Radiative corrections to muon decay spectrum

The XXVII International Scientific Conference of Young Scientists and Specialists

U.E. Voznaya, A.B. Arbuzov

BLTP JINR, State University "Dubna"

October 31, 2023

Outline

1. Motivation
2. Radiative corrections in QED
3. PDF approach
4. Results

Why muon?

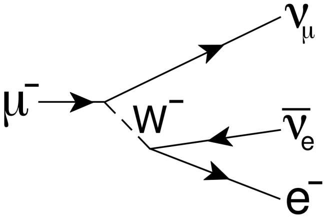
- Muon processes are very clean
- Experiments with high precision and high sensitivity are possible
- Small discrepancies from the SM can be detected
- Study of weak interaction
- Study of neutrino properties

μ decay, $\mu^+\mu^-$ annihilation, muonium atoms, μ scattering...
Processes with τ are decribed with the same diagrams due to lepton universality

Experiments

- TWIST (TRIUMF Weak Interaction Symmetry Test) – weak interaction symmetry test
- Fermilab Muon g-2 – magnetic moment of muon
- MUonE – cross-section of muon on the electron scattering
- Mu2E – μ decay with e radiation, new physics - decay without ν – 2026
- Mu3e – μ decay to 3 electrons (search for decays forbidden by SM)
- MEG, MEG II - lepton universality violation $\mu^+ \to e^+\gamma$

Muon decay



- Muon decay - pure weak process
- Nature of neutrino - Dirac or Majorana?
- Michel parameters determined with high precision

Michel parameters

ρ, δ, ξ, η – describe angular and energy distribution of the particles

Muon decay

Born cross-section:

$$f_0(x) = 6x\left(1+\frac{m_e^2}{m_\mu^2}\right)^4\sqrt{1-\frac{m_e^2}{E_e^2}}\Big[x(1-x)+\frac{2}{9}\rho(4x^2-3x-x_0^2)$$
$$+\eta x_0(1-x)\Big]$$

$$g_0 = -2x^2\xi\left(1+\frac{m_e^2}{m_\mu^2}\right)^4\left(1-\frac{m_e^2}{E_e^2}\right)\left[1-x+\frac{2}{3}\delta\left(4x-3-\frac{m_e^2}{m_\mu^2 x_0}\right)\right]$$

Radiative corrections in QED

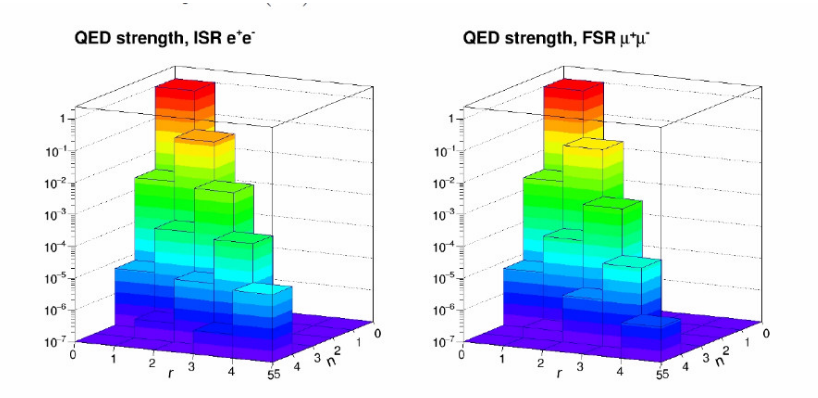


Fig. from Jadach, Skrzypek, arXiv: 1903.09895

Radiative corrections to muon decay

- Corrections to the muon lifetime are known $\mathcal{O}(\alpha^2)$ from 1950-s (Behrends, Finkelstein, Sirlin(1956); Berman, Sirlin (1962); Kinoshita, Sirlin, (1959); Berman, (1958));
- Results of the lowest order radiative corrections to the muon weak decay width received by Berman in 1958 reproduced in parton language and generalized to all orders of perturbation theory by Bartos (Bartos(2008));
- In the work of Arbuzov and Melnikov (2002) radiative corrections to unpolarized muon decay spectrum to the order $\alpha^2 L$, and in (Arbuzov, (2002)) radiative corrections to the polarized muon decay spectrum in LO to $\alpha^3 L^3$ and NLO to $\alpha^2 L$ orders were calculated;
- TWIST experiment required corrections to the $\mathcal{O}(\alpha^2)$ order.

