Lectures 1&2: Linear Optics

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Objectives

- Design of linear optics is a very important step in the design of an accelerator
- It determines all major parameters and properties
- In majority of cases the optics design does not require accounting of coupling between different degrees of freedom
  - And coupling can be considered in the perturbation theory
- However, in the recent years, machines, where different degrees of freedom are strongly coupled, were considered
  - Examples: Electron and Ionization cooling (including both linear and the ring-based machines), Optical stochastic cooling
- In this lecture we consider basics of linear optics for the coupled and uncoupled optics
  - We shortly refresh uncoupled optics
  - Then, having some experience, we consider x-y coupled optics

Uncoupled Betatron Motion

Equations for Uncoupled Motion

- Linearized equation of motion

$$x'' + \left(K_x^{\ 2} + k\right)x = 0$$

where: $K_x(s) \equiv K_x = eB_y(s)/Pc$, $\quad k(s) \equiv k = eG(s)/Pc$

- In Hamiltonian form

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{\partial H}{dp} \\ \dfrac{dp}{ds} = -\dfrac{\partial H}{dx} \end{cases} \qquad \text{with} \qquad H = \frac{p^2}{2} + \left(K_x^{\ 2} + k\right)\frac{x^2}{2}$$

- General solution of 2-nd order linear equation

$$x(s) = C(s)x(0) + S(s)\theta(0)\,, \quad \theta(s) \equiv dx/ds$$

where **C**(s) and **S**(s) two linear independent solutions

We can rewrite it in matrix form

$$\begin{bmatrix} x(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{bmatrix} x(0) \\ \theta(0) \end{bmatrix} \quad \text{or} \quad \boxed{\mathbf{x}(s) = \mathbf{M}(s)\mathbf{x}(0)}$$

Conservation of the Phase Space Volume

Jacobian does not depend on time

$$\frac{d}{ds}\left(\frac{\partial(p,q)}{\partial(p_0,q_0)}\right)=\frac{d}{ds}\left(\begin{vmatrix}\frac{\partial}{\partial p_0}\left(p_0+\frac{dp}{ds}ds\right) & \frac{\partial}{\partial p_0}\left(x_0+\frac{dx}{ds}ds\right)\\ \frac{\partial}{\partial x_0}\left(p_0+\frac{dp}{ds}ds\right) & \frac{\partial}{\partial x_0}\left(x_0+\frac{dx}{ds}ds\right)\end{vmatrix}\right)=\frac{d}{ds}\left(\begin{vmatrix}1-\frac{\partial^2 H}{\partial s\partial p}ds & \frac{\partial^2 H}{\partial p^2}ds\\ \frac{\partial^2 H}{\partial^2 x}ds & 1+\frac{\partial^2 H}{\partial x\partial p}ds\end{vmatrix}\right)=0$$

where we used $\begin{cases}\dfrac{dx}{ds}=\dfrac{\partial H}{dp}\\ \dfrac{dp}{ds}=-\dfrac{\partial H}{dx}\end{cases}$

⇨ The phase space volume is conserved in the course of motion and, consequently, $|\mathbf{M}|=1$

- The conservation of the phase space volume is also justified for multidimensional motion.
  It is called **Liouville theorem**

Betatron Motion in a Ring

- Arbitrary turn-by-turn betatron motion at a given place may be presented through eigen-vectors

$$\mathbf{x}_n = \mathrm{Re}\left(\Lambda_1^{\ n}\left(A_1\mathbf{v}_1\right)+\Lambda_2^{\ n}\left(A_2\mathbf{v}_2\right)\right) \quad \text{where} \quad \mathbf{M}\mathbf{v}_k = \Lambda_k\mathbf{v}_k\ , \quad k=1,2$$

  - Stable betatron motion requires $|\Lambda_k|=1$ => $\Lambda_2 = \Lambda_1^{\ *}$ (since real **M**)
  - Introduce betatron frequencies so that $\Lambda_{1,2} = e^{\pm i\mu}$

    Corresponding betatron tune (fractional part): $Q = \mu/2\pi$

- Description of betatron motion for the entire ring
  - The eigen-vector $\mathbf{v}(s) = \mathbf{M}(0,s)\mathbf{v}$ is the eigen-vector for the total ring transfer matrix for coordinate *s*.
  - Then we normalize the eigen-vectors so that

    $\mathbf{v}(s) = \mathbf{M}(0,s)\mathbf{v}(0)e^{-\mu(s)}$

    and require $\mathrm{Im}(\mathrm{v}_{1'}(s))=0$ and $\mathbf{v}^{+}(s)\mathbf{S}\mathbf{v}(s) = -2i$, where $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

    Then we can describe the entire ring betatron motion

$$\boxed{\mathbf{x}(s) = \sqrt{2I}\,\mathrm{Re}\left(e^{i(\psi-\mu(s))}\mathbf{v}\right)}$$

where the action *I* and the betatron phase ψ determine initial part. pos.

