# *Dispersion relations in Kerr-AdS/CFT holography*

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1. Motivation
   - Closed strings
   - Open strings
   - Branes and their sources
   - AdS/CFT correspondence

2. Elements of the dictionary between dual theories
   - Anomalous dimensions and dispersion relations
   - The Pulsating string in $AdS_5 \times S^5$

3. A new approach to the problem of dispersion relations
   - Classical pulsating string solutions in the 5d Kerr-$AdS$ background
   - Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

4. Concluding

# Closed string action

⊙ Polyakov string action (string sigma-model form):

$$S_P = S_G + S_B + S_\Phi$$

• The $S_G$, $S_B$, $S_\Phi$ actions: the most general sigma model string actions preserving the symmetries of the theory and renormalizability:

- $G$-coupling

$$S_G = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{g}\,g^{\alpha\beta}\partial_\alpha X^\mu(\tau,\sigma)\partial_\beta X^\nu(\tau,\sigma)G_{\mu\nu}(X^\mu)$$

- $B$-coupling

$$S_B = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\varepsilon^{\alpha\beta}\partial_\alpha X^\mu(\tau,\sigma)\partial_\beta X^\nu(\tau,\sigma)B_{\mu\nu}(X^\mu)$$

- $\Phi$-coupling

$$S_\Phi = -\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}\,\Phi(X^\mu)R^{(2)}(\tau,\sigma).$$

• Low-energy limit. Requiring conformal and Weyl symmetries of the $\sigma$-model string action, one ends up with the conditions ensuring the vanishing of the corresponding $\beta$-functions:

$$\beta^G_{\mu\nu}: \quad R_{\mu\nu} - \frac{1}{4} H_{\mu\sigma\lambda} H^{\sigma\lambda}_{\nu} + 3 D_\mu \partial_\nu \Phi = 0,$$

$$\beta^B_{\mu\nu}: \quad -\frac{1}{2} D^\sigma H_{\sigma\mu\nu} + H_{\sigma\mu\nu} D^\sigma \Phi = 0,$$

$$\beta^\Phi: \quad \frac{1}{6}\left[d - 10\right] - \frac{\alpha'}{2}\left[D^2\Phi - 2(\nabla\Phi)^2 - \frac{1}{12}H^2\right] = 0.$$

**The effective 10d action from closed strings: SUPERGRAVITY**

$$S = \frac{1}{2\kappa}\int d^{10}X\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H^2\right)$$

$\surd$ should be understood as a universal one, i.e. any superstring background must satisfy the above equations

$\surd$ the equations of motion following from this action coincide with the conditions ensuring vanishing of the $\beta$-functions.

# Open strings

- adding open string sector
  In the case of boundaries of the world sheet one can write the action as

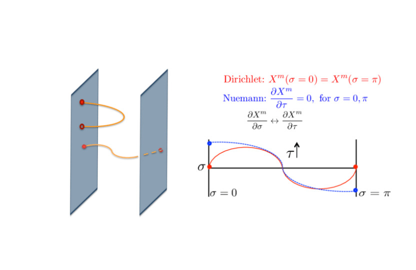
$$S = \frac{1}{1\pi\alpha'}\left(\int_\Sigma d^2\sigma \frac{1}{2}(\partial_\alpha X_\mu \partial^\alpha X^\mu + \varepsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu) + \int_{\partial\Sigma} d\sigma^1 A_\mu(X^\nu)\partial_1 X^\mu\right)$$

- expanding the above action we get model dependent terms

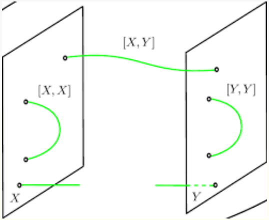
$$S^{II}_{open} = -\frac{1}{2\kappa^2}\int d^{10}x \sum_p \frac{1}{2(p+2)!}F^2_{p+2},$$

where $F_{p+2}$ is the field strength of a $p+1$ form gauge field. The couplings $A_\mu$ get promoted to gauge fields on the subspace where string endpoints live.

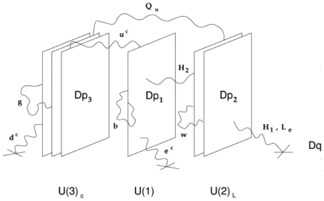
# Brane degrees of freedom and effective spacetime geometry




Open strings ending on D-branes.