Parton distribution functions approach

- Based on perturbation theory and R. Feynman parton theory
- Came from QCD to QED
- Allows to calculate the most significant (logarithmic) corrections
- Expansion in powers of coupling constant and the large logarithm

Large logarithm

$$L = \ln \frac{\mu^2}{\mu_0^2}$$

μ - factorization scale, μ_0 - renormalization scale (in QED $\mu_0 = m_e$)

- Leading logarithmic approximation (LL or leading order - LO) - $\alpha^n L^n$
- Next-to-leading logarithmic approximation (NLL or next-to-leading order - NLO) - $\alpha^n L^{n-1}$

Parton distribution functions approach

- Partons in QED - electron, positron or photon
- Solve evolution equation of PDFs by iterations to get the desired approximation
- PDF evolution equation is an analog to DGLAP equation in QCD
- Make convolution of process independent PDFs with the functions determining the process

Parton distribution functions approach

Parton distribution functions (PDFs)

A function $D_{ia}(x, s)$ describes the density of the distribution of the massles parton of type i in the initial massive parton a. x is the energy fraction of the parton relative to the total energy of the particle which emitted it. Structure functions correspond to transition of a massive particle to a massless, and fragmentation functions to transition from massless particle to a massive one, which can be observed.

Splitting functions

$P_{ij}(x)$ describes the probability density of a transition of a parton j to a parton i with the energy fraction x.

Splitting functions and PDFs are independent of the process

Massless Wilson coefficients

$\sigma_{ij}(x)$ contain information of the process.

PDF evolution equation in QED

$$D_{ba}(x,\mu^2,\mu_0^2) = \delta(1-x)\delta_{ba} + \sum_{i=e,\bar{e},\gamma} \int\limits_{\mu_0^2}^{\mu^2} \frac{dt\alpha(t)}{2\pi t} \int\limits_x^1 \frac{dy}{y} D_{ia}(y,t,\mu_0) P_{bi}\left(\frac{x}{y}\right)$$

Equations are solved using iterative method:

$$D_{ee}^{(k)} = D_{ee}^{(0)} + \frac{\alpha}{2\pi}\left(P_{ee}\otimes D_{ee}^{(k-1)} + P_{e\gamma}\otimes D_{\gamma e}^{(k-1)} + P_{e\bar{e}}\otimes D_{\bar{e}e}^{(k-1)}\right)$$
$$D_{e\gamma}^{(k)} = D_{e\gamma}^{(0)} + \frac{\alpha}{2\pi}\left(P_{ee}\otimes D_{e\gamma}^{(k-1)} + P_{e\bar{e}}\otimes D_{\bar{e}\gamma}^{(k-1)} + P_{e\gamma}\otimes D_{\gamma\gamma}^{(k-1)}\right)$$
$$P_{ji}(x) = P_{ji}^{(0)}(x) + \frac{\alpha}{2\pi}P_{ji}^{(1)}(x) + \mathcal{O}(\alpha^2)$$

Initial conditions:

$$D_{ee}^{(0)}(x,\mu^2) = \delta(1-x) + \frac{\alpha}{2\pi}d_{ee}^{(1)}, \quad D_{\bar{e}\gamma}^{(0)}(x,\mu^2) = 0, \quad D_{\gamma\gamma}^{(0)}(x,\mu^2) = \delta(1-x)\frac{\beta}{3}$$
$$D_{e\gamma}^{(0)}(x,\mu^2) = \frac{\alpha}{2\pi}d_{e\gamma}^{(1)}(x), \quad D_{e\bar{e}}^{(0)}(x,\mu^2) = 0$$

PDF evolution equation in QED

Convolution operation

$$(f \otimes g)(x) \equiv \int_0^1 dz \int_0^1 dy\ f(z)g(y)\delta(x - yz) = \int_x^1 \frac{dz}{z} f(z)g(\frac{x}{z})$$

$$f(x) = \lim_{\Delta \to 0} \Big(f_\Theta(x)\Theta(1 - x - \Delta) + f_\Delta \delta(1 - x) \Big)$$

$$\int_z^1 dx [f(x)]_+ g(x) = \int_0^1 dx f(x) \Big[g(x)\Theta(x - z) - g(1) \Big]$$

$$f_\Delta = -\int_0^{1-\Delta} f_\Theta(z) dz$$

$$\Big(f \otimes g \Big)_\Theta (z) = \lim_{\Delta \to 0} \Big\{ \int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x} f_\Theta(x) g_\Theta \left(\frac{z}{x} \right) + f_\Delta g_\Theta(z) + f_\Theta(z) g_\Delta \Big\}$$