The Eigen-vector Parameterization

- Parametrize the eigen-vector

$$\mathbf{v} \equiv \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix} \equiv \mathbf{v}(s) = \begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} \quad , \quad \begin{cases} \mathbf{v}_1 = \mathbf{v} \\ \mathbf{v}_2 = \mathbf{v}^* \end{cases}$$

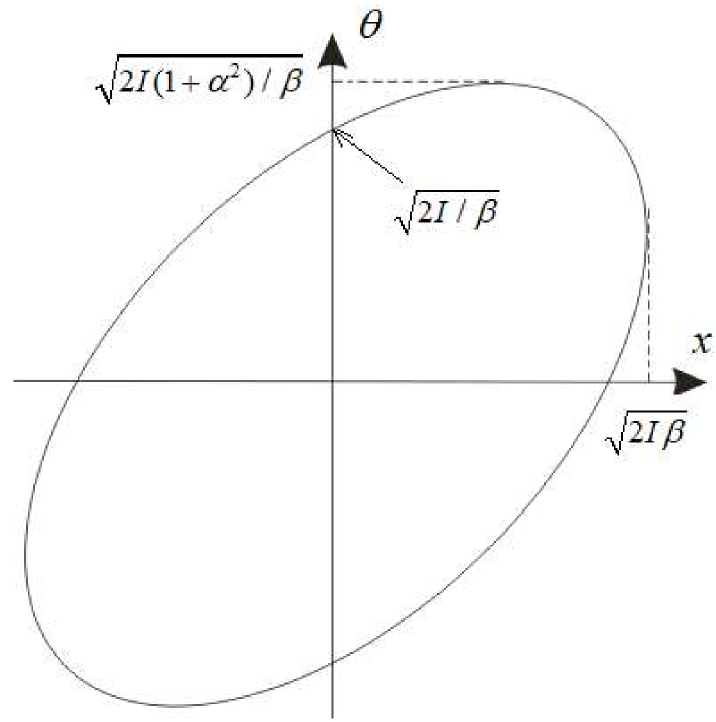
  - we define that $\mathrm{Im}(\mathrm{v}_1(s)) = 0$
  - The eigen-vectors are orthogonal and correctly normalized

$$\begin{cases} \mathbf{v}^{+}\mathbf{S}\mathbf{v} = \begin{bmatrix} \sqrt{\beta(s)} & \dfrac{i-\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} = -2i \\ \mathbf{v}^{T}\mathbf{S}\mathbf{v} = 0 \quad \text{or} \quad \mathbf{v}_2^{+}\mathbf{S}\mathbf{v}_1 = 0 \end{cases}$$

Courant-Snider Invariant

The betatron amplitude (maximum particle displacement) $= \sqrt{2I\beta}$

The maximum angle $= \sqrt{\dfrac{2I}{\beta}\left(1+\alpha^2\right)}$



The maximum angle for x=0 is achieved when

$$\sqrt{2I}\,\mathrm{Re}\left(\begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} e^{i\pi/2}\right) = \sqrt{2I}\,\mathrm{Re}\left(\begin{bmatrix} i \\ \dfrac{1-i\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix}\right)$$

⇨ Local angular spread: $\theta_m = \sqrt{\dfrac{2I}{\beta}}$

- Finding action from the known x and θ

$$\mathbf{v}^+\mathbf{S}\left[\mathbf{x} = \sqrt{2I}\left(\frac{e^{i\psi}\mathbf{v}+CC}{2}\right)\right] \xrightarrow[\text{condition}]{\text{orthogonality}} \mathbf{v}^+\mathbf{S}\mathbf{x} = -i\sqrt{2I} \rightarrow I = \frac{1}{2}\left|\mathbf{v}^+\mathbf{S}\mathbf{x}\right|^2$$

