

Strong-Coupling Extrapolation of Gell-Mann-Low Functions

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Gell-Mann-Low functions can be calculated by means of perturbation theory and expressed as truncated series in powers of asymptotically small coupling parameters. However, it is necessary to know their behavior at finite values of the parameter and, moreover, their behavior at asymptotically large coupling parameters is also important. The problem of extrapolation of weak-coupling expansions to the region of finite and even infinite coupling parameters is considered. A method is suggested allowing for such an extrapolation. The basics of the method are described and illustrations of its applications are given for the examples where its accuracy and convergence can be checked. It is shown that in some cases the method allows for the exact reconstruction of the whole functions from their weak-coupling asymptotic expansions. Gell-Mann-Low functions in multicomponent field theory, quantum electrodynamics, and quantum chromodynamics are extrapolated to their strong-coupling limits.

PACS: 44.25.+f; 44.90.+c

Introduction

There exists a general problem, where, because of complexity, the sought solution can be calculated solely by means of perturbation theory resulting in truncated asymptotic series in powers of a coupling parameter or some variable. However, one needs to know the behavior of the solution at finite values of the parameter and, moreover, even for asymptotically large values of this parameter. There are several methods of extrapolation of asymptotic series to finite parameters but, it seems, there has been no reliable methods allowing for the extrapolation to infinite values of the parameters, when just a weak-coupling expansion is available and not so many terms of perturbation theory are known.

Consider, for instance, an expansion in powers of a variable $x \in [0, \infty)$. The popular Padé approximation [1]

$$P_{M/N}(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_Mx^M}{1 + b_1x + b_2x^2 + \dots + b_Nx^N}$$

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can provide reasonable accuracy for finite values of x , but for asymptotically large x , it behaves as

$$P_{M/N}(x) = \frac{a_M}{b_N} x^{M-N} \quad (x \rightarrow \infty) .$$

Since M and N can be any integers, the large-variable behavior is not defined, being spread, depending on M and N , between $-\infty$ and ∞ . Such ambiguities are, actually, common for all methods of extrapolation, because of which the problem of finding the behavior of functions in the limit of $x \rightarrow \infty$ is so much complicated.

In the present communication, we describe a general method that makes it straightforward to extrapolate a weak-coupling expansion, containing just a few terms, to the whole range of the variable, including the limit $x \rightarrow \infty$. This approach is based on self-similar approximation theory [2–6]. In the following section, we give a justification of the method we shall use.

1. Method of Self-Similar Approximations

Suppose, we are interested in the solution of a problem that can be treated only by means of perturbation theory with respect to an asymptotically small parameter or variable, resulting in an expansion

$$f_k(x) = \sum_{n=0}^k a_n x^n \quad (x \rightarrow 0) . \quad (1)$$

For concreteness, let us keep in mind a real function of a real variable $x \in [0, \infty)$. Our goal is to extrapolate this asymptotic expansion to finite and even to infinite values of the variable. As far as the region of large x is the most difficult for treatment, at the same time being often the most interesting, we shall concentrate on this limit.

The basic idea is to consider the transition from one approximation order to another as a motion of a dynamical system with respect to discrete time played by the approximation order. Then, knowing the law of motion, it could be possible to study the tendency of the dynamical system for approaching an effective limit. The formation of such a dynamical system consists of the following steps.

