Professor Valery Burov and relativistic few-body problem

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Dedicated to memory of Professor Valery Burov

Bethe-Salpeter approach with separable kernel:

- NN-scattering, dispersion relations for separable T matrix: S.Bondarenko, V.Burov, S.Dorkin
- elastic eD-scattering: S.Bondarenko, V.Burov, S.Dorkin, A.Bekzhanov, M.Beyer, H.Toki, A.Hosaka, N.Hamamoto, Y.Manabe
- deep-inelastic scattering: V.Burov, A.Molochkov, G.Smirnov
- deuteron photodisintegration at threshold: S.Bondarenko, V.Burov, K.Kazakov, D.Shulga
- partial-wave analisys of NN-scattering: S.Bondarenko, V.Burov, E.Rogochaya, P.Hwang
- exclusive deuteron electrodisintegration: S.Bondarenko, V.Burov, E.Rogochaya
- three-nucleon systems, elastic electron-3N scattering: S.Bondarenko, V.Burov, S.Yurev NPA 1004 (2020) 122065; 1014 (2021) 122251.

Why a relativistic approach?

- Elastic electron-deuteron scattering experiments
  "Large Momentum Transfer Measurements of the Deuteron Elastic Structure Function A(Q^2) at Jefferson Laboratory"
  JLab Hall A Collaboration, Phys.Rev.Lett.82:1374-1378,1999
  Q^2=0.7-6.0 $(\text{GeV/c})^2$

  Lorentz transformation: $\eta_{LOR} = -Q^2/4M_d^2 \sim 0.43$, $\sqrt{1+\eta_{LOR}} \sim 1.19$
- Exclusive disintegration of the deuteron experiments
  JLab Hall C Deuteron Electro-Disintegration at Very High Missing Momenta (E12-10-003) proposal
  https://www.jlab.org/exp_prog/proposals/10/PR12-10-003.pdf:
  "We propose to measure the D(e,e'p)n cross section at $Q^2 = 4.25$ $(\text{GeV/c})^2$ and xbj = 1.35 for missing momenta ranging from pm = 0.5 GeV/c to pm = 1.0 GeV/c expanding the range of missing momenta explored in the Hall A experiment (E01-020)"

  Lorentz transformation: $\eta_{LOR} = -Q^2/4s_{np} \sim 0.30$, $\sqrt{1+\eta_{LOR}} \sim 1.14$

Bethe-Salpeter equation for the nucleon-nucleon T matrix

$$T(p',p;P) = V(p',p;P) + \frac{i}{4\pi^3} \int d^4k \, V(p',k;P) \, S_2(k;P) \, T(k,p;P)$$

p', p - the relative four-momenta
P - the total four-momentum

$V(p',p;P)$ - the interaction kernel

$S_2^{-1}(k;P) = \left(\frac{1}{2} \, P\cdot\gamma + k\cdot\gamma - m\right)^{(1)} \left(\frac{1}{2} \, P\cdot\gamma - k\cdot\gamma - m\right)^{(2)}$
free two-particle Green function

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable ansatz for the kernel

$$V(p';p;s) = \sum_{m,n=1}^{N} \lambda_{mn}(s) g_m(p') g_n(p)$$

Solution for the T matrix

$$T(p';p;s) = \sum_{m,n=1}^{N} \tau_{mn}(s)\, g_m(p')\, g_n(p)$$

where

$$\Big[\tau_{mn}(s)\Big]^{-1} = \Big[\lambda_{mn}(s)\Big]^{-1} + h_{mn}(s),$$

$$h_{mn}(s) = -\frac{i}{(2\pi)^4} \int d^4k\, g_m(k) g_n(k) S_2(k;P)$$

g - the model functions, $\lambda(s)$ - a matrix of model parameters.

What is a separable kernel?

The integral equations in the nuclear physics (Lippmann-Schwinger, Bethe-Salpeter) are similar to the Fredholm (first or second) type of equations.
The separable kernel of the integral equation is the degenerated kernel.
Fredholm integral equation of the second type:

$$\phi(x) = f(x) + \lambda \int dy\ K(x,y)\phi(y)$$

Degenerated kernel of the equation:

$$K(x,y) = \sum_i a_i(x) b_i(y)$$

Solution of the equation:

$$\phi(x) = f(x) + \lambda \sum_i c_i a_i(x)$$

Constants c_i can be found by solving the system of linear equations

$$c_i - \lambda \sum_j k_{ij} c_j = f_i$$

Matrix k_{ij} and f_i are: $k_{ij} = \int dy\ b_i(y) a_j(y), \qquad f_i = \int dy\ f(y) b_i(y)$

