Differential equations method for expansion of hypergeometric functions

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Hypergeometric functions and Feynman integrals

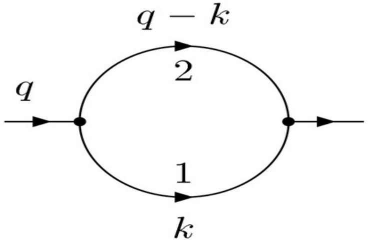
$$
{}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right) = \sum_{n=0}^{\infty} c_n z^n, \qquad \frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\ldots(n+a_p)}{(n+b_1)\ldots(n+b_1)(n+1)}.
$$

Methods for calculating Feynman integrals in terms of hypergeometry

Mellin-Barnes

dimensional reduction

Exact Frobenius



$$
= \int \frac{d^{4-2\varepsilon}k}{i\pi^{2-\varepsilon}} \frac{1}{(k^2-m^2)((k-q)^2-m^2)} = m^{\varepsilon}\Gamma(\varepsilon)\ {}_2F_1\left(\begin{array}{c} 1,\varepsilon \\ 3/2 \end{array}\middle| \frac{q^2}{4m^2}\right)
$$

Polylogarithms

$$G(a_1,...,a_n;x)=\int\limits_0^x \frac{G(a_2,...,a_n;x')}{x'-a_1}dx', \qquad n>0, \qquad G(;x)=1, \qquad G(\vec{0}_n;x)=\frac{\log^n x}{n!}$$

A. B. Goncharov, Mathematical Research Letters 5, 497 (1998).
A. B. Goncharov, arXiv preprint math/0103059 (2001).

$$G(a;b)=\log\left(1-\frac{b}{a}\right), \qquad a\neq 0$$

$$\mathrm{Li}_n(x)=-G\left(\vec{0}_{n-1},\frac{1}{x};1\right)=\int\limits_0^x \frac{dx'}{x'}\mathrm{Li}_{n-1}(x')$$

$$dG(a_1,...,a_n;x)=\sum_{i=1}^n G(a_1,...,a_{i-1},a_{i+1},...,a_n;x)d\log\left(\frac{a_{i-1}-a_i}{a_{i+1}-a_i}\right)$$

Polylogarithms form Hopf algebra

Differential equation

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n,\ldots,(a_p)_n}{(b_1)_n,\ldots,(b_q)_n}\frac{z^n}{n!} $$

$$ \Big[z(z\frac{d}{dz}+a_1)(z\frac{d}{dz}+a_2)\ldots(z\frac{d}{dz}+a_p) \\ -z\frac{d}{dz}(z\frac{d}{dz}+b_1-1)(z\frac{d}{dz}+b_2-1)\ldots(z\frac{d}{dz}+b_q-1)\Big]\ {}_pF_q = 0 $$

$$ J = \{f_0, f_1, \ldots, f_q\}, \qquad f_0 = {}_pF_q, \qquad f_n = z^n\frac{d^n f_0}{dz^n}. $$

$$ \frac{d}{dz}J = \left(\frac{\mathbf{A}}{z} + \frac{\mathbf{B}}{z-1}\right)J $$

$$ \mathbf{C} = -\mathbf{A} - \mathbf{B} $$

$$ \mathbf{A}:\quad \{0, 1-b_1, \ldots, 1-b_q\}, $$

$$ \mathbf{B}:\quad \Big\{\underbrace{0,\ldots,0}_{q}, -q - \sum_{i=1}^{p} a_i + \sum_{i=1}^{q} b_i\Big\}, $$

$$ \mathbf{C}:\quad \{a_1, a_2, \ldots, a_p\}. $$

DE solution and Lee algorithm

$$\frac{dJ}{dx} = \mathcal{M}(x,\varepsilon)J, \qquad J = T(x,\varepsilon)\tilde{J}, \qquad \frac{d\tilde{J}}{dx} = \tilde{\mathcal{M}}\tilde{J} = \left[T^{-1}\mathcal{M}T - T^{-1}\frac{d}{dx}T\right]\tilde{J}$$

$$\tilde{\mathcal{M}}(x,\varepsilon) = \varepsilon\sum_r \frac{\mathcal{M}_r}{x - x_r} = \varepsilon\bar{\mathcal{M}}(x)$$

$$U(x) = \mathrm{Pexp}\left[\varepsilon\int_{x_0}^{x}\bar{\mathcal{M}}(x')dx'\right] = \sum_n \varepsilon^n \int\limits_{x>x_n\cdots>x_1>x_0}\cdots\int dx_n\ldots dx_1\left[\bar{\mathcal{M}}(x_n)\ldots\bar{\mathcal{M}}(x_1)\right].$$

