# Symplectic extension of the Haag-Araki axiomatics and its applications in the physics of causal geodesic structures.

**Goal of the work:** Create an axiomatic equivalent to the axiomatics of Haag-Araki and Whiteman, based on the principle of causality for sticky sets.

**Axiomatics of causality set on local geodesic structure:**

**Assumption(Axiom) 1.**

$$IntI^+(q)IntI_j \subset \overline{IntI^-(p) \subset IntI^+(p)}$$

Proof of the Assumption 1.

$$IntI^+(q)\overline{IntI(p)}IntI_j \subset IntI^-(p) \subset IntI(p)$$
$$= IntI^+(q)\overline{IntI^-(p) \subset IntI^+(p)} \subset IntI^-(p) \subset IntI(p)$$
$$= IntI^+(q)I^+(p)\overline{IntI^-(p)\subset} IntI^-(p) \subset IntI(p) \subset IntI_j$$
$$= IntI^+(q)I^+(p) \subset IntI_O^+(P) \subset I^-(q) \cup IntI^+(p) \subset IntI_j$$
$$= IntI^+(q)I^+(p) \subset I^-(q) \cup IntI^+(p) \cup IntI^-(p)$$
$$= IntI^+(q) \cap IntI^-(p)$$

$I_o^+ \subset I^-(q), I_O^+(P) \neq \emptyset.$

**Axiom 2. This axiom defines the complement of the past set in a globally hyperbolic neighborhood of an eventually periodic point.**

$$IntI^-(p) \subset I^-(q) \subset IntI_i \overline{\subset IntI(p)} I^+(p)$$
$$= IntI_i\ I_o^+ \subset I^-(q) IntI^-(p)\ IntI^+(q)$$
$$\cup\ IntI^+(q) \subset IntI_j \bigcup \subset \overline{IntI^+(p)} \cap IntI^-(p) IntI^+(p)$$
$$\cup\ IntI^-(q)\ \ IntI^+(q) \subset IntI^+(p) \subset IntI_j$$
$$= \bigcup \subset \overline{IntI^+(p)} \cap IntI^-(p) IntI^+(p)$$
$$\cup\ IntI^-(q)\ \ IntI^+(q) \subset IntI^+(p) \subset IntI_j\, IntI^+(q)$$
$$\cap\ IntI^-(p)$$
$$= \subset \overline{IntI^+(p)} \cap IntI^-(p) IntI^+(p) \subset IntI_j\, IntI^+(q)$$
$$\cap\ IntI^-(p)$$
$$= \overline{IntI^-(q)} \subset IntI_j \subset \overline{IntI^+(p)} \cap IntI^-(p) \cup IntI^+(q)$$

$$\overline{IntI^-(p)} \cap I^+(q) \neq \emptyset \qquad IntI_{U \ni V} \bigcup Int_{V \ni N}$$

**Axiom 3.**

$$\overline{IntI(p)} \subset IntI^+(q) \cap IntI^-(p) \bigcup T\ IntI^-(q)\overline{IntI(p)} \subset IntI^-(q) \cap IntI^+(p)$$

**Corollaries from Assumption 1 and properties of nested sets:**

1. Time is a Markov process
2. Controlled packed globally hyperbolic sets are completely simple

$$IntI_{U \ni v} \subset I^-(q) \subset IntI_j T$$
$$\cup\ Int\left[x_p^* \ldots, x_{p+1}\right] \subset IntI^+(q)I^+(p) \subset I^-(q) \subset I_i^- \subset IntI_j$$
$$= IntI_{U \ni v} \subset \overline{I^-(q)}IntI^-(p)IntI^+(q)T \cap IntI_i$$
$$\coloneqq [x \ldots, x_i]^p \subset IntI_j \cup IntI\left[x_p^* \ldots, x_{p+1}\right] = IntI_j$$
$$\coloneqq [x^* \ldots, x_i] \subset IntI_j \cup T_\omega^* \overline{\subset IntI^-(p) \subset IntI^+(p)}$$
$$= IntI_{U \ni v} \subset I^+(q) \subset IntI_j \cup T_\omega^* \overline{\subset IntI^-(p) \subset IntI^+(p)}$$
$$= IntI_{U \ni v} \subset I^+(q) \subset IntI_j \cup T_{\omega \ldots, \omega^*{}_0}$$
$$\cap\ I_{\omega_0 \ldots, \acute{\omega}} \overline{\subset IntI^-(p) \subset IntI^+(p)}$$

