METHOD OF AVERAGING IN THE INVESTIGATION OF THE INFORMATION CHARACTERISTICS OF FIBER CHANNELS WITH SMALL SECOND DISPERSION

Alexey V. Reznichenko *, Vyacheslav O. Guba *

* Budker Institute of Nuclear Physics SB RAS, Novosibirsk State University

The XXVII International Scientific Conference of Young Scientists and Specialists, October 29–November 3, 2023

Abstract

We present a method to obtain approximations to solutions of differential equations with slowly varying coefficients. So far we considered three different cases as applications of our method: a linear partial differential equation, time-dependent Schrodinger equation, and the linear equation which emerges from the nonlinear Schrodinger equation in the large SNR limit.

Demonstration of the method

The simplest case: a linear partial differential equation:

$$\partial_z\psi(z,t) + B\partial_t\psi(z,t) + a(t)\psi(z,t) = \eta(z,t), \tag{1}$$

where $\psi(z=0,t) = X(t)$.

The function $a(t)$ **is slowly varying** in the following sense:

$$\tilde{B} \equiv \frac{BL}{T_a} \ll 1,$$

where L is the maximum value of the variable z, and T_a is the time the function $a(t)$ takes to change considerably. The equation (1) can be solved exactly, but we consider it, because it is a good way to demonstrate the work of our method:

1. Solve Eq. (1) with $a(t) = const = a$, which is easier than the original equation. In the resulting solution return time dependence to the function $a(t)$, i.e. put $a(t)$ instead of a. This is the zero-order approximation:

$$\psi^{(0)}(z,t;\{a(t)\}) = X(t-Bz)e^{-a(t)z} + \int_0^z dz'\eta(z',t-B(z-z'))e^{-a(t)(z-z')}.$$

2. Put $\psi = \psi^{(0)} + \psi^{(1)}$ (with $\psi^{(n)} \propto \tilde{B}^n$) in the Eq. (1) and neglect terms of order $O(\tilde{B}^2)$ if they appear. The terms of order $\tilde{B}$ come from $\dot{a}(t)$. In the resulting equation put $a(t) = const$ and $\dot{a}(t) = const$:

$$\partial_z\psi^{(1)} + B\partial_t\psi^{(1)} + a\psi^{(1)} = -B\dot{a}\partial_a\psi^{(0)}$$

and solve it for $\psi^{(1)}$ returning time dependence to a and $\dot{a}$ in the solution.

3. Finding the second-order correction is similar: put $\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)}$ in the Eq. (1) and neglect terms of order $O(\tilde{B}^3)$. Solve the resulting equation for $\psi^{(2)}$ putting $a(t), \dot{a}(t), \ddot{a}(t) = const$.

Application to the Schrodinger equation

The Schrodinger equation:

$$i\partial_t\psi(t,z) + B\partial_z^2\psi(t,z) - V(z)\psi(t,z) = 0 \tag{2}$$

with $\psi(t=0,z) = X(z)$ can be investigated with our method in the following regime:

$$\frac{BT}{L_V^2} \ll \frac{BT}{L_V L_X} \ll 1, \ \frac{BT}{L_X^2} \sim 1,$$

where L_V is the characteristic scale of change of the potential $V(z)$ and L_X is the characteristics scale of change of the initial condition $X(z)$.

Considering the potential $V(z)$ to be a slowly varying function we can apply our algorithm just as we did for the previous equation: we write the solution as the series $\psi = \sum_n \psi^{(n)}$ in which $\psi^{(n)} \propto (BT/L_X L_V)^n$ and obtain following equation for $\psi^{(n)}$:

$$i\partial_t\psi^{(n)}(t,z) + B\partial_z^2\psi^{(n)}(t,z) - V(z)\psi^{(n)}(t,z) = R^{(n)}(t,z), \tag{3}$$

where the r.h.s $R^{(n)}$ depends on derivatives of $V(z)$ and on the $V(z)$ itself. The equation (3) can be solved putting $V(z)$ and its derivatives to be constant and using Fourier transform with respect to the variable z. Using the solution we can calculate, for instance, the quantum-mechanical partition function in form of the perturbative expansion in B:

$$Z_{QM}(T) \equiv \int dz\,\langle z|\,e^{-i\hat{H}T}\,|z\rangle = \int dz \frac{e^{-iV(z)T}}{\sqrt{4\pi iTB}}\Big\{1 + \frac{B}{12}T^2V_2 + \frac{iB^2}{288}T^3V_4 + \frac{7B^2}{1440}T^4V_2^2 + \frac{31B^3}{120960}T^6V_2^3 + \frac{iB^3}{45360}T^5V_3^2 + \frac{iB^3}{1920}T^5V_2V_4 - \frac{B^3}{10368}T^4V_6 + \frac{127B^4}{9676800}T^8{V_2}^4 + \frac{247iB^4}{4838400}T^7{V_2}^2V_4 + \frac{11iB^4}{2721600}T^7V_2{V_3}^2 - \frac{11B^4}{622080}T^6V_2V_6 - \frac{B^4}{272160}T^6V_3V_5 - \frac{11B^4}{645120}T^6{V_4}^2 - \frac{iB^4}{497664}T^5V_8 + \dots\Big\},$$

where $V_n = \partial_z^n V(z)$.

Application to a nonlinear noisy communication channel

The nonlinear Schrodinger equation (NLSE) with Gaussian noise $\eta(z,t)$ which describes propagation of the signal $\psi(z,t)$ through fiber:

$$\partial_z\psi(z,t) + i\beta\partial_t^2\psi(z,t) - i\gamma|\psi(z,t)|^2\psi(z,t) = \eta(z,t)$$

can be converted to a linear problem in the large SNR limit:

$$\partial_z F_{(a)} + i\beta\partial_t^2 F_{(a)} - 2\beta\partial_t\theta_0\partial_t F_{(a)} + i\beta F_{(a)}\left(i\partial_t^2\theta_0 - (\partial_t\theta_0)^2\right) + i\frac{\mu}{L}F_{(a)} - 2i\gamma|\tilde{\Phi}|^2F_{(a)} - i\gamma\tilde{\Phi}^2\bar{F}_{(a)} = \eta_{(a)}.$$

where $\psi(z,t) = \Phi(z,t) + F(z,t)e^{i\theta_0(z,t)}$, and functions θ_0 and $\mu = \gamma L|X|^2$ are related to the input signal $\psi(z=0,t) = X(t)$. The function Φ is the solution to NLSE without noise. The subscript (a) corresponds to the averaged signal, which in our model is observed by the receiver and is defined as follows:

$$F_{(a)}(z,t) = \frac{1}{2\tau_a}\int_{-\tau_a}^{+\tau_a} dt' F(z,t').$$

The characteristic time of change of an averaged function is τ_a. Our method can be used to solve the equation above in the following regime:

$$\tilde{\beta} \ll \tilde{\beta}_a \ll 1, \ \frac{\beta L}{\tau_a^2} \sim 1,$$

with $\tilde{\beta} = \beta L/T_X^2$ and $\tilde{\beta}_a = \beta L/T_X\tau_a$. The functions which are considered to be slowly varying are θ_0 and μ. The solution has the form of perturbative expansion in the parameter $\tilde{\beta}_a$. So far we have obtained the solution up to the first order in $\tilde{\beta}$ (the second order in $\tilde{\beta}_a$). It can be used to calculate correlators of the output signal $\psi(z=L,t) = Y(t)$ and then information characteristics (mutual information, channel capacity, etc.).