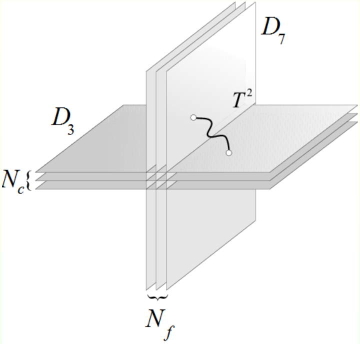

Degrees of freedom of Open strings.

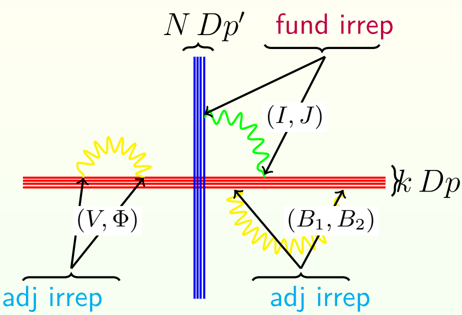



A stack of multiple coincident D-branes possess non-commutative gauge degrees of freedom.

# How gauge degrees of freedom appears?

- The string endpoints on the same $D_p$ branes transform under adjoint representation of the gauge group. The large number of these branes gives the background geometry.
- The string endpoints ending on different $D_p - D_{p'}$ branes transform in the fundamental. This is the way we introduce flavors in the theory.



# Four-dimensional world: D3 branes

• The example of D3-branes.
The solution for the $D3$ branes can be obtained from solving the combined system

$$S = \frac{1}{2\kappa^2}\int_M d^{10}\mathcal{L}_{(10)} + \int_{V_{D3}} d^4\mathcal{L}_{D3}. \tag{1}$$

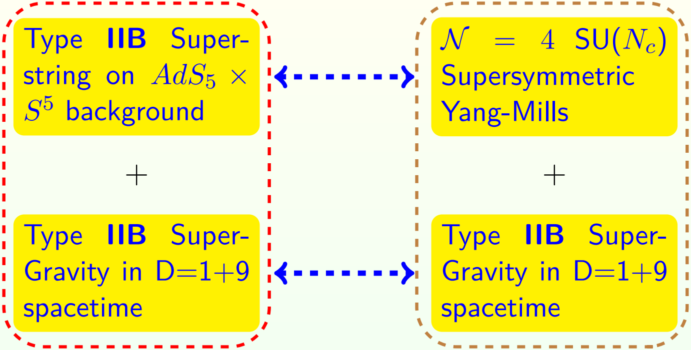
The explicit solution for embedded $D3$-brane:

$$D3\text{-brane} \Rightarrow \begin{cases} ds^2 = H^{-\frac{1}{2}}dx^2_{(4)} + H^{\frac{1}{2}}\left(dy^2 + y^2 d\Omega^2_{(5)}\right) \\ H(y) = 1 + \left(\dfrac{R}{y}\right)^4, \\ F_{(5)} = dx^4 \wedge dH^{-1} + \star d^4x \wedge dH^{-1}, \\ e^{\Phi} = g_s, \quad R^4 = 4\pi g_s N_c (\alpha')^2. \end{cases} \tag{2}$$

• For small $y$ the elementary brane solution has a warp factor which in the near-horizon limit determines the geometry of the space-time.

# AdS/CFT correspondence

- In the near-horizon limit, $y/R \to 0$, the theory in the bulk and that on the stack of D3 brane decouple.



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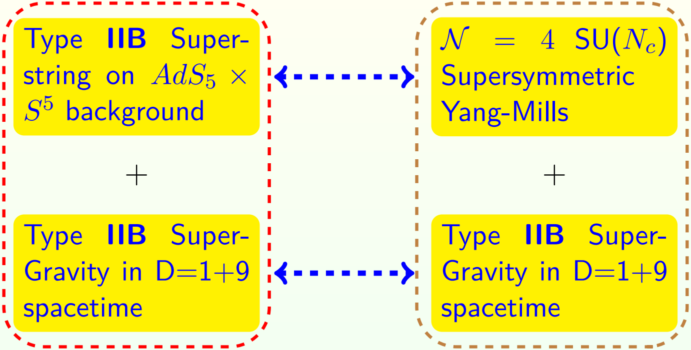
- The geometry becomes that of $AdS_5 \times S^5$



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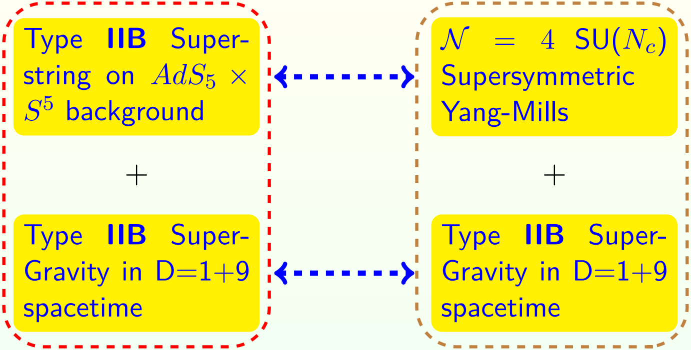
$\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in $3+1$ dimensions is the same as (or dual to) type IIB superstring theory on $AdS_5 \times S^5$ spacetime.