- Courant-Snyder invariant

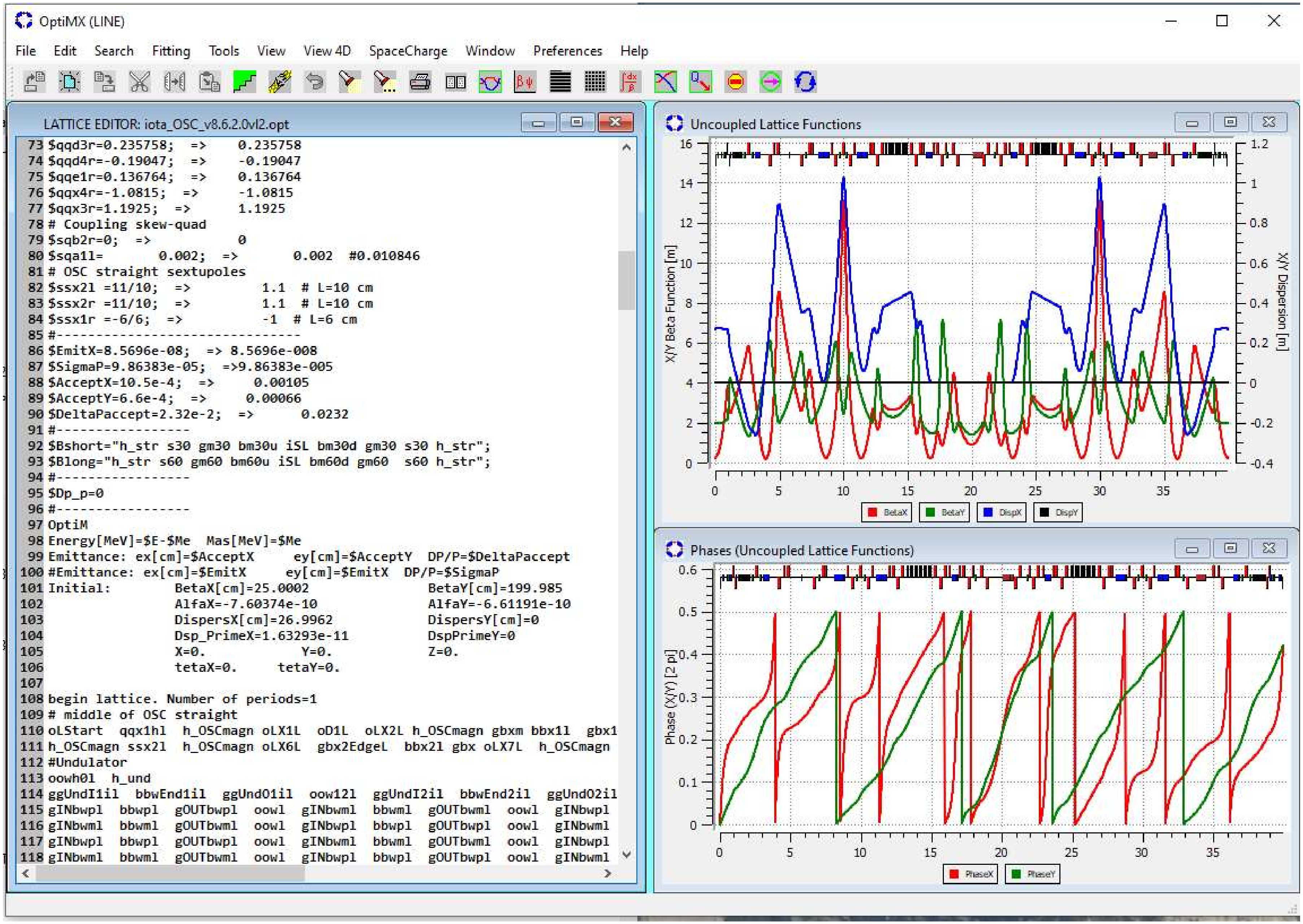
Remember that: $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$2I = \left|\mathbf{v}^+\mathbf{S}\mathbf{x}\right|^2 = \beta\theta^2 + 2\alpha x\theta + \frac{1+\alpha^2}{\beta}x^2$$

Computation of Machine Optics

Software for Computation of machine optics

- There are many computer codes allowing one to compute beam optics
- I mention 3 of them
  1. MAD -> MAD-8 -> MADX - supported by CERN
     https://mad.web.cern.ch/mad/
  2. Elegant - supported by ANL
     https://www.aps.anl.gov/Accelerator-Operations-Physics/Software#elegant
  3. OptiMX - supported by Fermilab
     https://home.fnal.gov/~ostiguy/OptiM/ (temporary link because of Fermilab security:
     https://www.dropbox.com/s/56l4nctnwegf7w7/OptimX64-20210526-setup.exe?dl=0)
- In this course we will be using OptiM
  - Interactive, GUI driven, easy to learn
  - Operates on major computer platforms: Windows, Unix, MAC
  - Free installation, Easy to install
  - Online help (documentation)
- Input file consists of:
  - Math header
  - Main body starting from keyword OptiM. It includes: (1) beam parameters, (2) element sequence, (3) parameters of elements, (4) service blocks



