The Gorishny-Isaev vacuum integrations and UV(IR)-regime

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The XXVth International Baldin-Burov Seminar on High Energy Physics Problems
September, 21st 2023
Dubna

$$\boxed{Effective\ Potential\ Methods\ with\ masses}$$

$$\Big\Downarrow \ m^2\ in\ vertices$$

$$\boxed{Vacuum\ Massless\ Integrations}$$

$$Generated\ func. \to singular\ distrib. \Big\Downarrow Sequential\ Approach$$

$$\boxed{\delta(0) \equiv \lim_{\varepsilon \to 0} 1/\varepsilon \Rightarrow UV(IR) - divergencies}$$

- We present the important details regarding the vacuum massless integrations which are not outlined in the literature. In particular, it has been shown how the delta-function represents either UV-regime or IR-regime.
- In the case of vacuum massless integration, we advocate the use of sequential approach to the singular generated functions (distributions). The sequential approach is extremely useful for many practical applications, in particular, in the effective potential method.

- Due to the dimensional analysis, the vacuum massless integration gives

$$\boxed{\int \frac{d^D k}{[k^2]^n} = 0 \quad \text{for} \quad \forall\; n}$$

- However, if $n = D/2$ (the dim. analysis arguments do not work!) we can check that the zero is achieved owing to the cancellation, each other, of UV- and IR-divergencies.

  Hence, focusing on the only UV(IR)-divergency we deal with the non-zero contribution which is a object of our consideration.

Every singular functional is being a limit of the regular functional, *i.e.*

$$\delta(x) = \lim_{\varepsilon\to 0} \delta_\varepsilon(x) \quad \text{in the function space}$$
$$\Rightarrow \lim_{\varepsilon\to 0} \int d\mu(x)\delta_\varepsilon(x) \quad \text{in the functional space}$$

where $d\mu(x) = dx\,\phi(x)$ with the restricted (finite) function $\phi(x)$ and

$$\delta_\varepsilon(x) = \Big\{ \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2 + x^2}; \quad \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}}; \quad \frac{1}{\pi}\frac{\sin x/\varepsilon}{x} \Big\}$$
$$\overset{x=0}{\sim} \Big\{ [\frac{1}{\varepsilon}]; \quad [\frac{1}{\sqrt{\varepsilon}}]; \quad [\frac{1}{\varepsilon}] \Big\} \Rightarrow \text{parametrization of infinity if } \varepsilon \to 0.$$

P.S. Here, $\overset{x=0}{\sim}$ means "behaves as" if $x = 0$.

Using the Fourier transform, the propagator $\Delta_F(0)$ can be write as (In what follows +i0 is omitted in the denominators)

$$\Delta_F(0) = \int \frac{(d^D k)}{k^2} = \Gamma(D/2-1) \int (d^D z)\, \frac{\delta(z)}{\left(z^2\right)^{D/2-1}},$$

If we assume that $D/2 - 1 = 0$, then

$$\Delta_F(0) = \Gamma(0) \int (d^D z)\, \delta(z) \Rightarrow \Gamma(0),$$

where

$$\Gamma(0) = \lim_{\epsilon\to 0} \Gamma(\epsilon) = \lim_{\epsilon\to 0} \left\{ \frac{1}{\epsilon} + ... \right\}.$$

P.S. The condition given by $D/2 - 1 = 0$ has to be applied before the integration over $(d^D k)$ in order to avoid the uncertainty.

On the other hand, see Gorishny-Isaev:1984, the vacuum integration results in the delta-function.