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| | | |
|---|---|---|
| Type **IIB** Superstring on $AdS_5 \times S^5$ background | ↔ | $\mathcal{N} = 4$ SU($N_c$) Supersymmetric Yang-Mills |
| + | | + |
| Type **IIB** Super-Gravity in D=1+9 spacetime | ↔ | Type **IIB** Super-Gravity in D=1+9 spacetime |

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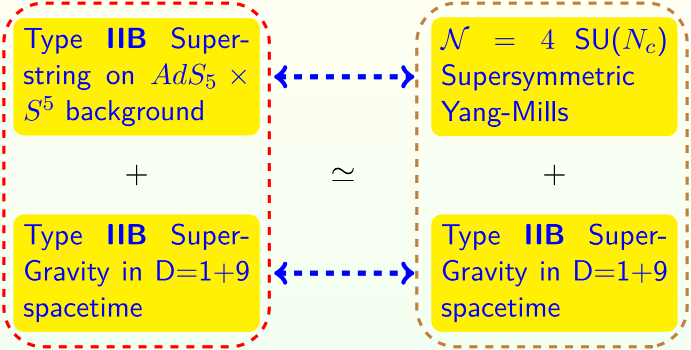

## How does the correspondence work?

- Consider a spacetime $M$ supplied with a boundary $\partial M = \Sigma$; bulk fields$\Phi_i$; boundary fields $\phi_k$.
- The metric of $M$ is $g_{\mu\nu}$ while the induced metric on the boundary $\partial M$ is $g_{|\partial M} = \gamma$.
- Consider a radial slice of the spacetime $\Sigma_\rho$ at fixed $\rho$. The fields on this slice will be also denoted by $\phi_k$.
- At given slice $\Sigma_\rho$ one can consider the amplitude

$$\Psi_{\Sigma_\rho}[\phi_i] = \int_{\Phi_i|\Sigma_\rho = \phi_i} D\Phi_i e^{iS_{B,M}(\Phi_i)}, \qquad (3)$$

  where $S_{B,M}$ is the bulk action and the integral is evaluated with Dirichlet boundary conditions for the fields on $\Sigma_\rho$.
- For any given CFT one can define the generating functional of connected correlation functions

$$Z_{CFT}[\phi_i] = \langle e^{\int_\Sigma \phi_i \mathcal{O}_i} \rangle. \qquad (4)$$

### AdS/CFT correspondence

Thinking of $\Psi_{\Sigma_\rho}[\phi_i] = e^{iS(\phi_i)}$ as *on-shell* amplitude defining connected S-matrix elements, *AdS/CFT correspondence* states the equality of

$$\boxed{Z_{CFT}[\phi_i] = \Psi_{\Sigma_0}[\phi_i]}, \qquad (5)$$

where $\Sigma_0$ is the (asymptotic) boundary of the spacetime.

⊙ In what sense holographic correspondence?

- In principle $\Psi_{\Sigma_\rho}$ *represents the quantum spacetime* but only trough the dependence on the boundary metric!
- Changing the *radial slice* changes the induced metric on $\Sigma_\rho$! Thus, knowing $\Sigma_\rho$ for all $\rho$ (i.e. all possible $\gamma$) allows to reconstruct the semi-classical spacetime!
- On the other hand, assuming the correspondence, the variation of the boundary means moving the radial slice in the bulk!

# Anomalous dimensions

⊙ On any CFT$_d$ , there is a mapping between operators $\mathcal{O}_a(x)$ and states on a Hilbert space of he theory on $R \times S^{d-1}$,

$$\mathcal{O}_a(x) \leftrightarrow |\mathcal{O}_a\rangle_{S^{d-1}}.$$

⊙ The eigenvalue of the translation generator along the holography direction is $\Delta_a =$ scaling dimension of $\mathcal{O}_a$ such that,

$$\mathcal{O}_a(\lambda x) = \lambda^{-\Delta_a}\mathcal{O}_a(x) \quad \Longrightarrow \quad e^{\tau \mathcal{H}_\tau}|\mathcal{O}_a\rangle = e^{-i\Delta_a}|\mathcal{O}_a\rangle,$$

where $\mathcal{H}_\tau$ is the Hamiltonian corresponding to the dilatation operator in radial quantization.