PDF evolution equation in QED

$$\alpha(q^2) = \frac{\alpha_0}{1 + \overline{\Pi}(\frac{-q^2}{\mu^2}, \frac{\overline{m}}{\mu}, \alpha_0)}$$

$$\overline{\Pi} = 2\alpha_0 \left(\left(\frac{5}{9} - \frac{L}{3} \right) + 4\alpha_0^2 \left(\frac{55}{48} - \zeta_3 - \frac{L}{4} \right) + 8\alpha_0^3 \left(\frac{-L^2}{24} \right) \right) + \dots$$

P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn and C. Sturm, Nucl. Phys. B **867** *(2013), 182-202*
$\alpha_0 = \frac{1}{137}$

Corrections to muon decay spectrum

Leading logarithmic approximation: $\alpha^k L^k$
Next-to-leading logarithmic approximation: $\alpha^k L^{k-1}$

$$
\begin{aligned}
[D_{ee}^{(\mathrm{III})}]_T &= \delta(1-x) + \frac{\alpha}{2\pi} d_{ee}^{(1)}(x) + \frac{\alpha}{2\pi} L P_{ee}^{(0)} + \left(\frac{\alpha}{2\pi}\right)^2 L\Big(\mathbf{d}_{\gamma\mathbf{e}}^{(\mathbf{1})}(\mathbf{x}) \otimes \mathbf{P}_{\mathbf{e}\gamma}^{(\mathbf{0})} + P_{ee}^{(1)} \\
&- \frac{10}{9} P_{ee}^{(0)} + P_{ee}^{(0)} \otimes d_{ee}^{(1)}(x)\Big) + \left(\frac{\alpha}{2\pi}\right)^2 L^2 \Big(\frac{1}{2} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{3} P_{ee}^{(0)} + \frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)}\Big) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^2 \Big(\frac{1}{2} P_{\gamma e}^{(1)T} \otimes P_{e\gamma}^{(0)} + \frac{1}{2} P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(1)T} + \frac{1}{3} d_{\gamma e}^{(1)} \otimes P_{e\gamma}^{(0)} \\
&+ \frac{1}{2} d_{\gamma e}^{(1)} \otimes P_{\gamma\gamma}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{2} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(1)T} - \frac{10}{9} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{2}{3} P_{ee}^{(1)T} \\
&+ \frac{1}{2} d_{ee}^{(1)} \otimes P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} - \frac{13}{54} P_{ee}^{(0)} + \frac{1}{2} P_{ee}^{(0)} \otimes d_{\gamma e}^{(1)} \otimes P_{e\gamma}^{(0)} + P_{ee}^{(0)} \otimes P_{ee}^{(1)T} \\
&+ \frac{1}{3} P_{ee}^{(0)} \otimes d_{ee}^{(1)} s - \frac{10}{9} P_{ee}^{(0)} \otimes P_{ee}^{(0)} + \frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes d_{ee}^{(1)}\Big) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^3 \Big(\frac{4}{27} P_{ee}^{(0)} + \frac{1}{6} P_{\gamma\bar{e}}^{(0)} \otimes P_{\bar{e}e}^{(0)} \otimes P_{e\gamma}^{(0)} + \mathbf{\frac{2}{9} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}} \\
&+ \frac{1}{3} P_{ee}^{(0)} \otimes P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{3} P_{ee}^{(0)} \otimes P_{ee}^{(0)} + \frac{1}{6} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes P_{ee}^{(0)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
[D_{e\gamma}^{\mathrm{II}}]_T &= \frac{\alpha}{2\pi} d_{e\gamma}^{(1)} + \frac{\alpha}{2\pi} L(P_{e\gamma}^{(0)}) + \left(\frac{\alpha}{2\pi}\right)^2 L\Big(P_{e\gamma}^{(1)T} - \frac{10}{9} P_{e\gamma}^{(0)} + \\
&+ P_{ee}^{(0)} \otimes d_{e\gamma}^{(}1)\Big) + \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{3} P_{e\gamma}^{(0)} + \frac{1}{2} P_{\gamma\gamma}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{2} P_{ee}^{(0)} \otimes P_{e\gamma}^{(0)}\right)
\end{aligned}
$$