(i) *Introduction of control functions.* The initially given sequence (1), of course, cannot represent a stable dynamical system, since such sequences, as a rule, are divergent. Therefore the formation of a stable dynamical system has to start with the introduction of control parameters or control functions governing the sequence convergence. This process can be schematically denoted as a transformation

$$F_k(x, u_k) = \hat{T}[u_k] f_k(x) \quad (2)$$

enjoying the inverse transformation

$$f_k(x) = \hat{T}^{-1}[u_k] F_k(x, u_k) . \quad (3)$$

Control functions have to satisfy one of the two general conditions. Either they have to govern the transformed sequence convergence, or have to be defined from a training set of empirically known data. The first way is analogous to the introduction of controls for realizing controlled dynamical systems [7,8]. The second way is similar to the learning procedure in machine learning [9,10]. The convergence condition is regulated by the Cauchy criterion saying that for a given ε there exists k_ε such that

$$| F_{k+p}(x, u_{k+p}) - F_k(x, u_k) | < \varepsilon \quad (4)$$

for any $k > k_\varepsilon$ and $p > 0$. The learning procedure assumes that, for a training set $\{z_k\}$, there are the known empirical data $\{F_k\}$, for which

$$F_k(z_k, u_k) = F_k. \quad (5)$$

(ii) *Definition of approximation endomorphisms.* This requires for the sequence of approximants $F_k(x, u_k)$ to put into correspondence a bijective sequence of endomorphisms $y_k(f)$ acting on the approximation space

$$\mathcal{A} = \overline{\mathcal{L}}\{y_k(f) : k = 0, 1, 2, \dots\}, \quad (6)$$

which is a closed linear envelope of the approximation endomorphisms. The latter, by imposing a rheonomic constraint

$$f = F_0(x, u_k(x)), \quad x = x_k(f), \quad (7)$$

are defined as

$$y_k(f) = F_k(x_k(f), u_k(x_k(f))). \quad (8)$$

(iii) *Formulation of evolution equation.* The Cauchy criterion (4), in terms of the approximation endomorphisms (8), reads as

$$| y_{k+p}(f) - y_k(f) | < \varepsilon. \quad (9)$$

The existence of a sequence limit implies that, with the increasing approximation order k , one comes to a limiting value

$$y_{k+p}(f) \simeq y^*(f). \quad (10)$$

The sequence limit, for the approximation endomorphisms, is a fixed point, satisfying the condition

$$y_k(y^*(f)) = y^*(f). \quad (11)$$

From these conditions, it follows that, in the vicinity of a fixed point, the self-similar relation

$$y_{k+p}(f) = y_k(y_p(f)) \quad (12)$$

is valid.

In that way, we have a family of the approximation endomorphisms $y_k(f)$, acting on the approximation space (6), with the approximation order k playing the role of discrete time, and satisfying the self-similar evolution equation (12). Such a dynamical system is called cascade [11], or in our case this is an *approximation cascade*

$$\{y_k(f) : \mathbb{Z}_+ \times \mathcal{A} \mapsto \mathcal{A}\} . \quad (13)$$

The points

$$f \mapsto y_1(f) \mapsto y_2(f) \mapsto \dots \mapsto y_k(f) \mapsto y_k^*(f) \quad (14)$$

form the cascade trajectory. Finding a fixed point of the cascade gives us the sequence limiting value $F_k^*(x, u_k)$. Performing the inverse transformation results in the self-similar approximant

$$f_k^*(x) = \hat{T}^{-1}[u_k] F_k^*(x, u_k) . \quad (15)$$

More details on the described techniques can be found in the recent reviews [12, 13].

2. Self-Similar Factor Approximants

Control functions or control parameters can be introduced by means of different transformations. Presenting expansion (1) in the form

$$f_k(x) = \prod_{j=1}^k (1 + b_j x) , \quad (16)$$

we employ the fractal transform [14]

$$F_k(x, \{n_j\}) = \prod_{j=1}^k x^{-n_j} (1 + b_j x) . \quad (17)$$

Following the steps explained in the previous section, we come [15, 16] to the self-similar factor approximants

$$f_k^*(x) = \prod_{j=1}^{N_k} (1 + A_j x)^{n_j} , \quad (18)$$

where

$$N_k = \begin{cases} k/2, & k = 2, 4, \dots \\ (k+1)/2, & k = 3, 5, \dots \end{cases} . \quad (19)$$

Expression (18) shows that, in order to define a real function $f_k^*(x)$, either A_j and n_j have to be real or, if they are complex valued, they need to enter

the factor approximant in complex conjugate pairs, so that their product be real, taking into account the property

$$|z^\alpha|^2 = |z|^{2\Re\alpha} \exp\{-2(\Im\alpha)\arg z\}.$$

Occasional complex approximants are to be discarded.