**Anomalous dimensions**

$$\boxed{\Delta_a\,(\mathrm{inCFT}) = E_a\,(\mathrm{globalAdS}).}$$

The *anoumalous dimensions* are determined by the *dispersion relations* from string side!

# A simple pulsating string in $AdS_5 \times S^5$

**The set up:** we consider a circular string which pulsates expanding and contracting on $S^5$ part of $AdS_5 \times S^5$ ($R^2 = 2\pi\alpha'\sqrt{\lambda}$)

$$ds^2 = R^2 \left( \cos^2\theta d\Omega_3^2 + d\theta^2 + \sin^2\theta d\psi^2 + d\rho^2 - \cosh^2 dt^2 \right).$$

- The ansatz $\psi = m\sigma$ (string stretched along $\psi$ direction), $\theta = \theta(\tau)$, $\rho = \rho(\tau)$ and $g_{ij}$ the metric of $S^3$.
- The reduced Nambu-Goto action in this case is

$$S = -m\sqrt{\lambda} \int dt \sin\theta \sqrt{1 - \dot{\theta}^2 - \cos^2\theta g_{ij}\dot{\phi}^i\dot{\phi}^j},$$

- The Hamiltonian has the form

$$H = \sqrt{\Pi_\theta^2 + \frac{g^{ij}\Pi_i\Pi_j}{\cos^2\theta} + \qquad\qquad}.$$

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# A simple pulsating string in $AdS_5 \times S^5$

- The wave function for the unperturbed theory ($w = \cos^2\theta$)

$$-\frac{4}{w}\frac{d}{dw}w^2(1-w)\frac{d}{dw}\Psi(w) + \frac{J(J+2)}{w}\Psi(w) = L(L+4)\Psi(w).$$

- $\phi_i$ - cyclic, $g^{ij}\Pi_i\Pi_j \to J(J+2)$

-$L$ is eigenvalue of the $S^5$ angular momentum

- The solution to the above equation is ($\ell = L/2,\ j = J/2$)

$$\Psi(w) = \frac{\sqrt{2(\ell+1)}}{(\ell-j)!}\frac{1}{w^{j+1}}\left(\frac{d}{dw}\right)^{\ell-j} w^{\ell+j}(1-w)^{\ell-j},$$

- To first order in $\lambda$ the energy is given by

$$E^2 = L(L+4) + m^2\lambda\frac{L^2-J^2}{2L^2}.$$

- To first order in $\lambda$ the anomalous dimension of the corresponding YM operators (in notations of hep-th/0310188)

$$\gamma = \frac{m^2\lambda}{4L}\alpha(2-\alpha), \qquad \alpha = 1 - J/L.$$

# Classical pulsating string solutions in the 5d Kerr-$AdS$ background

This section is based on our joint work with A. A. Golubtsova, D. D. Hristo, O. V. Geytota and R.R. Rashkov in Journal General Relativity and Gravitation, [arXiv:2108.12621]

- For the pure AdS case the energy of the string can be related to the anomalous dimensions of single trace operators in $\mathcal{N} = 4$ SYM theory. In the black hole case at finite temperature we cannot establish this connection, since the notion of the anonymous dimension is defined in the conformal point.
- The interpretation of results from holographic point of view is not straightforward since the dual theory is at finite temperature. Nevertheless, near or at conformal point the expressions can be thought of as the dispersion relations of stationary states.
- One can think on the relevance of the dispersion relations of the states in the thermal ensemble of $\mathcal{N} = 4$ SYM theory on $\mathbb{S}^1 \times \mathbb{S}^3$.

# Classical pulsating string solutions in the 5d Kerr-$AdS$ background

$$
\begin{aligned}
ds^2 &= -(1+y^2\ell^2)dT^2 + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) && (6)\\
&+ \frac{2M}{y^2\Xi^3}(dT - a\sin^2\Theta d\Phi - a\cos^2\Theta d\Psi)^2 && (7)\\
&+ \frac{y^4 dy^2}{y^4(1+y^2\ell^2) - \frac{2M}{\Xi^2}y^2 + \frac{2Ma^2}{\Xi^3}}, && (8)
\end{aligned}
$$

where

$$\Xi = 1 - a^2\ell^2, \tag{9}$$

$M$ is the mass of the black hole, $a$ is a rotational parameter and we use the Hopf coordinates to parametrize the metric on the sphere with $0 \leq \Theta \leq \frac{\pi}{2}$, $0 \leq \Phi, \Psi \leq 2\pi$.