Computations can be done in a ring and beam line modes

X-Y Coupled Betatron Motion

Equations for X-Y Coupled Motion

- Linearized equations of motion

$$\begin{cases} x'' + \left(K_x^{\ 2} + k\right)x + \left(N - \dfrac{1}{2}R'\right)y - Ry' = 0 \\ y'' + \left(K_y^{\ 2} - k\right)y + \left(N + \dfrac{1}{2}R'\right)x + Rx' = 0 \end{cases}$$

where: $K_{x,y}(s) \equiv K_{x,y} = eB_{y,x}(s)/Pc$, $k(s) \equiv k = eG(s)/Pc$, $N = eG_s/Pc$, $R = eB_s/Pc$

- In Hamiltonian

$$H = \frac{p_x^{\ 2} + p_y^{\ 2}}{2} + \left(K_x^{\ 2} + k + \frac{R^2}{4}\right)\frac{x^2}{2} + \left(K_y^{\ 2} - k + \frac{R^2}{4}\right)\frac{y^2}{2} + Nxy + \frac{R}{2}\left(yp_x - xp_y\right)$$

the corresponding canonical momenta are: $\begin{cases} p_x = x' - \dfrac{R}{2}y \\ p_y = y' + \dfrac{R}{2}x \end{cases}$

In matrix form: $\hat{\mathbf{x}} = \mathbf{R}\mathbf{x}$

$$\hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Form of Equations for X-Y Coupled Motion

$$H = \frac{1}{2}\hat{\mathbf{x}}^T \mathbf{H}\hat{\mathbf{x}}$$ where

$$\mathbf{H} = \begin{bmatrix} K_x^{\,2} + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^{\,2} - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix}$$

Then the motion equations are

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{U}\mathbf{H}\hat{\mathbf{x}}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Properties of matrix $\mathbf{U}$ (called unit symplectic matrix)
  $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{U}\mathbf{U} = -\mathbf{I}$, where $\mathbf{I}$ is the identity matrix
- Similar to the single dimensional motion we introduce 4-dimensional transfer matrix, $\hat{\mathbf{x}} = \hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0$, for the 2-dimensional motion
  The cap here and below denotes that we consider the transfer matrix which uses canonical momenta instead of angles

Motion Symplecticity

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{UH}\hat{\mathbf{x}}$$

- Lagrange invariant

$$\frac{d}{ds}\left(\hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2\right) = \frac{d\hat{\mathbf{x}}_1{}^T}{ds}\mathbf{U}\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1{}^T\mathbf{U}\frac{d\hat{\mathbf{x}}_2}{ds} = \hat{\mathbf{x}}_1{}^T\mathbf{H}^T\mathbf{U}^T\mathbf{U}\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1{}^T\mathbf{UUH}\hat{\mathbf{x}}_2 = 0$$

$$\Rightarrow \quad \hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2 = \mathrm{const}$$

- Motion symplecticity

  Substituting $\hat{\mathbf{x}} = \hat{\mathbf{M}}\hat{\mathbf{x}}_0$ into above equation one obtains

$$\hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1{}^T\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_2 = \mathrm{const}$$

  As the above equation is satisfied for any $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ it yields

$$\hat{\mathbf{M}}^T\mathbf{U}\hat{\mathbf{M}} = \mathbf{U}$$

  This property is called symplecticity and matrix $\hat{\mathbf{M}}$ symplectic

- $\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)$ is antisymmetric

  ⇨ Only six ($(n^2-n)/2 = 6$) of these equations are independent (4 diagonal ones are identities). Thus, out of 16 matrix elements of matrix **M** the motion symplecticity leaves only 10 elements linearly independent

Symplecticity of Eigen-Vectors

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_k = \lambda_k \hat{\mathbf{v}}_k \ , \quad k = 1,..,4$$