The parameters A_j and n_j are control parameters that can be defined from a training set played by the coefficients a_n of the initial expansion (1). The training conditions are

$$\lim_{x \rightarrow 0} \frac{1}{n!} \frac{d^n}{dx^n} f_k^*(x) = a_n. \quad (20)$$

As is clear, these conditions guarantee the asymptotic equality

$$f_k^*(x) \simeq f_k(x) \quad (x \rightarrow 0). \quad (21)$$

By comparing $\ln f_k^*(x)$ and $\ln f_k(x)$, the training conditions (20) can be represented as the equations

$$\sum_{j=1}^{N_k} n_j A_j^m = L_m \quad (m = 1, 2, \dots, k), \quad (22)$$

where

$$L_m = \frac{(-1)^{m-1}}{(m-1)!} \lim_{x \rightarrow 0} \frac{d^m}{dx^m} \ln f_k(x). \quad (23)$$

Being interested in the large-variable limit, we find from (18)

$$f_k^*(x) \simeq B_k x^{\nu_k} \quad (x \rightarrow \infty), \quad (24)$$

with the large-variable amplitude

$$B_k = \prod_{j=1}^{N_k} A_j^{n_j} \quad (25)$$

and the large-variable exponent

$$\nu_k = \sum_{j=1}^{N_k} n_j. \quad (26)$$

Equations (22) uniquely define all control parameters for even orders $k = 2, 4, \dots$. For odd orders of $k = 3, 5, \dots$, the number of equations k is smaller than the number $k + 1$ of the parameters A_j and n_j . To make the system of equations well defined, it is necessary to add one more equation. For this purpose, one can employ the diff-log transformation of expansion (1), construct the factor approximant for the transformed expansion, and to find the exponent ν_k , thus getting one more equation [17]. The convergence can be improved resorting to Borel transformation of series (1).

3. Zero-Dimensional Field Theory

Before going to complicated problems, whose solutions are not known, let us first consider the simpler cases, where it would be possible to check the convergence and accuracy of the method. The popular test-horse is the so-called zero-dimensional φ^4 field theory, with the generating functional

$$Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\varphi^2 - g\varphi^4) d\varphi, \quad (27)$$

with the coupling $g \in [0, \infty)$. The weak-coupling expansion

$$Z_k(g) = \sum_{n=0}^k a_n g^n \quad (g \rightarrow 0) \quad (28)$$

strongly diverges for any finite coupling g , since the expansion coefficients

$$a_n = \frac{(-1)^n}{\sqrt{\pi} n!} \Gamma(2n+1) \quad (29)$$

factorially grow with n .

We construct the self-similar factor approximants (18) for expansion (28) up to the 16-th order and obtain the large-variable behavior of the functional

$$Z_{16}^*(g) \simeq 0.828g^{-0.187} \quad (g \rightarrow \infty),$$

whose accuracy is within about 20%. Borel transformation improves convergence and accuracy, yielding in the 14-th order

$$Z_{14}^*(g) \simeq 0.973g^{-0.242} \quad (g \rightarrow \infty),$$

whose error is around 3%. This should be compared with the exactly known asymptotic behavior

$$Z(g) \simeq 1.023g^{-0.25} \quad (g \rightarrow \infty).$$

Although the accuracy of these results may seem to be not extremely impressive, however it is not as bad as well. In addition, one has to remember that no information has been used, except the bare weak-coupling expansion (28). Moreover, calculations prove that even the low-order approximants give reasonable accuracy.