# A pulsating string in the 5d Kerr-AdS black hole

We can consider general pulsating string ansatz, such as

$$\Theta \equiv \xi_1 = \xi_1(\tau)\,,\, y \equiv \xi_2 = \xi_2(\tau)\,, \tag{10}$$

$$T \equiv X_0 = x_0(\tau) + m_0\sigma, \qquad x_0(\tau) = \kappa\tau, \qquad m_0 = 0\,, \tag{11}$$

$$\Phi \equiv X_1 = m_1\sigma + x_1(\tau)\,, \quad \Psi \equiv X_2 = m_2\sigma + x_2(\tau)\,. \tag{12}$$

For convenience, we use the following notations of the Kerr-AdS metric

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} d\xi_i d\xi_j + \sum_{k,p=0}^{2} \hat{G}_{kp} dX_k dX_p\,, \tag{13}$$

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

The starting point is the Polyakov string action in the conformal gauge written as follows

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\{\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}\},$$

The Virasoro constraints are given by

$$\text{Vir1:}\quad \sum_{M,N} G_{MN}\left(\partial_\tau X^M\partial_\tau X^N + \partial_\sigma X^M\partial_\sigma X^N\right) = 0,$$

$$\text{Vir2:}\quad \sum_{M,N} G_{MN}\partial_\tau X^M\partial_\sigma X^N = 0.$$

The ansatz for the pulsating string configuration, involving the $y$-direction, which is consistent with the equations of motion is

$$T = \kappa\,\tau, \qquad y = y(\tau), \qquad \Theta = \Theta^* = const,$$

$$\Phi = m_\phi\sigma, \qquad \Psi = m_\psi\sigma\,.$$

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

From Virasoro constraints we have an equation

$$\dot{y}^2 = \left(\kappa^2\left(1+\ell^2y^2-\frac{2M}{\Xi^3y^2}\right)-y^2K^2\right)\left(1+\ell^2y^2-\frac{2M}{\Xi^2y^2}+\frac{2Ma^2}{\Xi^3y^4}\right),$$

or

$$\dot{y}^2 = \kappa^2\left(\ell^2-\frac{K^2}{\kappa^2}\right)\frac{\ell^2}{y^6}(y-y_-)(y-y_+)\prod_{j=1}^{4}(y-y_j)(y-y_j^*),$$

where $y_-$, $y_+$ are real zeros of blackening function, thus $y_+$ is the horizon, $y_1^*, y_2^*, y_3^*, y_4^*$ are complex zeros of the blackening function while $y_1$, $y_2$, $y_3$ and $y_4$ are zeros of the first multiplier of equation for $\dot{y}^2$ following from Virasoro constraints. So we can always find appropriate conditions on the right-hand side for the existence of a periodic solution

$$y_- < 0 < y_+ < y_2 < y(\tau) < y_1,$$

Therefore, there exists a pulsating string configuration, expanding and contracting between the turning points $y_1$ and $y_2$.

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

In Fig. below we plot the potential of the effective mechanical system. We are able to tune parameters $M$, a,$K$ and $\kappa$ in such a way that there will be two real positive roots $y_1$ and $y_2$. One can see that it has three positive real zeroes $y_+$, $y_1$, $y_2$. The evolution of the pulsating string is defined in the region $y_1 < y(\tau) < y_2$ .

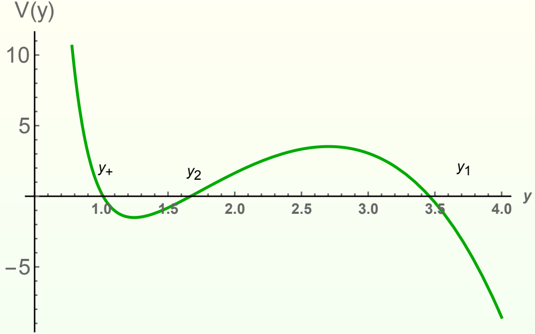


Figure: The behaviour of the effective potential of the system. The parameters

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

The Nambu-Goto action becomes

$$S_{NG} = -\frac{1}{\alpha'}\int d\tau \, \frac{\sqrt{\left(\sum_{k,p=1}^{2}\hat{G}_{kp}\, m_k \dot{x}_p + \sum_{k=1}^{2}\hat{G}_{k0}\, m_k\, \kappa\right)^2 +}}{-\|\vec{m}\|^2 \left(\sum_{i,j=1}^{2} g_{ij}\,\dot{\xi}_i\dot{\xi}_j + \sum_{k,p=1}^{2}\hat{g}_{kp}\,\dot{x}_k\dot{x}_p + \hat{G}_{00}\,\kappa^2 + 2\sum_{k=1}^{2}\hat{G}_{0k}\,\kappa\dot{x}_k\right)}\,, \tag{14}$$