- For any two eigen-vectors the symplecticity yields the identity

$$0 = \lambda_j \hat{\mathbf{v}}_j^T \mathbf{U}\left(\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_i \hat{\mathbf{v}}_i\right) = \left(\hat{\mathbf{M}}\hat{\mathbf{v}}_j\right)^T \mathbf{U}\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_j \hat{\mathbf{v}}_j^T \mathbf{U}\lambda_i \hat{\mathbf{v}}_i = \left(1 - \lambda_j \lambda_i\right)\hat{\mathbf{v}}_j^T \mathbf{U}\hat{\mathbf{v}}_i$$

where we substituted: $\left(\hat{\mathbf{M}}\hat{\mathbf{v}}_j\right)^T \mathbf{U}\hat{\mathbf{M}}\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_i^T \hat{\mathbf{M}}^T \mathbf{U}\hat{\mathbf{M}}\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_i^T \mathbf{U}\hat{\mathbf{v}}_i$

It determines that the eigen-values of stable motion always appear in two reciprocal pairs, and, consequently, the four eigen-values split into two complex conjugate pairs: λ_1, λ_1^* and λ_2, λ_2^*

For $\lambda_1 \neq \lambda_2$ (non-degenerate case) we obtain the orthogonality condition

$$\hat{\mathbf{v}}_1^+ \mathbf{U}\hat{\mathbf{v}}_1 \neq 0 \ ,$$

$$\hat{\mathbf{v}}_2^+ \mathbf{U}\hat{\mathbf{v}}_2 \neq 0 \ ,$$

$$\hat{\mathbf{v}}_i^+ \mathbf{U}\hat{\mathbf{v}}_j = 0 \qquad \text{if } i \neq j,$$

$$\hat{\mathbf{v}}_i^T \mathbf{U}\hat{\mathbf{v}}_j = 0 \ ,$$

Normalizing eigen-vectors we obtain:

$$\hat{\mathbf{v}}_1^+ \mathbf{U}\hat{\mathbf{v}}_1 = -2i \quad , \quad \hat{\mathbf{v}}_2^+ \mathbf{U}\hat{\mathbf{v}}_2 = -2i \quad ,$$

$$\hat{\mathbf{v}}_1^T \mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , \quad \hat{\mathbf{v}}_2^T \mathbf{U}\hat{\mathbf{v}}_2 = 0 \quad ,$$

$$\hat{\mathbf{v}}_2^T \mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , \quad \hat{\mathbf{v}}_2^+ \mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad .$$

Out of 2 complex conjugated vectors we choose one which satisfies the normalization condition. Normalization of CC vector has different sign.

Parameterization of Eigen-vectors

- Betatron motion is described similar to 1D case:

$$\hat{\mathbf{x}}(s) == \mathrm{Re}\left(\sqrt{2I_1}\,\hat{\mathbf{v}}_1(s)e^{-i(\psi_1+\mu_1(s))} + \sqrt{2I_2}\,\hat{\mathbf{v}}_2(s)e^{-i(\psi_2+\mu_2(s))}\right)$$

- There are 2 popular parameterizations: Edwards-Teng and Mais-Ripken
  - Here we shortly consider the extended Mais-Ripken

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}\,e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1} \end{bmatrix} \qquad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}\,e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

- The betatron motion is described by 10 linearly independent functions: 4 β-functions, 4 α-functions, and 2 betatron phase advances
- Symplecticity allows one to compute functions u, ν_1 & ν_2 from known α's & β's. However, there are 4 solutions for their values and additional information is required to choose α's and β's.
  In practice, first, we find the eigen-vectors from known transfer matrix, and, then unique solutions for all 4D-Twiss functions

4D Ellipsoid in the Phase Space

$$\hat{\mathbf{x}} = \mathrm{Re}\left(A_1 e^{-i\psi_1}\hat{\mathbf{v}}_1 + A_2 e^{-i\psi_2}\hat{\mathbf{v}}_2\right) = A_1\left(\hat{\mathbf{v}}_1{}' \cos\psi_1 + \hat{\mathbf{v}}_1{}'' \sin\psi_1\right) + A_2\left(\hat{\mathbf{v}}_2{}' \cos\psi_2 + \hat{\mathbf{v}}_2{}'' \sin\psi_2\right)$$

- Rewrite it in matrix form

$$\boxed{\hat{\mathbf{x}} = \hat{\mathbf{V}}\mathbf{A}\boldsymbol{\xi}_A} \quad \text{where} \quad \hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}', -\hat{\mathbf{v}}_1{}'', \hat{\mathbf{v}}_2{}', -\hat{\mathbf{v}}_2{}''\right] \qquad \mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \qquad \boldsymbol{\xi}_A = \begin{bmatrix} \cos\psi_1 \\ -\sin\psi_1 \\ \cos\psi_2 \\ -\sin\psi_2 \end{bmatrix}$$