In any case, we have to accept that some hidden information is contained in the small-variable expansions of type (1), and self-similar approximants do decode the hidden information, so that, even having nothing, except the small-variable expansion, the sought function can be restored for arbitrary variables, including the most difficult and important limit of asymptotically large variables.

4. One-Dimensional Anharmonic Oscillator

The other touch-stone for calculational methods is the one-dimensional anharmonic oscillator characterized by the Hamiltonian

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + gx^4, \quad (30)$$

with $x \in (-\infty, \infty)$ and the coupling $g > 0$. The weak-coupling expansion of the ground-state energy

$$E_k(g) = \sum_{n=0}^k a_n g^n \quad (g \rightarrow 0) \quad (31)$$

strongly diverges because of the factorially growing coefficients that can be found in Ref. [18].

Directly applying self-similar factor approximants, we have in 14-th order

$$E_{14}^*(g) \simeq 0.739g^{0.294} \quad (g \rightarrow \infty),$$

within the error of 10%. Involving Borel transformation, in the same order, we get

$$E_{14}^*(g) \simeq 0.688g^{0.327} \quad (g \rightarrow \infty),$$

with the error of about 2%. The exact asymptotic behavior is

$$E(g) \simeq 0.668g^{1/3} \quad (g \rightarrow \infty).$$

Again, being based solely on the weak-coupling expansion, self-similar approximants restore the sought function for all couplings, including asymptotically large.

5. Supersymmetric Yang-Mills Theory

In some cases, self-similar approximants can restore the whole function exactly, provided we can compare the obtained approximants with the known exact expression [19, 20]. Let us consider the $N = 1$ symmetric pure Yang-Mills theory, where the weak-coupling expansion for the Gell-Mann-Low function reads as [21–24]

$$\beta_k(g) = -\frac{3g^3 N_c}{16\pi^2} \sum_{n=0}^k b_n g^{2n} \quad (g \rightarrow 0), \quad (32)$$

with the coefficients $b_n = (N_c/8\pi^2)^n$.

The self-similar approximant of second order is

$$\beta_2^*(g) = -\frac{3g^3 N_c}{16\pi^2} (1 + A_1 g^2)^{n_1}. \quad (33)$$

From the training set of the given b_n , we have the control functions

$$A_1 = - \frac{N_c}{8\pi^2}, \quad n_1 = -1.$$

Thus the self-similar factor approximant takes the form

$$\beta_2^*(g) = - \frac{3g^3 N_c}{16\pi^2} \left(1 - \frac{N_c}{8\pi^2} g^2\right)^{-1}, \quad (34)$$

which coincides with the exact known beta function [21–24]. All higher orders of the approximants are also reduced to the exact expression (34),

$$\beta_k^*(g) = \beta_2^*(g) \quad (k = 2, 3, \dots). \quad (35)$$

6. Multicomponent Field Theory

The weak-coupling expansion of the Gell-Mann-Low function in the N -component ϕ^4 theory, is

$$\beta_k(g) = g^2 \sum_{n=0}^k b_n g^n \quad (g \rightarrow 0), \quad (36)$$

with the coefficients within a minimal subtraction scheme, in the six-loop approximation, given in Ref. [25]. For $N = 1$, the Gell-Mann-Low function is known in the seven-loop approximation [26].

The self-similar factor approximants, for $N = 1, 2, 3, 4$, in the strong-coupling limit yield

$$\begin{aligned} \beta_5^*(g) &\simeq 1.698g^{1.764} & (N = 0), \\ \beta_6^*(g) &\simeq 1.857g^{1.750} & (N = 1), \\ \beta_5^*(g) &\simeq 2.017g^{1.735} & (N = 2), \\ \beta_5^*(g) &\simeq 2.178g^{1.719} & (N = 3), \\ \beta_5^*(g) &\simeq 2.340g^{1.702} & (N = 4). \end{aligned} \quad (37)$$