Separable NN kernels for BS equation

- NN scattering with spinor nucleon propagators

  G. Rupp and J. A. Tjon "Relativistic contributions to the deuteron electromagnetic form factors" Phys. Rev. C41. 472 (1990)
  Nonrelativistic Graz-II $\rightarrow$ relativistic Graz-II (only ${}^3S_1 -{}^3 D_1$ partial-wave states)

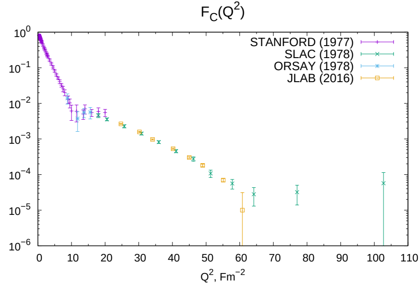
- NN scattering with scalar nucleon propagators

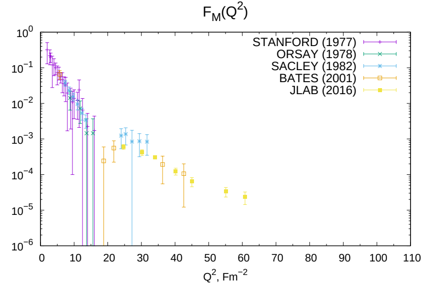
  K. Schwarz, J. Haidenbauer, J. Frohlich "A Separable Approximation of the NN Paris Potential in the Framework of the Bethe-Salpeter Equation" Phys.Rev. C33 456-466 (1986)
  partial-wave states with $J = 0, 1$ for Paris meson-exchange potentials

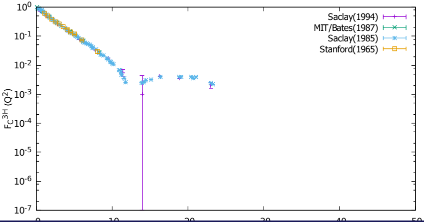
  G. Rupp, J.A. Tjon "Bethe-Salpeter calculation of three-nucleon with multirank observables separable interactions" Phys.Rev. C45 2133 (1991)
  ${}^1S_0$ and ${}^3S_1 -{}^3 D_1$ partial-wave states for Paris and Bonn meson-exchange potentials

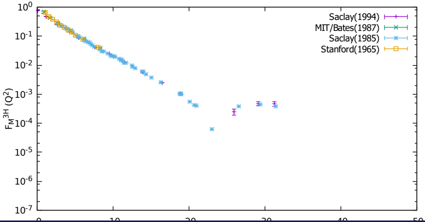
Elastic $e +^3 He(^3H) \to e +^3 He(^3H)$ scattering

Experimental data for ^3He (first line) and ^3H (second line)

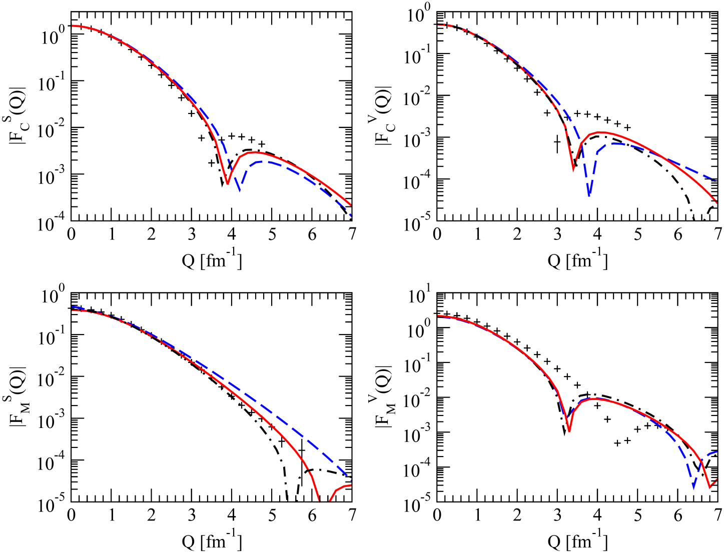








Relativistic covariant spectator theory (F. Gross, M. Pena *et al.*)



The relativistic three-particle equation for T matrix

is considered in the Faddeev form with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction is separable
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so $^3He \equiv T$

Bethe-Salpeter-Faddeev equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

Three-particles T matrix $T = \sum_i T^{(i)}$

Two-particle Green functions G_i is the free two-particles (j and n)