- To obtain a 4D ellipsoid which includes all particles we need to account that the mode amplitudes are interdependent. To account it we put:

$$\boldsymbol{\xi} = \begin{bmatrix} \cos\psi_1 \cos\psi_3 \\ -\sin\psi_1 \cos\psi_3 \\ \cos\psi_2 \sin\psi_3 \\ -\sin\psi_2 \sin\psi_3 \end{bmatrix}$$

  so that vector ξ stays at 3D surface with unit radius, i. e. $(\boldsymbol{\xi},\boldsymbol{\xi}) = 1$

  - Substituting x in this equation we obtain: $\hat{\mathbf{x}}^T\left(\left(\hat{\mathbf{V}}\mathbf{A}\right)^{-1}\right)^T\left(\hat{\mathbf{V}}\mathbf{A}\right)^{-1}\hat{\mathbf{x}} = 1$
  - Matrix symplecticity yields $\hat{\mathbf{V}}^{-1} = \mathbf{U}^T\hat{\mathbf{V}}^T\mathbf{U}$ using this equation we finally obtain: $\boxed{\hat{\mathbf{x}}^T\hat{\boldsymbol{\Xi}}\hat{\mathbf{x}} = 1, \quad \hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T, \quad \hat{\boldsymbol{\Xi}}' = \mathbf{A}^{-1}\mathbf{A}^{-1}}$

1D and 2D Emittances

- We define the beam emittance as a product of the ellipsoid semi-axes (omitting the factor $\pi^2/2$ correcting for the real 4D volume of the ellipsoid):
$$\varepsilon_{4D} = \frac{1}{\sqrt{\hat{\Xi}'_{11}\hat{\Xi}'_{22}\hat{\Xi}'_{33}\hat{\Xi}'_{44}}} = \frac{1}{\sqrt{\det(\hat{\mathbf{\Xi}}')}}$$

- Consequently:
$$\varepsilon_1\varepsilon_2 = \varepsilon_{4D}\ , \qquad \hat{\mathbf{\Xi}}' = \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}$$

- Gaussian distribution:
$$f(\hat{\mathbf{x}}) = \frac{1}{4\pi^2\varepsilon_1\varepsilon_2}\exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{\Xi}}\hat{\mathbf{x}}\right)$$

- Second order moments
$$\hat{\Sigma}_{ij} \equiv \overline{\hat{x}_i\hat{x}_j} = \int \hat{x}_i\hat{x}_j f(\hat{\mathbf{x}})d\hat{x}^4 = \frac{1}{4\pi^2\varepsilon_1\varepsilon_2}\int \hat{x}_i\hat{x}_j \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{\Xi}}\hat{\mathbf{x}}\right)d\hat{x}^4$$

