019)]

Modified TOV equations

- Modified TOV equations in combined Rainbow-Rastall theory are

$$\frac{d\tilde{p}}{dr} = -\frac{G\tilde{M}\tilde{\rho}}{r^2}\frac{\left(1+\frac{\tilde{p}}{\tilde{\rho}}\right)\left(1+\frac{4\pi r^3\tilde{p}}{\tilde{M}}\right)}{1-\frac{2G\tilde{M}}{r}},$$

$$\frac{d\tilde{M}}{dr} = 4\pi\tilde{\rho}\, r^2.$$

- Effective pressure and energy density are given by

$$\tilde{\rho} = \frac{1}{\Sigma(x)^2}\left[\alpha_1\rho + 3\alpha_2 p\right], \qquad \tilde{p} = \frac{1}{\Sigma(x)^2}\left[\alpha_2\rho + (1-3\alpha_2)p\right];$$

where

$$\alpha_1 = \frac{1-3\kappa}{2(1-2\kappa)}; \qquad \alpha_2 = \frac{1-\kappa}{2(1-2\kappa)}.$$

[C. E. Mota, L. C. N. Santos et al., PRD **100**, 024043 (2019)]

Bose Einstien Condensate stars

- At very low temperatures, system of bosons can occupy the same quantum ground state which can be represented by a single wavefunction - Bose Einstein Condensate (BEC)
- A coherent massive object such as boson star can be realized within astrophysical scales formed of BECs, confined by the self-generated gravitational interaction of bosonic particles
- Self-gravitating BECs are possible candidates for dark matter [C. G. Boehmer & T. Harko, JCAP **06**, 025 (2007); T. Harko, JCAP **05**, 022 (2011) *etc.*]
- Pure BEC stars - nucleons in the star can exist in superfluid phase, with nucleons forming Cooper pairs and can be treated as bosons having mass $m = 2m_n$ [P. Chavanis & T. Harko, PRD **86**, 064011 (2012)]

Equation of State (EoS)

- Lagrangian for scalar field with ϕ^4 interactions

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \phi^*_{;\mu} \phi^*_{;\nu} - \frac{1}{2} m^2 |\phi|^2 - \frac{1}{4} \lambda |\phi|^4$$

- Relativistic BEC EoS at zero temperature, known as the Colpi-Shapiro-Wasserman (CSW), is derived from Einstein-Klein-Gordon equations

$$p(\rho) = \frac{1}{36K'} \left[\left(1 + 12K^{'} \rho \right)^{1/2} - 1 \right]^2 ,$$

with $K^{'} = \lambda \hbar^3 / (4m^4)$ and $\lambda = (9.523 \times 8\pi) \left(\frac{a}{1\,\mathrm{fm}} \right) \left(\frac{m}{2m_n} \right)$.

[M. Colpi et al., PRL **57**, 2485-2488 (1986)]

Equation of State

- Gross-Pitaevskii (GP) equation describing self-gravitating BEC with short-range interactions is given by

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+\Phi(\mathbf{r},t)+gn(\mathbf{r},t)\right]\Psi(\mathbf{r},t),$$

  where $\Psi(\mathbf{r},t)$: condensate wavefunction, $\Phi(\mathbf{r},t)$: Newtonian gravitational potential, $g=4\pi a\hbar^2/m$ represents the strength of the repulsive contact interaction with a being the s-wave scattering length of bosons in the system
- Zero temperature BEC EoS based on the Newtonian GP equation:

$$p(\rho)=g\rho^2/2m^2=K\rho^2.$$

Equation of State

- Generalized GP equation at finite temperature

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \Phi(\mathbf{r},t) + gn(\mathbf{r},t) + 2gn_{\mathsf{th}}(\mathbf{r},t)\right]\Psi(\mathbf{r},t)$$
$$+ gn_a(\mathbf{r},t)\Psi^\dagger(\mathbf{r},t) + g\left\langle\hat{\Psi}^\dagger_{\mathsf{th}}(\mathbf{r},t)\hat{\Psi}_{\mathsf{th}}(\mathbf{r},t)\hat{\Psi}_{\mathsf{th}}(\mathbf{r},t)\right\rangle.$$

- EoS describing finite temperature BEC matter:

$$p(\rho,T) = \frac{g\rho^2}{2m^2} + \frac{2g\rho}{m\lambda_{th}^3}\,\zeta_{3/2}\left[e^{-g\rho/(mk_BT)}\right]$$
$$+ \frac{2k_BT}{\lambda_{th}^3}\zeta_{5/2}\left[e^{-g\rho/(mk_BT)}\right] - \frac{2k_BT}{\lambda_{th}^3}\,\zeta_{5/2}\left[1\right].$$

- $\lambda_{th} = \sqrt{2\pi\hbar^2/mk_BT}$ is the thermal de Broglie wavelength and $\zeta_\nu[z] = \sum_{n=1}^{\infty} z^n/n^\nu$ represents the polylogarithmic function of order ν.
- For $T \longrightarrow 0$, $p(\rho,T) \longrightarrow p(\rho) = K\rho^2$