where $1/\alpha' = \sqrt{\lambda}$ is the 't Hooft coupling constant and

$$\|\vec{m}\|^2 = \sum_{k,h=0}^{2}\hat{G}_{kh} m_k m_h \equiv \sum_{k,h=1}^{2}\hat{g}_{kh}(\xi_1,\xi_2) m_k m_h \, > \, 0\,. \tag{15}$$

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

The problem reduces again to the dynamics of an effective point-particle with Lagrangian

$$L_{eff} = -\sqrt{\lambda}\,\sqrt{\ldots\ldots}\,. \tag{16}$$

where

$$\sqrt{\ldots\ldots} \equiv \sqrt{\left(\sum_{k,p=1}^{2} \hat{g}_{kp}\, m_k \dot{x}_p + \sum_{k=1}^{2} \hat{G}_{k0}\, m_k\, \kappa\right)^2 - \|\vec{m}\|^2 \left(\sum_{i,j=1}^{2} g_{ij}\, \dot{\xi}_i \dot{\xi}_j + \right.}$$
$$\overline{\left. + \sum_{k,p=1}^{2} \hat{g}_{kp}\, \dot{x}_k \dot{x}_p + \hat{G}_{00}\, \kappa^2 + 2 \sum_{k=1}^{2} \hat{G}_{0k}\, \kappa \dot{x}_k \right)}\;. \tag{17}$$

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

Consider the Hamiltonian formulation → the canonical momenta

$$\Pi_{\xi_i} = \sqrt{\lambda}\,\frac{\|\vec{m}\|^2\, g_{ij}\,\dot{\xi}_j}{\sqrt{\ldots\ldots}}\,, \tag{18}$$

$$\Pi_{x_p} = \sqrt{\lambda}\,\frac{\|\vec{m}\|^2\,\hat{g}_{pq}\,\dot{x}_q - (\hat{g}_{kq}\,m_k\dot{x}_q)\,\hat{g}_{pq}\,m_q + \|\vec{m}\|^2\,\hat{G}_{p0}\,\kappa - (\hat{G}_{k0}\,m_k\kappa)\,\hat{g}_{pq}\,m}{\sqrt{\ldots\ldots}} \tag{19}$$

which also implies the constraint

$$\sum_{p=1}^{2} m_p\,\Pi_{x_p} = 0. \tag{20}$$

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

By Legendre transformation → (square of) the pulsating string Hamiltonian

$$\frac{H^2}{\kappa^2} = K^2(y) \left\{ \sum_{i,j=1}^{2} \Pi_{\xi_i}\, g^{ij}\, \Pi_{\xi_j} + \sum_{i,j=1}^{2} \Pi_{x_i} \left[ \frac{h_2^2(y)}{K^2(y)}\, \hat{\delta}^{ij} + \hat{g}^{ij} \right] \Pi_{x_j} + \lambda\, \|\vec{m}\|^2 \right. \tag{21}$$

where

$$\left(\hat{\delta}^{ij}\right) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ i, j = 1, 2\,. \tag{22}$$

The above Hamiltonian can be considered as the effective Hamiltonian of an effective point-particle on the Kerr-AdS.
⇒ The effective potential is:

$$-\lambda\, \kappa^2\, \|\vec{m}\|^2 \left( \hat{G}_{00} + \sum_{l,s=1}^{2} \hat{G}_{0l}\, \hat{g}^{ls}\, \hat{G}_{s0} \right) \equiv \lambda\, \kappa^2\, \|\vec{m}\|^2\, K^2(y) \equiv \lambda\, U(\Theta,\, y)\,. \tag{23}$$

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

The square of the Hamiltonian on the reduced subspace $y = const$ is:

$$H^2 = \kappa^2 K^2 \left\{ \Pi_\Theta \, g^{\Theta\Theta} \, \Pi_\Theta + \sum_{i,j=1}^{2} \Pi_{x_i} \, \hat{g}^{ij} \, \Pi_{x_j} + \lambda \, \|\vec{m}\|^2 \right\} . \qquad (24)$$