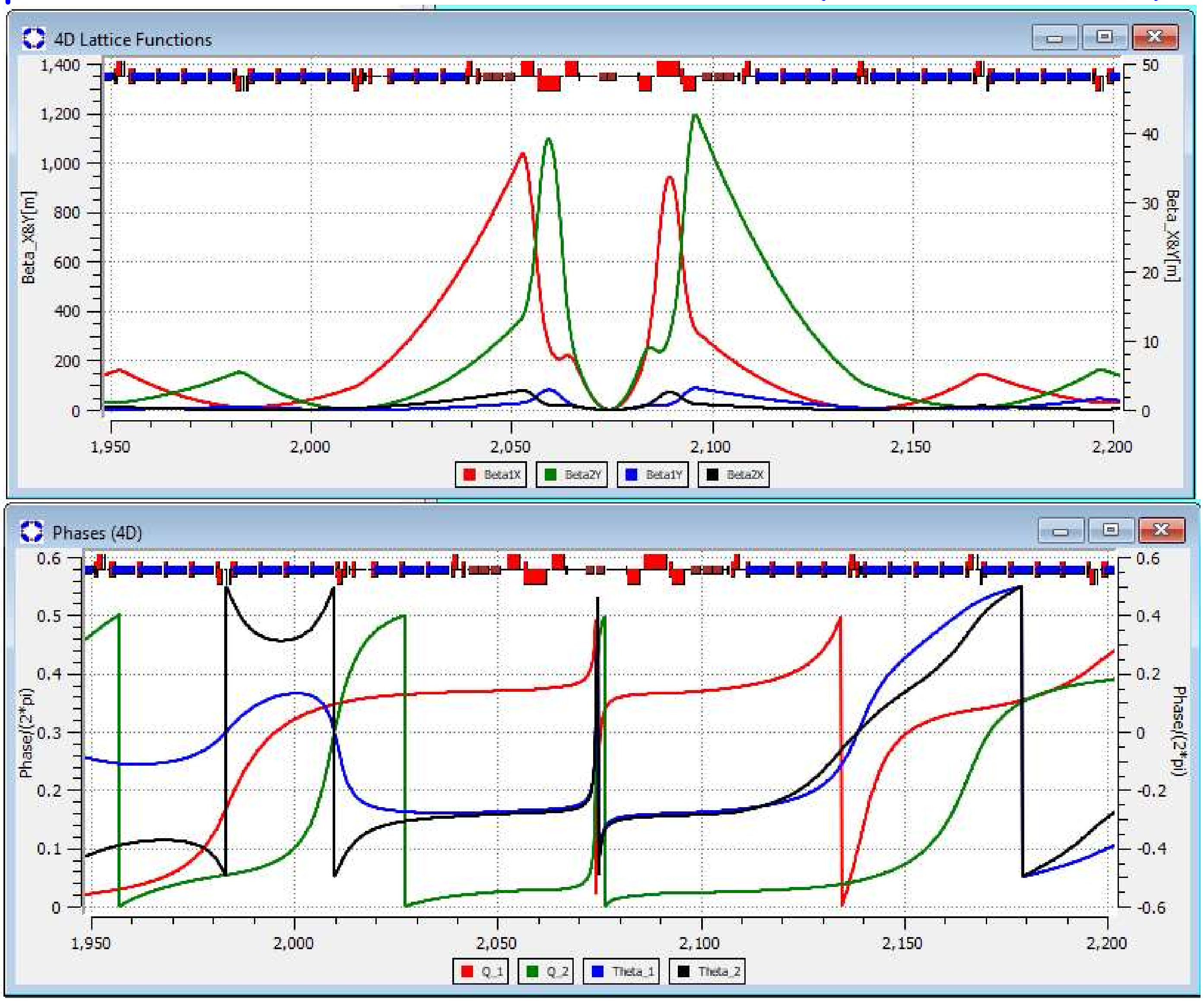
  To carry out the integration we use a transform $\hat{\mathbf{y}} = \hat{\mathbf{V}}^{-1}\hat{\mathbf{x}}$. It reduces matrix to the diagonal form. => $\hat{\mathbf{\Sigma}} = \hat{\mathbf{V}}\hat{\mathbf{\Xi}}'^{-1}\hat{\mathbf{V}}^T = \hat{\mathbf{\Xi}}^{-1}$

General Remarks

- ε_1 and ε_2 are the motion invariants – they are conserved
- In practical applications the longitudinal magnetic field at boundaries of elements is zero. Consequently, the difference between variables with and without caps disappears.

OptiMX 4D Calculations

- 4D Twiss parameters for Tevatron near B0 (CDF detector)



Perturbations of Uncoupled Betatron Motion

Perturbed Betatron Motion in Uncoupled Case

- To simplify equations, we transit to new variables

$$x = \frac{X}{\sqrt{\beta}}, \quad p = \beta \frac{d}{ds}\frac{X}{\sqrt{\beta}} = \beta\left(\frac{1}{\sqrt{\beta}}\frac{dX}{ds} - \frac{X}{2\beta^{3/2}}\frac{d\beta}{ds}\right) = \sqrt{\beta}\theta + \alpha\frac{X}{\sqrt{\beta}}$$

- In the new variables the motion description is greatly simplified. Accounting that $d\mu / ds = 1/\beta$ we obtain

$$\frac{d^2X}{d\mu^2} = -X, \quad \mathbf{M} = \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix}$$

  Consequently, the unperturbed solution is characterized by $\beta = 1, \alpha = 1$

- Choose perturbed initial particle coordinates as following:

$$\mathbf{X}_0 = \sqrt{\varepsilon}\,\mathrm{Re}\left(\begin{bmatrix} \sqrt{\hat{\beta}} \\ -\dfrac{i+\hat{\alpha}}{\sqrt{\hat{\beta}}} \end{bmatrix} e^{i\psi}\right)$$