A key moment of Gorishny-Isaev's method goes as below. Using the spherical system, $\Delta_F(0)$ can be represented as

$$\Delta_F(0) = \int \frac{(d^D k)}{k^2} = \frac{1}{2} \int d\Omega \int_0^\infty d\beta \, \beta^{D/2-2}.$$

The replacement $\beta = e^y$ leads to the following expression

$$\Delta_F(0) = \frac{1}{2} \int d\Omega \int_{-\infty}^{\infty} (dy) \, e^{iy\left[(-i)(D/2-1)\right]} = \frac{1}{2\,|i|} \delta\big(D/2 - 1\big) \int d\Omega$$

or, restoring all coefficients, it reads

$$\Delta_F(0) = -\,2i\,\pi^{1+D/2}\,\delta(1 - D/2)\Big|_{D=2} = -\,2i\,\pi^2\,\delta(0).$$

So, for the case of $D = 2$, the matching gives the following representation

$$(-i)\,\Delta_F(0) = \Gamma(0) = -\,2\,\pi^2\,\delta(0).$$

With this, we may conclude that $\delta(0)$-singularity can be treated as the singularity of $\Gamma(0)$. The same inference has been reached by the different method, see Anikin:2020. Notice that the physical (UV or IR) nature of the mentioned singularity has been somewhat hidden.

In the dimensional regularization, the remaining UV- and IR-divergencies are associated with the (small) positive ($\varepsilon > 0$) and negative ($\varepsilon < 0$) regularized parameter ε, respectively. In α-parametrization, we have

$$\Delta_F(0) = \int \frac{(d^D k)}{k^2} = \Gamma(D/2-1) \int (d^D z)\, \frac{\delta(z)}{\left(z^2\right)^{D/2-1}}$$

$$= \int (d^D z)\, \delta(z) \left\{ \int_0^\infty d\alpha\, \alpha^{D/2-2}\, e^{-\alpha z^2} \right\} = \int_0^\infty (d\alpha)\, \alpha^{D/2-2}.$$

Hence, one gets

$$\Delta_F(0) = \int \frac{(d^D k)}{k^2} = \int_0^\infty (d\alpha)\, \alpha^{D/2-2} \Longrightarrow \frac{1}{D/2-1}\Big\{ \lim_{\alpha\to\infty} \alpha^{D/2-1} - \lim_{\alpha\to 0} \alpha^{D/2-1} \Big\}.$$

One can see that the first term corresponds to the UV-divergency, while the second term – to the IR-divergency:

$$\lim_{\alpha\to\infty} \alpha^{D/2-1} = [\infty]_{\text{UV}} \quad \text{if } D > 2,$$
$$\lim_{\alpha\to 0} \alpha^{D/2-1} = [\infty]_{\text{IR}} \quad \text{if } D < 2.$$

In other words, if ε in $D = d - 2\varepsilon$ is small one, $|\varepsilon| < 1$, and it varies from the negative to positive variables, we have

$$\Delta_F(0)\Big|_{d=2} \overset{\text{stand.}}{\Longrightarrow} \frac{1}{D/2-1}\Big\{\Theta(D>2\,|\,\varepsilon<0)\lim_{\alpha\to\infty}\alpha^{D/2-1}$$

$$-\ \Theta(D<2\,|\,\varepsilon>0)\lim_{\alpha\to 0}\alpha^{D/2-1}\Big\} = 0$$

$$\overset{\text{G-Is}}{\Longrightarrow} \delta\big(1-D/2\big)\Big|_{D\neq 2} = 0.$$

Thus, every of the methods gives the same final conclusion !

P.S. In the dimensional regularization, the positive small ε is regularizing the UV-divergency but not IR-divergency.

We remind the other useful representation given by

$$\Delta_F(0) = \lim_{z^2 \to 0} \Delta_F(z^2) = \lim_{z^2 \to 0} \frac{1}{4\pi} \delta_+(z^2) = \delta(0), \quad z \in \mathbb{E}^4$$

which is in agreement with the above mentioned eqns.

In the dimensional regularization procedure, we begin with two-point 1PI massless Green function given by

$$\mathcal{I}(p^2) = \int \frac{(d^D k)}{k^2(k^2+p^2)} = (c.c.)\,(p^2)^{D/2-2}\,G(1,1),$$

where $(c.c)$ implies the coefficient constant and

$$G(1,1) = \frac{\Gamma(-D/2+2)\Gamma^2(D/2-1)}{\Gamma(D-2)}.$$

(See for example [Chetyrkin-Kataev-Tkachov:1980; Grozin:2005])

Using $D = 4 - 2\epsilon$, we get

$$\mathcal{I}(p^2) = \int \frac{(d^D k)}{k^2(k^2 + p^2)} = (c.c.)\,(p^2)^{-\epsilon}\,\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}.$$

Here, the scale dependence of μ^2 is hidden as irrelevant one.