Balance transformation

$$\mathcal{B}(\mathcal{P},x_1,x_2;x) = \mathcal{I} - \mathcal{P} + \frac{x - x_2}{x - x_1}\mathcal{P}$$

J. M. Henn, Physical review letters, vol. 110, no. 25, p. 251601, 2013.

R. N. Lee, JHEP, vol. 04, p. 108, 2015.

Case	**A** eigenvalues	**B** eigenvalues	**C** eigenvalues	variable change
A	$m_a + q_a\varepsilon$	$m_b + q_b\varepsilon$	$m_c + q_c\varepsilon$	nun
B	$\frac{k_a}{n} + m_a + q_a\varepsilon$	$\frac{k_b}{n} + m_b + q_b\varepsilon$	$m_c + q_c\varepsilon$	$z \to \frac{z_1^n}{1+z_1^n}$
C	$\frac{k_a}{n} + m_a + q_a\varepsilon$	$m_b + q_b\varepsilon$	$\frac{k_c}{n} + m_c + q_c\varepsilon$	$z \to z_2^n$
D	$m_a + q_a\varepsilon$	$\frac{k_b}{n} + m_b + q_b\varepsilon$	$\frac{k_c}{n} + m_c + q_c\varepsilon$	$z \to 1 - z_3^n$
E	$\frac{1}{2} + m_a + q_a\varepsilon$	$\frac{1}{2} + m_b + q_b\varepsilon$	$\frac{1}{2} + m_c + q_c\varepsilon$	$z \to -\frac{4z_4^2}{(z_4^2-1)^2}$
F^*	$\frac{k_a}{n} + m_a + q_a\varepsilon$	$\frac{k_a}{n} + m_b + q_b\varepsilon$	$\frac{k_a}{n} + m_c + q_c\varepsilon$	$z \to \frac{z_1^n}{1+z_1^n}$

$$\mathrm{k} = \#\{\ \mathrm{a}_i | i = 1, \ldots, q; a_i \notin \mathbb{Z}\}, \qquad l = \#\{b_j | j = 1, \ldots, p; b_j \notin \mathbb{Z}\}$$

1. $n = 2$ and $|k - l| \geq 2$.

2. $n > 2$ and $(k \geq 2$ or $l \geq 2)$ and $\{a_i - a_1, b_j - a_1 | i = 1, q, j = 1, p\} \not\subset \mathbb{Z}$.

3. $n > 2$, $k = l = 1$ and $a_i - b_j \notin \mathbb{Z}$ where a_i and b_j are two non-integer indices.

A basis $J=\{f_0, f_1, \ldots, f_q\}$ with a DE $dJ/dx = MJ$ forms a vector bundle on the Riemannian sphere $\mathbb{C}\bigcup\infty$, M is the connection of the bundle.

According to the Birkgroff-Grothendieck theorem, any vector bundle on a Riemannian sphere can be represented as a sum of one-dimensional bundles, each of which characterized by an integer.
The entire bundle is characterized by a set of numbers
D = diag(d1,...,dp)(vector bundle splitting)

It can be shown that a system of differential equations can be reduced to epsilon form only if the splitting of the corresponding bundle is trivial

R. N. Lee and A. A. Pomeransky, arXiv:1707.07856 [hep-th]

$$ {}_2F_1\left(\begin{matrix} \frac{1}{2}+\alpha\varepsilon, -\frac{1}{2}+\beta\varepsilon \\ 1+\gamma\varepsilon \end{matrix} \middle| x\right) \qquad {}_3F_2\left(\begin{matrix} \alpha\varepsilon, \beta\varepsilon, \gamma\varepsilon \\ \frac{3}{2}+\psi\varepsilon, \frac{1}{2}+\omega\varepsilon \end{matrix} \middle| x\right) $$

$$ D=\mathrm{diag}(1,-1) \qquad D=\mathrm{diag}(1,0,-1) $$

Main steps of the algorithm

Determine a basis of p elements for the chosen hypergeometric functions (or with indices shifted by an integer) pFq and write down the corresponding DE system.