Unpolarized muon decay spectrum

$$F(x) = f_{Born}(x) + \frac{\alpha}{2\pi} f_1(x)$$
$$+ \left(\frac{\alpha}{2\pi}\right)^2 \left\{ \left[f_2^{(0,\gamma)}(x) + f_2^{(0,NS)}(x) + f_2^{(0,S)}(x) \right] L^2 \right.$$
$$\left. + \left[f_2^{(1,\gamma)}(x) + f_2^{(1,NS)}(x) + f_2^{(1,S)}(x) + f_2^{(1,int)}(x) \right] L \right\}$$
$$+ \left(\frac{\alpha}{2\pi}\right)^3 \left\{ \left[f_3^{(0,\gamma)}(x) + f_3^{(0,NS)}(x) + f_3^{(0,S)}(x) \right] L^3 \right.$$
$$\left. + \left[f_3^{(1,\gamma)}(x) + f_3^{(1,NS)}(x) + f_3^{(1,S)}(x) + f_3^{(1,int)}(x) \right] L^2 \right\}$$
$$= f_{Born} + \frac{\alpha}{2\pi} f_1(x) + \left(\frac{\alpha}{2\pi}\right)^2 \left(F_{22} L^2 + F_{21} L \right)$$
$$+ \left(\frac{\alpha}{2\pi}\right)^3 \left(F_{33} L^3 + F_{32} L^2 \right)$$

Corrections to muon decay spectrum α^3

$$F(x) = \left(f_e^{(0)}(z) + \frac{\alpha}{2\pi} f_e^{(1)}(z)\right) \otimes \left[D_{ee}^{(\mathrm{III})}\right]_T + \left(f_\gamma^{(0)}(z) + \frac{\alpha}{2\pi} f_\gamma^{(1)}(z)\right) \otimes \left[D_{e\gamma}^{(\mathrm{II})}\right]_T$$

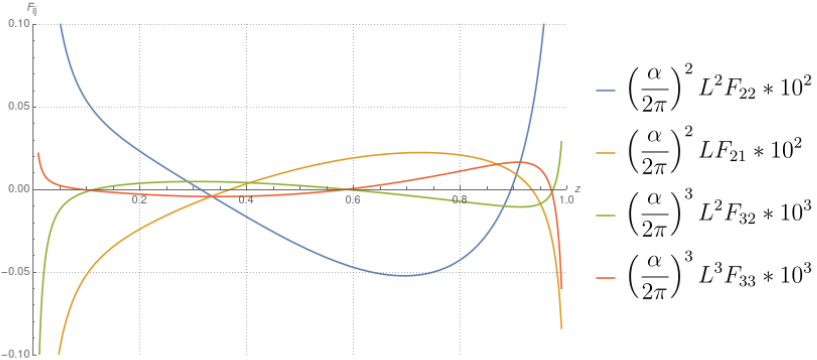
$$f_e^{(0)}(z) = z^2(3-2z), \quad f_\gamma^{(0)}(z) = 0$$

$$f_e^{(1)}(z) = 2z^2(2z-3)\big(4\zeta(2) - 4\mathrm{Li}_2(z) + 2\ln z^2 - 3\ln z \ln(1-z)$$
$$- \ln(1-z)^2\big) + \left(\frac{5}{3} - 2z - 13z^2 + \frac{34}{3}z^3\right)\ln(1-z)$$
$$+ \left(\frac{5}{3} + 4z - 2z^2 - 6z^3\right)\ln z + \frac{5}{6} - \frac{23}{3}z - \frac{3}{2}z^2 + \frac{7}{3}z^3$$

$$f_\gamma^{(1)}(z) = \ln z\left(-\frac{10}{3} + \frac{2}{z} + 4z\right) + \ln(1-z)\left(-\frac{5}{3} + \frac{1}{z} + 2z - 2z^2 + \frac{2}{3}z^3\right)$$
$$+ \frac{1}{3} - \frac{1}{z} + \frac{35}{12}z - 2z^2 - \frac{1}{4}z^3$$

Results

We recalculated correctons of the orders $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^3 L^3)$ and calculated correction of the order $\mathcal{O}(\alpha^3 L^2)$
Calculated using program in FORM, and HPL and MT Wolfram Mathematica packages



Conclusion

- Muon processes can be used to search for new physics
- Experiments on muon decay allow searching for decays forbidden by the SM
- New physics can be seen in small discrepansies from the SM
- To predict the results of high sensitivity and high precision experiments very accurate teoretical computations are required