The vacuum integration can be obtained from this eqn. with the help of the corresponding limit as

$$\mathcal{V}_2 \equiv \int \frac{(d^D k)}{(k^2)^2} = \lim_{p^2 \to 0} \mathcal{I}(p^2).$$

There are, however, some subtleties of the mentioned limit.

Indeed, having used the α-representation, let us calculate the following integral

$$\mathcal{I}(p^2) = (c.c.) \int_0^\infty d\alpha d\beta \frac{e^{-p^2 \frac{\alpha\beta}{\alpha+\beta}}}{[\alpha+\beta]^{D/2}} = (c.c.) \int_0^\infty \lambda\lambda^{1-D/2} \int_0^1 dx e^{-p^2\lambda x\bar{x}},$$

where

$$\alpha = \lambda x_1, \quad \beta = \lambda x_2, \quad \lambda \in [0, \infty].$$

The next stage of calculations is to make a replacement as

$$\boxed{\tilde{\lambda} = p^2 \lambda x \bar{x}, \quad d\tilde{\lambda} = p^2 x \bar{x} d\lambda}$$

in the exponential function. This replacement simplifies the integrals and it leads to the corresponding combination of Γ-functions denoted as $G(1,1)$ Grozin:2005, Grozin:2007.

- The first mathematical subtlety: if we suppose the limits $p^2 \to 0$ and $\epsilon \to 0$ are consequent ones, not simultaneous, that these limits are not commutative operations, *i.e.*

$$\left[\lim_{p^2 \to 0}, \lim_{\epsilon \to 0} \right] \neq 0.$$

P.S. On the other hand, if the limits are simultaneous ones we deal with the uncertainty of $[0]^0$ which should be somehow resolved.

- The second subtlety: to avoid the mentioned uncertainty, we have to implement the limit $p^2 \to 0$ before the possible replacement. In this case, the limit of $p^2 \to 0$ is well-defined operation and we finally obtain that

$$\lim_{p^2 \to 0} \mathcal{I}(p^2) = (c.c.) \int_0^\infty d\lambda \lambda^{1-D/2} =$$

$$\frac{1}{2-D/2} \Big\{ \lim_{\lambda \to \infty} \lambda^{2-D/2} - \lim_{\lambda \to 0} \lambda^{2-D/2} \Big\}$$

$$\equiv \int \frac{(d^D k)}{(k^2)^2} = \mathcal{V}_2.$$

Based on the dimensional analysis, we may conclude that

$$\mathcal{V}_n = \int \frac{(d^D k)}{[k^2]^n} = 0 \quad \text{for } n \neq D/2.$$

However, the case of $n = D/2$ (or $n = 2$ if $\varepsilon \to 0$) requires the special consideration because the dimensional analysis argumentation does not now work.

Nevertheless, the nullification of $\mathcal{V}_{D/2}$ takes still place but thanks to different reasons. It turns out, the ultraviolet and infrared divergencies are cancelled each other. Hence, if only the ultraviolet divergencies are under our consideration, $\mathcal{V}_{D/2}$ is not equal to zero.

To demonstrate, we dwell on the vacuum integration which is externally the IR-regularized one. In the spherical co-ordinate system, we write the following representation (μ^2 plays a role of IR-regularization)

$$\mathcal{V}_2 = \int_{UV} \frac{(d^D k)}{[k^2]^2} \equiv \frac{\pi^{D/2}}{\Gamma(D/2)} \int_{\mu^2}^{\infty} d\beta \beta^{D/2-3} \quad \text{with} \quad \beta = |k|^2,$$

Next, calculating β-integration, we reach the representation as

$$\boxed{\mathcal{V}_2 = \frac{\pi^{2-\varepsilon}\mu^{-2\varepsilon}}{\Gamma(2-\varepsilon)} \frac{1}{\varepsilon}\Big|_{\varepsilon\to 0},}$$

where the ϵ-pole corresponds to the UV-divergency only (the IR-divergency is absent by construction thanks for μ^2) [Grozin:2005, Grozin:2007].