Two-particle T_i matrix

G. Rupp, J. A. Tjon, Phys. Rev. C 37 (1988), 1729
G. Rupp, J. A. Tjon, Phys. Rev. C 45 (1992), 2133.

Bethe-Salpeter-Faddeev equation

Introducing the equal-mass Jacobi momenta

$$p_i = \frac{1}{2}(k_j - k_n), \quad q_i = \frac{1}{3}K - k_i, \quad K = k_1 + k_2 + k_3.$$

Amplitude of three-particle state as a projection of T matrix to the bound state:

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle,$$

with $\sqrt{s} = M_B = 3m_N - E_t$.

Partial-wave three-nucleon functions for a one-rank separable kernel

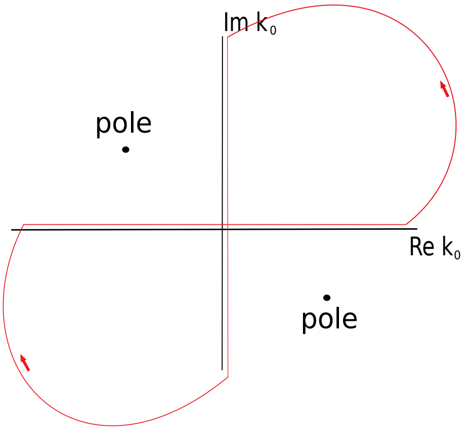
$$\Psi^{(a)}(p,q;s) = g^{(a)}(p)\tau^{(a)}[(\frac{2}{3}\sqrt{s}+q)^2]\,\Phi^{(a)}(q;s)$$

System of the integral equations

$$\Phi^{(a)}(q;s) = \frac{i}{4\pi^3}\sum_{a'}\int d^4q'\,Z^{(aa')}(q;q';s)\frac{\tau^{(a')}[(\frac{2}{3}\sqrt{s}+q')^2]}{(\frac{1}{3}\sqrt{s}-q')^2-m_N^2+i\epsilon}\Phi^{(a')}(q';s)$$

with effective kernels of equation

$$Z^{(aa')}(q;q';s) = C_{(aa')}\frac{g^{(a)}(-q/2-q')g^{(a')}(q+q'/2)}{(\frac{1}{3}\sqrt{s}+q+q')^2-m_N^2+i\epsilon}$$


Im k_0
pole
Re k_0
pole

Method of solution

- Wick-rotation procedure: $q_0 \to iq_4$
- The Gaussian quadrature with $N_1 \times N_2[q_4 \times |\mathbf{q}|]$ grid

$$q_4 = (1+x)/(1-x)$$

$$|\mathbf{q}| = (1+y)/(1-y)$$

- Iteration method to obtain the triton binding energy

$$\lim_{n\to\infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)}\Big|_{s=M_B^2} = 1$$

The convergence was investigated and $N_1 = 96$, $N_2 = 15$ was used in calculations

Rank-one Yamagichi kernels

Two-nucleon low-energy parameters for ${}^1S_0^+$ channel, ${}^3S_1^+ - {}^3D_1^+$ channels

	Exp.	${}^1S_0$
a_L (fm)	-23.748	-23.753
r_L (fm)	2.75	2.75

	Exp.	${}^3S_1 - {}^3D_1$ ($p_d = 4\%$)	${}^3S_1 - {}^3D_1$ ($p_d = 5\%$)	${}^3S_1 - {}^3D_1$ ($p_d = 6\%$)
a_L (fm)	5.424	5.454	5.454	5.453
r_L (fm)	1.756	1.81	1.81	1.80

Triton binding energy (MeV) with the one-rank Yamaguchi-type separable kernel

p_D	$^1S_0 - ^3S_1$	$^3D_1$	$^3P_0$	$^1P_1$	$^3P_1$
4	9.221	9.294	9.314	9.287	9.271
5	8.819	8.909	8.928	8.903	8.889
6	8.442	8.545	8.562	8.540	8.527
		Exp.	8.48		

- the main contribution is from S-states
- the D-state contribution is about 0.8 – 1.2 % depending on D-wave (pseudo)probability in deuteron
- the P-state contributions are alternating and give about -0.2%

Multi-rank kernels

Two-nucleon low-energy parameters for $^1S_0^+$ channel, $^3S_1^+ -^3 D_1^+$ channels

The relativistic generalization of the NR Graz-II and Paris separable kernel:

- Graz-II: $^1S_0^+$ – rank 2, $^3S_1^+ -^3 D_1$ – rank 3
- Paris-1,2: $^1S_0^+$ – rank 3, $^3S_1^+ -^3 D_1$ – rank 4

Results for $^1S_0^+$ channel

	Exp.	Graz-II	Paris-1	Paris-2
a (fm)	-23.748	-23.77	-23.72	-23.72
r_0 (fm)	2.75	2.683	2.810	2.817

Results for $^3S_1^+ -^3 D_1$ channels

	Exp.	Graz-II	Graz-II	Graz-II	Paris-1	Paris-2
p_d (%)		4	5	6	5.77	5.77
a (fm)	5.424	5.419	5.420	5.421	5.426	5.413
r_0 (fm)	1.759	1.780	1.779	1.778	1.775	1.765
E_d (MeV)	2.2246	2.2254	2.2254	2.2254	2.2246	2.2250

Results for multi-rank separable kernel

Table: Triton binding energy (MeV) with Graz-II kernel

Kernel	Nonrelativistic		Relativistic	
	$^1S_0, ^3S_1$	$^1S_0, ^3S_1, ^3D_1$	$^1S_0, ^3S_1$	$^1S_0, ^3S_1, ^3D_1$
GRAZ-II $p_D = 4$ %	8.372	8.334	8.628	8.617
GRAZ-II $p_D = 5$ %	7.964	7.934	8.223	8.217
GRAZ-II $p_D = 6$ %	7.569	7.548	7.832	7.831

Table: Triton binding energy (MeV) with Paris-1(2) kernel

Kernel	E_t
Paris-1	7.535
Paris-2	7.474

Electromagnetic form factors of three-nucleon systems:

$$2F_{\rm C}(^3{\rm He}) = (2F^p_{\rm C} + F^n_{\rm C})F_1 - \frac{2}{3}(F^p_{\rm C} - F^n_{\rm C})F_2,$$

$$F_C(^3{\rm H}) = (2F^n_{\rm C} + F^p_{\rm C})F_1 + \frac{2}{3}(F^p_{\rm C} - F^n_{\rm C})F_2,$$

$$\mu(^3{\rm He})F_{\rm M}(^3{\rm He}) = \mu_n F^n_{\rm M} F_1 + \frac{2}{3}(\mu_n F^n_{\rm M} + \mu_p F^p_{\rm M})F_2 + \frac{4}{3}(F^p_{\rm M} - F^n_{\rm M})F_3,$$

$$\mu(^3{\rm H})F_{\rm M}(^3{\rm H}) = \mu_p F^p_{\rm M} F_1 + \frac{2}{3}(\mu_n F^n_{\rm M} + \mu_p F^p_{\rm M})F_2 + \frac{4}{3}(F^n_{\rm M} - F^p_{\rm M})F_3,$$

Electric and magnetic form factors of the proton and neutron $F^{p,n}_{\rm C,M}$.

Impulse approximation:

$$F_i(Q) = \int d^4p \int d^4q \;\; G'_1(\hat{k}'_1)\, G_1(\hat{k}_1)\, G_2(\hat{k}_2)\, G_3(\hat{k}_3)\, f_i(p,q,q';P,P')$$

Nucleon propagators:

$$G_i(\hat{k}_1) = \left[\hat{k}_i^2 - m_N^2 + i\epsilon\right]^{-1},$$

$$G'_1(q'_0, q') = \left[(\frac{1}{3}\sqrt{s} - q'_0)^2 - \mathbf{q}'^2 - m_N^2 + i\epsilon\right]^{-1},$$

Three-nucleon vertex functions:

$$f_1 = \sum_{i=1}^{3} \Psi_i^*(p,q;P)\, \Psi_i(p,q';P')$$
$$f_2 = -3\Psi_1^*(p,q;P)\, \Psi_2(p,q';P')$$
$$f_3 = \Psi_3^*(p,q;P)\, \Psi_3(p,q';P')$$

Functions Ψ_i are the definite combinations of the partial state functions.