↓

Determine whether a suitable variable replacement exists and whether a given function can be expanded in term of MPLs

↓

Use the Lee algorithm to reduce the DE system to epsilon form

↓

Integrate the resulting system in terms of polylogarithms

↓

If necessary, apply a sequence of differential operators to restore the original function

Example

$$
{}_3F_2\left(\begin{matrix}1,\frac{\varepsilon+1}{2},\frac{\varepsilon}{2}\\ \frac{1-\varepsilon}{2},\frac{\varepsilon+3}{2}\end{matrix}\middle| z\right)=1+\varepsilon\left(-\frac{\left(z_4^2+1\right)G(-1,z_4)}{2z_4}+\frac{\left(z_4^2+1\right)G(1,z_4)}{2z_4}+1\right)
$$

$$
+\varepsilon^2\left(\left(2z_4+\frac{2}{z_4}\right)G\left(f_4^1,-1,z_4\right)-\frac{2\left(z_4^2+1\right)G\left(f_4^1,1,z_4\right)}{z_4}+\ldots\right)+\mathcal{O}(\varepsilon^3).
$$

$$
{}_3F_2\left(\begin{matrix}1,\frac{\varepsilon+1}{2},\frac{\varepsilon+2}{2}\\ \frac{1-\varepsilon}{2},\frac{\varepsilon+3}{2}\end{matrix}\middle| z\right)=\left(\frac{2z}{\varepsilon}\frac{d}{dz}+1\right)\ {}_3F_2\left(\begin{matrix}1,\frac{\varepsilon+1}{2},\frac{\varepsilon}{2}\\ \frac{1-\varepsilon}{2},\frac{\varepsilon+3}{2}\end{matrix}\middle| z\right)=
$$

$$
\frac{\left(z_4^2-1\right)^2\left(G\left(-1,z_4\right)-G\left(1,z_4\right)\right)}{2\left(z_4^3+z_4\right)}+\frac{\varepsilon\left(z_4^2-1\right)}{2\left(z_4^3+z_4\right)}\Big(-4G\left(f_4^1,-1,z_4\right)+
$$

$$
+4G\left(f_4^1,1,z_4\right)+G\left(-1,z_4\right)-G\left(1,z_4\right)+G\left(-1,-1,z_4\right)+\ldots\Big)+\mathcal{O}(\varepsilon^2).
$$

Appell functions

$$F_1(\alpha, \beta_1, \beta_2, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n} m! n!} x^m y^n, \qquad |x| < 1,\ |y| < 1$$

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + y(1-x)\frac{\partial^2}{\partial x \partial y} + [\gamma - (\alpha + \beta_1 + 1)x]\frac{\partial}{\partial x} - \beta_1 y \frac{\partial}{\partial y} - \alpha\beta_1\right] F_1 = 0,$$

$$\left[y(1-y)\frac{\partial^2}{\partial y^2} + x(1-y)\frac{\partial^2}{\partial x \partial y} + [\gamma - (\alpha + \beta_2 + 1)y]\frac{\partial}{\partial y} - \beta_2 x \frac{\partial}{\partial x} - \alpha\beta_2\right] F_1 = 0.$$

$$J_1 = \left\{F_1, x\frac{\partial}{\partial x}F_1, y\frac{\partial}{\partial y}F_1\right\}, \qquad \frac{\partial}{\partial x}J_1 = \left(\frac{\mathbf{A}_0}{x} + \frac{\mathbf{A}_1}{x-1} + \frac{\mathbf{A}_y}{x-y}\right) J_1$$

Lauricella functions

$$F_A^{(n)}(\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\ldots(\gamma_n)_{m_n}m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n},$$

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\ldots(\alpha_n)_{m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n},$$

$$F_D^{(n)}(\alpha;\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n}.$$

$$\left\{\theta_{x_{j_1}}\ldots\theta_{x_{j_k}}F_i \;\middle|\; 0\le k\le n,\ j_1<j_2<\cdots<j_k\right\}, \qquad i=A,B, \qquad \theta_x = x\frac{d}{dx}$$

$$\left\{F_D,\ \theta_{x_j}F_D \;\middle|\; j=1,\ldots,n\right\},$$

Conclusions

In this work we study hypergeometric functions and and their expansions in terms of polylogarithms.

Of particular importance here are non-trivial variable changes

All algorithms described here for calculating the expansion of single and multiple hypergeometric series were implemented in the **Diogenes** package (Coming soon) written in Wolfram Mathematica language.

Thank you for your attention!