On the other hand, we are able to calculate the vacuum integration by Gorishny-Isaev's method Gorishnii-Isaev:1984. In this case, $\mathcal{V}_n$ reads

$$\mathcal{V}_n = \int \frac{(d^D k)}{[k^2]^n} = \frac{2i\,\pi^{1+D/2}}{(-1)^{D/2}\,\Gamma(D/2)}\delta(n - D/2).$$

Supposing $D = 4 - 2\varepsilon$, the only contribution is given by

$$\mathcal{V}_2 = \int \frac{(d^D k)}{[k^2]^2} = \frac{2i\,\pi^{3-\varepsilon}}{\Gamma(2-\varepsilon)}\delta(\varepsilon) \neq 0.$$

Hence, the delta-function of argument ϵ reflects the UV-divergency. We specially stress that the representations of $\mathcal{V}_2$ given by the above mentioned eqns. are equivalent.

The delta-function as a generated function (distribution) is a linear singular functional (which cannot be generated by any locally-integrated functions) defined on the suitable finite function space.
Also, the delta-function can be understood with the help of the fundamental sequences of regular functionals provided the corresponding weak limit, see for example Antosik:1973, Gelfand:1964. Besides, one of the delta-function representations is related to the following realization

$$\delta(t) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t) \equiv \lim_{\varepsilon\to 0} \frac{St.F.(-\varepsilon \leq t \leq 0)}{\varepsilon},$$

where $St.F.(-\varepsilon \leq t \leq 0)$ implies the well-known step-function without any uncertainties.

One can see that the treatment of $\delta(\varepsilon)$ as the linear (singular) functional on the finite function space with $d\mu(\varepsilon) = d\varepsilon\phi(\varepsilon)$ meets some difficulties within the dimensional regularization approach. Indeed, for the practical use, ε is not a convenient variable for the construction of the finite function space because we finally need to be focused on the limit as $\epsilon \to 0$.

Meanwhile, within the sequential approach Antosik:1973, Gelfand:1964, the delta-function might be considered as the usual singular (meromorphic) function and the $\delta(0)$-singularity/uncertainty can be treated as a pole of the first order Anikin:2020,

$$\boxed{\delta(0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(0) \equiv \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}.}$$

P.S. For the demanding mathematician, this representation should be understood merely as a symbol. That is, $\delta(0)$ denotes alternatively the limit of $1/\epsilon$. This representation is also backed by the obvious fact that the mentioned eqns. are equivalent ones.

It is worth to notice that representation of $\delta(0)$ through the pole of an arbitrary meromorphic function should be used very carefully. For example, if we suppose that (here, $z \in \mathbb{E}^4$ and the delta-function is assumed to be a functional on the finite function space)

$$\big[\delta(z)\big]^2 = \delta(0)\,\delta(z),$$

the representation given by

$$\delta(z) = \lim_{\epsilon\to 0} \delta_\epsilon(z), \quad \delta_\epsilon(z) = \frac{1}{\pi^2\epsilon^4} e^{-z^2/\epsilon^2} \Rightarrow \delta(0) \sim \delta_\epsilon(0) = \frac{1}{\pi^2\epsilon^4}$$

does not satisfy the necessary condition. Another informative example can be found in Efimov:1973.

To conclude, we have presented the important explanations regarding the massless vacuum integrations. In the note, we have demonstrated the preponderance of sequential approach where the singular generated functions (distributions) are treated as a fundamental sequences of regular functionals. Due to this treatment, the uncertainty as $\delta(0)$ can be resolved via the meromorphic function of first order. Also, it has been shown in detail how the delta-function represents either UV-regime or IR-regime.