The Breit reference system

$$Q = (0, \mathbf{Q}), \qquad P = (E_B, -\frac{\mathbf{Q}}{2}), \qquad P' = (E_B, \frac{\mathbf{Q}}{2}), \tag{1}$$

with $s = M_{3N}^2$, $\eta = \mathbf{Q}^2/4s$, $E_B = \sqrt{\mathbf{Q}^2/4 + s} = \sqrt{s}\sqrt{1+\eta}$.

$$\begin{array}{lll} P = \mathrm{L} P_{c.m.}, & p = \mathrm{L} p_{c.m.}, & q = \mathrm{L} q_{c.m.} \\ P' = \mathrm{L}^{-1} P'_{c.m.}, & p' = \mathrm{L}^{-1} p'_{c.m.}, & q' = \mathrm{L}^{-1} q'_{c.m.} \end{array}$$

The explicit form of the transformation L can be obtained by using (1). Let us assume the boost of the system to be along the Z axis:

$$\mathrm{L} = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix}. \tag{2}$$

Relation of the arguments of initial and final $3N$ functions:

$$q'_0 = (1+2\eta)\, q_0 - 2\sqrt{\eta}\sqrt{1+\eta}\, q_z + \frac{2}{3}\sqrt{\eta}\, Q, \qquad (3)$$

$$q'_x = q_x \qquad q'_y = q_y$$

$$q'_z = (1+2\eta)\, q_z - 2\sqrt{\eta}\sqrt{1+\eta}\, q_0 - \frac{2}{3}\sqrt{1+\eta}\, Q,$$

here $q_z = q\cos\theta_{qQ}$ is the projection of momentum $\mathbf{q}$ onto the Z axis

Static approximation (SA) [G. Rupp, J. A. Tjon]:

$$q_0' = q_0, \qquad \mathbf{q}' = \mathbf{q} - \frac{2}{3}\mathbf{Q}$$

Propagator and final function:

$$G_1'(q_0', q') \to \left[(\frac{1}{3}\sqrt{s} - q_0)^2 - \mathbf{q}^2 - \frac{2}{3}\mathbf{q}\cdot\mathbf{Q} - \frac{4}{9}\mathbf{Q}^2 - m_N^2 + i\epsilon \right]^{-1}$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|)$$

with $\mathbf{q}\cdot\mathbf{Q} = qQ\cos\theta_{qQ}$.
The poles of G_1' on q_0 do not cross the imaginary q_0 axis and always stay in the second and fourth quadrants. In this case, the Wick rotation procedure $q_0 \to iq_4$ can be applied.

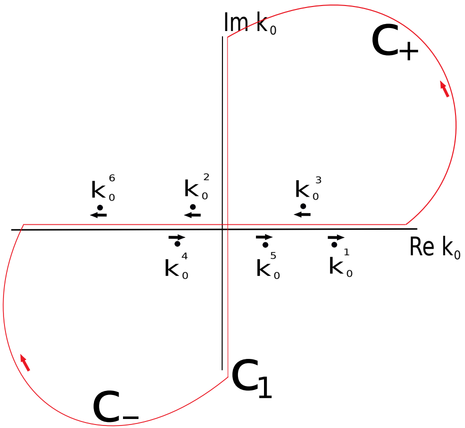
Beyond the SA:

1. Exact propagator

$$G_1' = \left[q_0^2 + \frac{2}{3}\sqrt{s}(1+6\eta)q_0 + 4\sqrt{1+\eta}\sqrt{s}\sqrt{\eta}q_z - \frac{8}{3}\eta s + \frac{1}{9}s - \mathbf{q}^2 - m_N^2 + i\epsilon\right]$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|).$$

For any $t = -Q^2 > -Q^2_{min} = 2/3\sqrt{s}(3m_N - \sqrt{s})$ the pole of G_1' on q_0 crosses the imaginary q_0 axis and appears in the third quadrant.


Im k_0
C_+
k_0^6
k_0^2
k_0^3
Re k_0
k_0^4
k_0^5
k_0^1
C_1
C_-

Beyond the SA:

2. Additional term from residue inside the countour of integration

Using the Cauchy theorem, one can transform the integrals over p_0, q_0 as follows:

$$\int d^4p \int d^4q \ldots f(p;q) = \tag{4}$$

$$- \int d^4p_E \int dq_E \, f(p_E; q_E)$$

$$+2\pi \underset{q_0=q_0^{(2)}}{\mathrm{Res}} \int d^4p_E \int_{q_{min}}^{q_{max}} dq \int_{y_{min}}^{1} dy \ldots f(p_E, ; q_0^{(2)}, q, y),$$

where $(\ldots)$ means the two-fold integral $\int_0^\infty dp \int_{-1}^1 dx$ and

$$q_0^{(1,2)} = \frac{\sqrt{s}}{3}(1+6\eta) \pm \sqrt{4\eta(1+\eta)s - 4\sqrt{s}\sqrt{\eta}\sqrt{1+\eta}qy + \mathbf{q}^2 + m_N^2} \tag{5}$$

are the simple poles of the propagator G_1'.