The effective potential is:

$$\lambda \, \kappa^2 \, K^2 \, \|\vec{m}\|^2 \equiv \lambda \, U(\Theta) \qquad (25)$$

The potential $\ll$ the kinetic part
$\Rightarrow$ calculate perturbatively quantum corrections to the energy

# Semi-classical quantisation of the pulsating string configuration in 5d Kerr-$AdS$ geometry

The kinetic term of the Hamiltonian (24) can be considered as a three dimensional Laplace-Beltrami operator of the Kerr-AdS subspace with $y = const$

$$\vec{P}^2 = \left\{ \Pi_\Theta \, g^{\Theta\Theta} \, \Pi_\Theta + \sum_{i,j=1}^{2} \Pi_{x_i} \, \hat{g}^{ij} \, \Pi_{x_j} \right\} \longrightarrow \triangle^{(3)}_{Kerr-AdS} \,, \tag{26}$$

which defines the eigen-functions of the Hamiltonian, satisfying the following Schrödinger equation

$$\triangle^{(3)}_{Kerr-AdS} \; F = - \frac{E^2}{\kappa^2 \, K^2} \, F \,. \tag{27}$$

It is convenient to define a new variable $z = \sin^2 \Theta \,, \; 0 \leq z \leq 1$. Then the equation can be written as

$$\left\{ \frac{d^2}{dz^2} + \frac{(1-2z)}{z(1-z)} \frac{d}{dz} - \frac{N^2}{z^2(1-z)^2} + \frac{\hat{E}^2}{z(1-z)} \right\} F(z) = 0 \,, \tag{28}$$

where $N = \frac{n^2}{\dots}$ and $\hat{E}^2 = \frac{y^2 \, E^2}{\dots}$ …

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In addition, we have to ensure that the solutions $F(\Theta)$ are square integrable with respect to the measure $\Theta$ (respectively $z$). The integrability condition leads to the following restriction on the parameters

$$2N + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\hat{E}^2} = -k\,, \qquad k \in \mathbb{N}\,. \tag{29}$$

This requirement imposes energy quantization:

$$E^2 = \kappa^2 \frac{K^2}{4y^2}\left[(4N + 1 + k)^2 - 1\right]. \tag{30}$$

The condition (29) converts the solution in terms of Jacobi ortogonal polynomials (in this case the solution of the equation can also be written directly in terms of Shifted Legendre polynomials)

$$F(z) = C\, z^{\alpha/2}\,(1 - z)^{\beta/2}\,\frac{k!\,\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)} P_k^{(\alpha,\beta)}(1 - 2z)\,, \qquad k \in \mathbb{N}\,, \tag{31}$$

where $\alpha = \beta = 2N \equiv \frac{n^2}{2}\,,\; n \in \mathbb{Z}\,.$

# Wave functions

It is more convenient to work in terms of $u \equiv 1 - 2z\,,\ -1 \leq u \leq 1$

$$F_{k,n}(u) = C\left(\frac{1-u}{2}\right)^{\alpha/2}\left(\frac{1+u}{2}\right)^{\alpha/2}\frac{k!\,\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)}P_k^{(\alpha,\beta)}(u) \qquad (32)$$

Then with respect to the measure

$$d\Omega = \sqrt{-\det G^{(4)}}\,d\Theta\,d\Phi\,d\Psi = -\sqrt{\frac{h_1\,h_2}{2aM\Xi^3\,y^2}}\,\frac{du}{4}\,d\Phi\,d\Psi, \qquad (33)$$

we find that the normalized wave function is

$$f_{k,n}(u) = \sqrt{\frac{(2\alpha+1+2k)\,k!\,\Gamma(2\alpha+1+k)}{2^{\alpha-1}\,\Gamma(\alpha+1+k)\,\Gamma(\alpha+1+k)}}\,(1-u)^{\alpha/2}\,(1+u)^{\alpha/2}P_k^{(\alpha,\alpha)}(u)\,. \qquad (34)$$

Finally, the total free wave functions have the form

$$f_{k,n}^{tot}(u,\,\Phi,\,\Psi) = \sqrt{\frac{(2\alpha+1+2k)\,k!\,\Gamma(2\alpha+1+k)}{\omega(y)\,2^{\alpha-1}\,\Gamma(\alpha+1+k)\,\Gamma(\alpha+1+k)}}\times$$

$$\times\ (1-u)^{\alpha/2}\,(1+u)^{\alpha/2}P_k^{(\alpha,\alpha)}(u)\,e^{in\Phi}\,e^{-in\Psi}\,. \qquad (35)$$