Using the self-similar Borel summation gives

$$\begin{aligned} \beta_4^*(g) &\simeq 1.42g^{1.83} & (N = 0), \\ \beta_6^*(g) &\simeq 1.70g^{1.77} & (N = 1), \\ \beta_5^*(g) &\simeq 2.18g^{1.67} & (N = 2), \\ \beta_5^*(g) &\simeq 2.50g^{1.63} & (N = 3), \\ \beta_5^*(g) &\simeq 2.81g^{1.61} & (N = 4). \end{aligned} \quad (38)$$

No exact results are known for the strong-coupling behavior of this function. In that sense, the behavior of the beta functions at large $g \rightarrow \infty$ is

a prediction. This prediction uses the self-similar factor approximant that is trained on the data from the weak-coupling region. This procedure is somewhat similar to the forecasting made by means of dynamical models trained on a set of initial data [27–29] and to machine-learning trained on these data [9, 10].

Some estimates for the value of the coupling $g \gtrsim 1$ have been done for $N = 1$ in Refs. [30, 31]. They used a Borel-type summation with conformal mapping, including three control parameters chosen so that to make the Borel transform self-consistent and satisfying the minimal-difference condition. The latter is known to improve convergence, being a part of optimized perturbation theory [32, 33]. The behavior of the beta function for $g \gtrsim 1$ was found to be proportional to g^ν , with $\nu \in [1.7, 2.2]$.

It is worth stressing that in the method of self-similar approximants not merely the strong-coupling limit can be evaluated, as in Refs. [30, 31], but the whole function is obtained. For example, in the case of $N = 1$, in the second and sixth order we have

$$\begin{aligned} \beta_2^*(g) &= \frac{3g^2}{(1 + 9.599g)^{0.197}} , \\ \beta_6^*(g) &= \frac{3g^2}{(1 + 5.357g)^{0.187}(1 + 13.72g)^{0.06}(1 + 22.096g)^{0.003}} \quad (N = 1) . \end{aligned} \quad (39)$$

From the renormalization group equation

$$\mu \frac{\partial g}{\partial \mu} = \beta(g) , \quad (40)$$

where μ is a scale parameter, it follows that, under $\nu > 1$, when the coupling rises, so that

$$\beta(g) \simeq Bg^\nu \quad (g \rightarrow \infty) , \quad (41)$$

then g increases as

$$g \simeq \frac{1}{[(\nu - 1)B \ln(\mu_0/\mu)]^{1/(\nu-1)}} \quad (\mu \rightarrow \mu_0 - 0) . \quad (42)$$

7. Quantum Electrodynamics

The Gell-Mann-Low function in quantum electrodynamics, in the renormalized minimal subtraction scheme, has the weak-coupling five-loop expansion

$$\beta_k(\alpha) = \left(\frac{\alpha}{\pi}\right)^2 \sum_{n=0}^k b_n \left(\frac{\alpha}{\pi}\right)^n . \quad (43)$$

One usually takes into account electrons, although the contributions of leptons with higher masses, such as muons and tau leptons, are neglected. The coefficients b_n are given in Ref. [34].

Self-similar factor approximants lead to the strong-coupling limit

$$\beta_4^*(\alpha) \simeq 0.476 \left(\frac{\alpha}{\pi}\right)^{2.096} \quad \left(\frac{\alpha}{\pi} \rightarrow \infty\right), \quad (44)$$

and the self-similar Borel summation results in the strong-coupling behavior

$$\beta_3^*(\alpha) \simeq 0.587 \left(\frac{\alpha}{\pi}\right)^{2.12} \quad \left(\frac{\alpha}{\pi} \rightarrow \infty\right). \quad (45)$$

The QED running coupling is the solution to the renormalization group equation

$$\mu^2 \frac{\partial}{\partial \mu^2} \left(\frac{\alpha}{\pi}\right) = \beta(\alpha). \quad (46)$$