Beyond the SA:

3. Final function arguments transformation

Remembering that the BSF solutions are known for real values of q_4 only, the following assumption was made:

$$\Psi(p_0, p, q_0', q') \to g(p_0, p)\,\tau[(\frac{2}{3}\sqrt{s} + q_0^{(2)})^2 - \bar{\mathbf{q}}'^2]\,\Phi(0, \bar{q}'),$$

where value $\bar{q}'$ is obtained using (3) with $q_0 = q_0^{(2)}$.

The expansion of the function $\Phi(q_4', q')$ up to the first order of the parameter η:

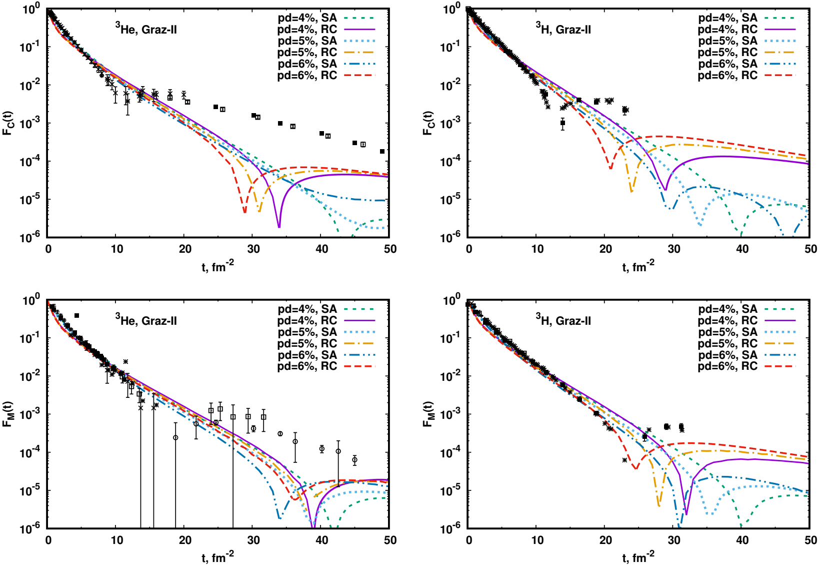
$$\Phi(iq_4', q') = \Phi(iq_4, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|) + \left[C_{q_4}\frac{\partial}{\partial q_4}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|} + \left[C_q\frac{\partial}{\partial q}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|},$$

The function Φ' is determined by the integral

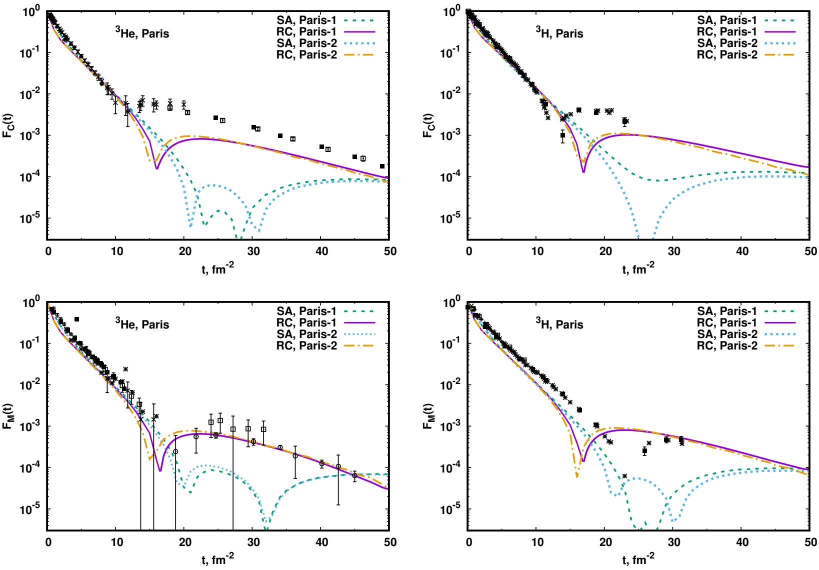
$$\Phi' = \int K'\Phi$$

where K' is a derivative of the kernel of the integral equation.

Graz-II relativistic kernel



Paris relativistic kernel



Summary

- the relativistic three-nucleon vertex functions were founs solving the BSF system of equations
- the charge and magnetic EM form factors of the 3N systems were calculated
- the static approximation and relativistic corrections were investigated
- the relativistic corrections were found to be significant in describing the experimental data