# Leading correction to the energy

The next step is to calculate perturbatively the corrections to the energy of the free ground states.
Perurbatively, the first correction to the energy reads

$$\delta E^2 = \lambda \langle f_{k,n}^{tot} \,|\, U \,|\, f_{k,n}^{tot} \rangle = \lambda \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2\pi} \left| f_{k,n}^{tot}(u,\, \Phi,\, \Psi) \right|^2 U(u,\, \Phi,\, \Psi) d\Omega(u, \Phi, \Psi) \tag{36}$$

The form of the potential :

$$U(\Theta) = \kappa^2 K^2 \|\vec{m}\|^2 = \kappa^2 K^2 \sum_{k,h=1}^{2} \hat{g}_{kh}(y,\, \Theta) m_k m_h \,, \tag{37}$$

where

$$\|\vec{m}\|^2 = y^2 m_1^2 \sin^2\Theta + y^2 m_2^2 \cos^2\Theta + \frac{2a^2 M}{y^2 \Xi^3} \left( m_1 \sin^2\Theta + m_2 \cos^2\Theta \right)^2 . \tag{38}$$

# Leading correction to the energy

Using the scalar product (36) and some properties of orthogonal Jacobi polynomials, one can compute the first correction to the energy

$$\begin{aligned}\delta E^2 = {} & \lambda\,\kappa^2\,K^2\;2^{\alpha-1}\;\Big\{y^2\,(m_1^2+m_2^2)+\\ & +\frac{a^2M}{y^2\Xi^3}\,(m_1^2+m_2^2)\;\Big[\frac{(k+2\alpha+1)(k+2\alpha+2)(k+\alpha+2)}{(2k+2\alpha+1)(k+\alpha+1)(2k+2\alpha+3)}+\\ & +\frac{(k+\alpha-1)(k-1)k}{(2k+2\alpha+1)(2k+2\alpha-1)(k+\alpha)}+\frac{(k+2\alpha+1)k}{(k+\alpha+1)(k+\alpha)}\Big]+\\ & +\frac{2a^2M}{y^2\Xi^3}\,m_1m_2\;\Big[\frac{(k+2\alpha+1)(k+2\alpha+2)}{(2k+2\alpha+1)(2k+2\alpha+3)}+\\ & +\frac{2k(k+2\alpha+1)}{(2k+2\alpha+1)^2}+\frac{k(k-1)}{(2k+2\alpha+1)(2k+2\alpha-1)}\Big]\Big\}\,. \qquad (39)\end{aligned}$$

# Leading correction to the energy

On other hand we need the limit of large values for the energy $E$ and tne quantum number $n$, which corresponds to the Killing directions. Since $\alpha = \beta = 2N \equiv \frac{n^2}{2}$, $n \in \mathbb{Z}$, $\hat{E}^2 = \frac{y^2 E^2}{\kappa^2 K^2}$ and taking into account the requirement (29) one has asymptotic behavior $k \sim \hat{E} - \alpha$ . Therefore, this gives the following approximation of the correction to the energy

$$\delta E^2 = \lambda \kappa^2 K^2 2^{\alpha - 1} \left\{ y^2 (m_1^2 + m_2^2) + \frac{3}{2} \frac{a^2 M}{y^2 \Xi^3} (m_1^2 + m_2^2) + 2 \frac{a^2 M}{y^2 \Xi^3} m_1 m_2 \right.$$
$$\left. - \frac{1}{2} \frac{a^2 M}{y^2 \Xi^3} (m_1^2 + m_2^2) \left( \frac{\alpha}{\hat{E}} \right)^2 \right\} . \quad (40)$$

# Summary and comments

- The semiclassical quantization of pulsating string dynamics gives us a powerful method for obtaining dispersion relations in different backgrounds with a compact subspace.
- We probe the five-dimensional Kerr-AdS space time by pulsating strings.First we find particular pulsating string solutions and then semi-classically quantize the theory.
- We obtain the wave function of the problem and thoroughly study the corrections to the energy, which according to the holographic dictionary are related to anomalous dimensions of certain operators in the dual gauge theory. The interpretation of results from holographic point of view is not straightforward since the dual theory is at finite temperature. Nevertheless, near or at conformal point the expressions can be thought of as the dispersion relations of stationary states.

THANK YOU