[C. Gruber & A. Pelster, EPJD **68**, 341 (2014)]

Results

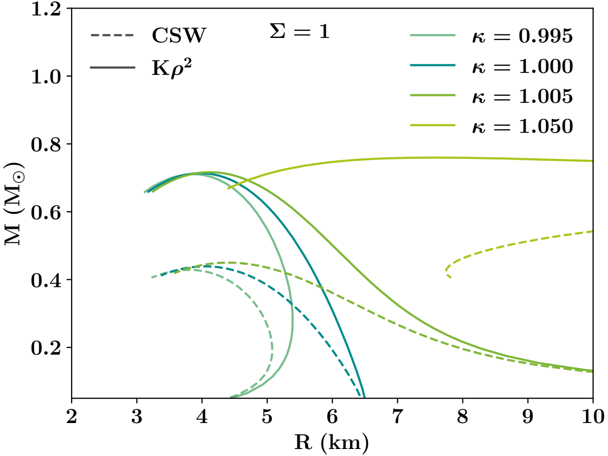

Figure: Stellar configurations of the BEC star at zero temperature within the Rastall gravity. Central density values corresponding to max. mass for $K\rho^2$: $(2.07 - 1.98) \times 10^{16}$ gcm^{-3}, for CSW : $(2.06 - 1.77) \times 10^{16}$ gcm^{-3}

Results

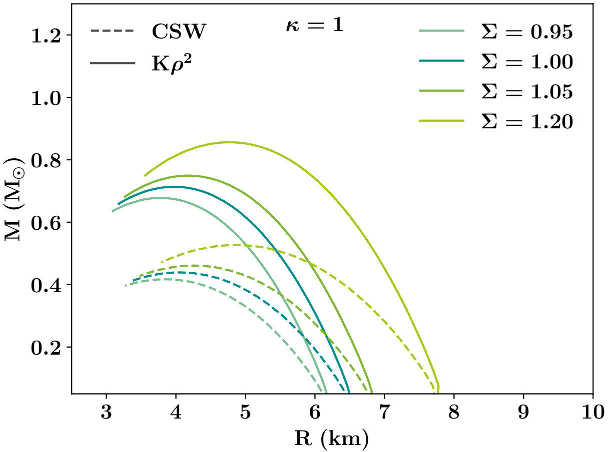

Figure: Stellar configurations of the BEC star at zero temperature within the Rainbow gravity. Central density values corresponding to max. mass for $K\rho^2$: $(2.25 - 1.42) \times 10^{16}$ gcm^{-3}, for CSW : $(2.13 - 1.34) \times 10^{16}$ gcm^{-3}

Results

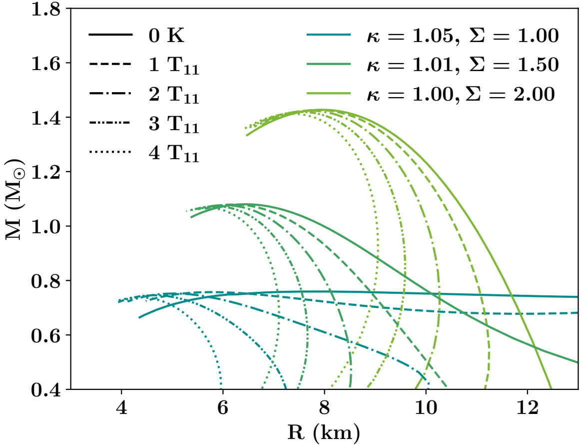

Figure: Effect of temperature ($T_{11} = 10^{11}$ K) in Rastall, Rainbow and Rastall-Rainbow BEC stars. With increase in temperature, stellar equilibria are achieved at reduced masses and radii.

Results (Zero Temperature)

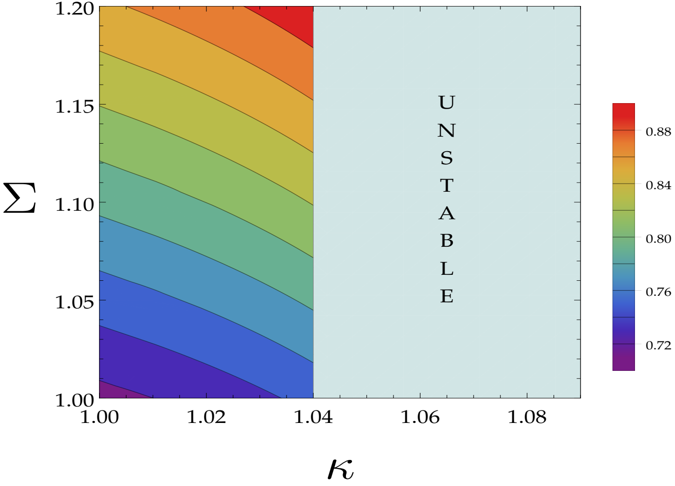

Figure: Maximum mass of BEC stars as a function of Rastall parameter κ and Rainbow function Σ for $T = 0$ K