  ⇨ Dependence of beam size on μ is

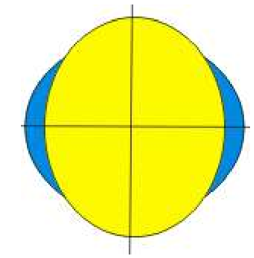
$$A(s) = \sqrt{\varepsilon}\,\mathrm{Re}\left(\left(\sqrt{\hat{\beta}}\cos\mu - \frac{i+\hat{\alpha}}{\sqrt{\hat{\beta}}}\sin\mu\right)e^{i\psi}\right)_{\substack{\max \\ \psi}} = \sqrt{\varepsilon\left(\left(\sqrt{\hat{\beta}}\cos\mu - \frac{\hat{\alpha}}{\sqrt{\hat{\beta}}}\sin\mu\right)^2 + \left(\frac{\sin\mu}{\sqrt{\hat{\beta}}}\right)^2\right)}$$

$$A(s) = \sqrt{\varepsilon\left(c^2\hat{\beta} + s^2\left(\frac{1+\hat{\alpha}^2}{\hat{\beta}}\right) - 2\hat{\alpha}cs\right)} = \sqrt{\varepsilon\left(\frac{1+\cos(2\mu)}{2}\hat{\beta} + \frac{1-\cos(2\mu)}{2}\left(\frac{1+\hat{\alpha}^2}{\hat{\beta}}\right) - \hat{\alpha}\sin(2\mu)\right)}$$

Perturbed Betatron Motion (2)

- The beam size oscillates at the double betatron frequency

$$A(s)=\sqrt{\frac{\varepsilon}{2}\left(\left(\hat\beta+\frac{1+\hat\alpha^2}{\hat\beta}\right)+\left(\hat\beta-\frac{1+\hat\alpha^2}{\hat\beta}\right)\cos(2\mu)-2\hat\alpha\sin(2\mu)\right)}$$



Consequently, the perturbed beta-function oscillates at the double betatron frequency as well. Here $\hat\beta=1+\Delta\beta/\beta$.

What is missed in the Lecture?

- Not all calculations are shown in detail
- Edwards-Teng parameterization
- How to find eigen-vectors from matrices Σ an Ξ and vice versa
- How to express a transfer matrix from known Twiss parameters or eigen vectors and betatron phase advances

- These details are not required to follow other lectures

- Look for details in:
  - V. A. Lebedev (Fermilab), S. A. Bogacz (Jefferson Lab), "Betatron motion with coupling of horizontal and vertical degrees of freedom", https://arxiv.org/abs/1207.5526
  - or "Accelerator Physics at the Tevatron Collider", edited by V. Lebedev and V. Shiltsev, Springer, 2014.

Problems

1. For uncoupled betatron motion prove that the normalization of eigen-vectors, $\hat{\mathbf{v}}_k{}^{+}\mathbf{S}\hat{\mathbf{v}}_k = -2i$, yields that $d\mu / ds = 1/\beta$. (For the proof use top Eq. of page 7)
2. Prove that if **v** is the eigen-vector for matrix **M** corresponding to the one turn matrix starting at *s*=0 (point 1) then the vector $\mathbf{M}_{12}\mathbf{v}$ will be the eigen vector of the transfer matrix corresponding to the point 2. Here $\mathbf{M}_{12}$ is the transfer matrix from point 1 to point 2.
3. Find 2D analog of Courant-Snyder invariant
4. Prove that matrix $\hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}', -\hat{\mathbf{v}}_1{}'', \hat{\mathbf{v}}_2{}', -\hat{\mathbf{v}}_2{}''\right]$ is symplectic
5. Fill missed calculations in computation $\hat{\mathbf{\Sigma}} = \hat{\mathbf{V}}\hat{\mathbf{\Xi}}'^{-1}\hat{\mathbf{V}}^T = \hat{\mathbf{\Xi}}^{-1}$
6. Prove that for a symplectic matrix, defined by the following equation $\hat{\mathbf{M}}^T\mathbf{U}\hat{\mathbf{M}} = \mathbf{U}$, its determinant is $\left|\hat{\mathbf{M}}\right| = 1$, $\hat{\mathbf{M}}^{-1} = \mathbf{U}^T\hat{\mathbf{M}}^T\mathbf{U}$ and the matrix also satisfies to $\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T = \mathbf{U}$.

7. Assuming that the motion after exit from KRION ion source is uncoupled and described uncoupled Twiss-parameters find equations describing the horizontal and vertical rms sizes in the downstream beam transport for two below cases.
(1) Ions exit at the axis of magnetic field. Beam parameters at the ion source center: magnetic field - B_0, ion rms beam size - σ, transverse temperature - T.
(2) Now add that the ions exit solenoid with offset r_0 directed at angle θ from the horizontal plane.