As an initial condition, it is possible to take α at the Z-boson mass,

$$\alpha(m_Z) = 0.007815, \quad m_Z = 91.1876 \text{ GeV}.$$

Then the behavior of the coupling in the vicinity of the scale point

$$\mu_0 = 8.58 \times 10^{260} \text{ GeV}, \quad (47)$$

where the beta function sharply increases as

$$\beta(\alpha) = B \left(\frac{\alpha}{\pi}\right)^\nu \quad \left(\frac{\alpha}{\pi} \rightarrow \infty\right), \quad (48)$$

can be written in the form

$$\alpha \simeq \frac{\pi}{[(\nu - 1)2B \ln(\mu_0/\mu)]^{1/(\nu-1)}}. \quad (49)$$

Substituting here numerical values gives

$$\alpha \simeq \frac{2.743}{[\ln(\mu_0/\mu)]^{0.682}} \quad (\mu \rightarrow \mu_0 - 0). \quad (50)$$

As is seen, the point of divergence (47) is drastically shifted from the Landau pole that is of order 10^{30} GeV [35].

8. Quantum Chromodynamics

The weak-coupling expansion of the Gell-Mann-Low function in quantum chromodynamics reads as

$$\beta_k(\alpha_s) = - \left(\frac{\alpha_s}{\pi}\right)^2 \sum_{n=0}^k b_n \left(\frac{\alpha_s}{\pi}\right)^n \quad \left(\frac{\alpha_s}{\pi} \rightarrow 0\right), \quad (51)$$

where α_s is the quark-gluon coupling. The coefficients, within the minimal subtraction scheme, can be found in Refs. [36–38]. We keep in mind the

physically realistic case of $N_c = 3$ colors and $n_f = 6$ flavors. Bound states are not considered.

Self-similar approximation results in the strong-coupling behavior of the beta function

$$\beta_2^*(\alpha_s) \simeq -2.277\alpha_s^{2.598} \quad \left(\frac{\alpha_s}{\pi} \rightarrow \infty\right). \quad (52)$$

Self-similar Borel summation leads to

$$\beta_2^*(\alpha_s) \simeq -1.89\alpha_s^{2.75} \quad \left(\frac{\alpha_s}{\pi} \rightarrow \infty\right). \quad (53)$$

The QCD running coupling satisfies the renormalization group equation

$$\mu^2 \frac{\partial}{\partial \mu^2} \left(\frac{\alpha_s}{\pi}\right) = \beta(\alpha_s). \quad (54)$$

As an initial condition, one can take the value of the coupling at the mass of the Z-boson,

$$\alpha_s(m_Z) = 0.1184, \quad m_Z = 91.1876 \text{ GeV}.$$

Then the running coupling in the vicinity of $\mu_c = 0.1 \text{ GeV}$ behaves as

$$\alpha_s \simeq \frac{0.907}{[\ln(\mu/\mu_c)]^{0.626}} \quad (\mu \rightarrow \mu_c + 0). \quad (55)$$

Again, the point of divergence is essentially shifted from the Landau pole that is close to 0.9 GeV [39].

9. Conclusion

Self-similar approximation theory is presented allowing for the extrapolation of functions from weak-coupling expansions to the whole range of their variables, including the limit of asymptotically large variables tending to infinity. The justification of the approach is given. Briefly speaking, the approach is based on discovering self-similarity in the coefficients of the considered expansion, which then allows for its effective extrapolation to higher orders.

The use of self-similar factor approximants is demonstrated for the problems, whose exact behavior is known. It is shown that in some cases, e.g. in the case of supersymmetric Yang-Mills theory, the beta function can be reconstructed exactly. The strong-coupling extrapolation for the Gell-Mann-Low functions is accomplished for the N -component φ^4 theory, quantum electrodynamics, and quantum chromodynamics.

Conflict of interest

The authors declare that they have no conflicts of interest.

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