RESULTS (Finite Temperature)

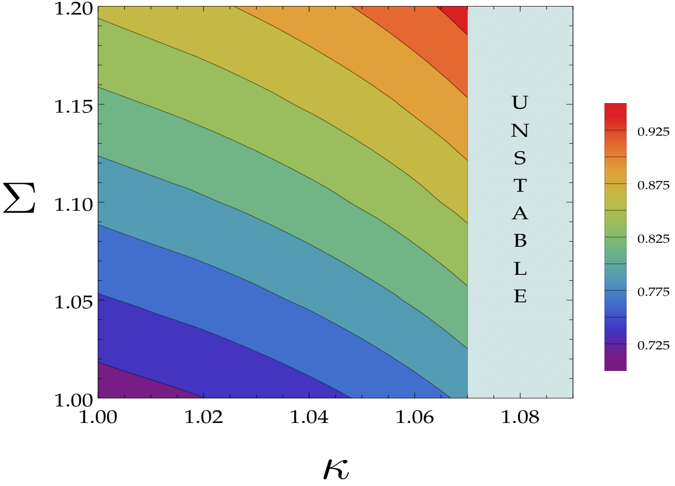

Figure: Maximum mass of BEC stars as a function of Rastall parameter κ and Rainbow function Σ for $T = 2 \times 10^{11}$K.

Observations

- For CSW EoS, within Einstein's GR : $M = 0.43$ $\mathrm{M}_{\odot}$ and $R = 4.05$ km using $a = 1$ fm and $m = 2m_n$

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MSP J0740+6620	∼2.14 M$_\odot$	[Nature Astron. **4**, 72–76 (2019)]
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- Within the Rastall-Rainbow theory, BEC star configurations for the CSW EoS can be obtained with good agreement with observational data

$K^{'} \times 10^5$ (cm^5/g/s^2)	κ	Σ	$M(M_\odot)$	R (km)	$\rho_c \times 10^{15}$ (g/cm^3)
2	0.95	1.8	2.09	16.21	1.02
2.1	0.95	1.7	2.02	15.75	1.06
2.2	0.95	1.8	2.04	15.8	1.07

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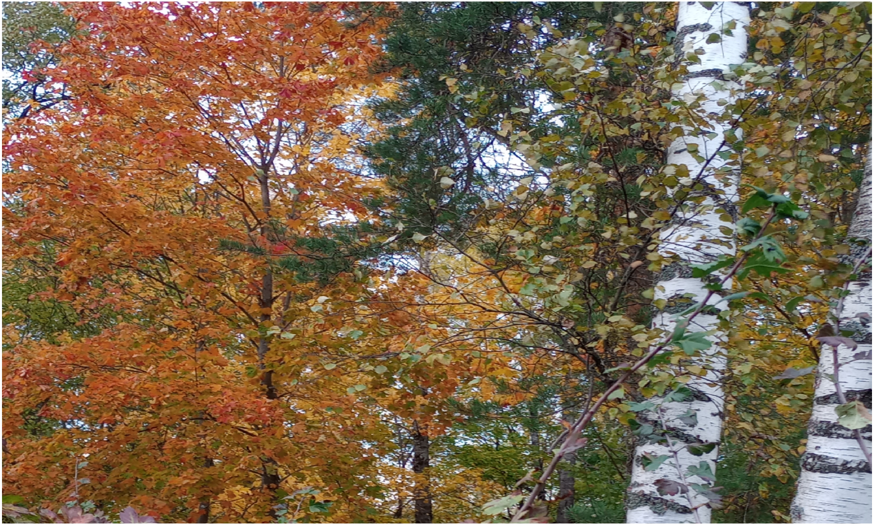
- BEC star configurations for the EoS $K\rho^2$ has agreement with observations in GR as well as RR gravities

Summary

- Static pure BEC stars within the Rastall, Rainbow and combined Rastall-Rainbow theories of gravity have been considered
- We employ zero temperature and finite temperature EoSs to describe the BEC star matter
- The modified TOV equations are solved to obtain the global properties of BEC star
- Effect of temperature is to reduce the masses and radii of the stellar equilibria
- Within the Rastall theory, stable BEC star configurations are obtained for the Rastall parameter $\kappa < 1.04$ for the zero temperature EoSs
- For $\kappa > 1.04$, stable solutions of BEC stars can be obtained in the Rastall gravity with the increase of temperature
- Within the Rainbow formalism, the increase in Rainbow function Σ results in larger maximum mass and radius of the star
- Within the Rastall-Rainbow theory, BEC star configurations for the CSW EoS can be obtained with good agreement with observational data

THANKS



"You probably think that my future landscapes will be soaked in pessimism, so to speak? Don't worry, I love nature too much." - Isaac Levitan [1860-1900]