**Consider,the causal geodesic structure. In the symplectic case, gets**

$$\frac{\partial}{\partial\tau} g\left(\frac{\partial}{\partial\tau}.\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial\tau} g\left(\frac{\partial}{\partial\tau}.\frac{\partial}{\partial t}\right)_j = \frac{\partial}{\partial\tau} g_j\big(J(u)\big)\partial s\partial t = \frac{1}{2}\frac{\partial}{\partial t_j}\ g\big(J(u)\big)\partial s\partial\tau = g\left(J(u)\frac{\partial}{\partial\tau}\frac{D}{\partial s}\frac{\partial}{\partial t_j}\right)$$
$$= g(u,v)expx(q)\ \partial s\partial u \frac{D}{\partial s}\frac{\partial}{\partial t_j} = g(u-v)\frac{D}{\partial s}\frac{\partial}{\partial t_j} g^{ba}{}_{ab} = g_j \frac{D}{\partial t}\left(\frac{\partial t}{\partial t_j}\right)\partial s\partial\tau$$

$$\bar{P}d(U,V) = max_{supp\ a^c \epsilon m_\xi(\alpha \in \acute{A})}(\min[x-y]) + max_{\alpha_c}\left(\min\|x-y\|^2\right) = \left(\bar{P}diamU + \overline{PdiamV}\right)$$
$$= \overline{Pdim_H(\acute{A})}_{supp\ a^c \epsilon m_\xi(\alpha \in \acute{A})} + \overline{PdiamV}_{supp\ a^c \epsilon m_\xi(\alpha \in \acute{A})}$$
$$= Pdim_H\left(U^{\psi_i}\right) + \overline{PdiamV}_{supp\ a^c \epsilon m_\xi(\alpha \in \acute{A})}$$

In tangent space the equation takes the form:

$$J\big(g(u - v_{.b}^a)\big)\frac{D}{\partial s}\frac{\partial}{\partial t_j} g_{ab}^{ba} = J\left(g_j(u-v)\partial t/\partial t \frac{D}{\partial\tau}^a\right)_b g^{ba}{}_{ab}\partial s = J\left((u)(v-s)g_{ab}^{ba}\right)\partial_\tau s$$
$$= J\left(\frac{1}{2}u\left(v_{.b}{}^a - \frac{D\partial}{\partial t}_j\right) g_{ab}^{ba} expx(q)\right)\partial_\tau\partial u = Jg_j\left(-\frac{1}{2}\right)\left(\frac{v_{.b}^a}{\frac{D\partial}{\partial t}} g_{ab}^{ba}(u-w)expx(q)\right)\partial_\tau\partial u$$
$$= J_j g\left(\frac{v_{.b}^a}{\frac{D\partial}{\partial t}} g_{ab}^{ba}(u-w)\right)_b expx(q)\partial_\tau\partial u\partial s = J_j g * s\left(\left(\frac{v_{.b}^a}{\frac{D\partial}{\partial t}} g_{ab}^{ba}(u-w)\right)\right)_b$$
$$= \frac{1}{2}\frac{\partial s}{\partial\tau}\frac{\partial u}{\partial\tau} J(u)\acute{\partial} u\partial t = \frac{D}{d\tau}\left(\frac{\partial}{\partial s_j}\frac{\partial u}{\partial\tau}J(u)\partial s\partial u\right) = \frac{1}{2}\frac{\partial}{\partial t}J(u)\partial s\partial u$$

For periodically point on the elliptic set

$$U(\psi: A \to \acute{A})_i = U\left(sup\psi\bar{B}(x^*(A,\alpha_c),\varepsilon_0) \to Usup\psi\bar{B}(x^{\acute{*}}(A,\alpha_c),\varepsilon_0)\right)uC^\gamma \to$$
$$C^p\left(\overline{Int^-}_p p\acute{q}_n(2^n)\#\langle\langle\pi_L^c: T_\lambda{}^*\ \bar{N}(p) \cap_{k=0}^n \acute{N} \subset Int^p I_{-i}\rangle\rangle\left[x_p{}^*\ x_{p+1}\right] - q_0\right)_{i=0}^{p-1}\left[q^{*:i,s}\right]^{2p+1}\Bigg)\Bigg]$$

Equivalent operator-valued function $\qquad T_\lambda^*\bar{N} = \boldsymbol{A}(\varphi)e^{-n\varepsilon_{\acute{\omega}}} = A(\varphi)e^{\frac{k^2}{4}}$

$$\overline{(1)T_{\lambda\ m}^* supBNJ^\pm(O)}_{\ Int_{V \ni N} \cap IntI^-{}_{U \ni v}}: expx_p^{(q)} = T_\lambda^{*,m}{}_{:m} \sup\left(Bd_0^4 I_O^\pm(x^*(A,\alpha_c),\varepsilon_0)\right)_{IntI^-{}_{U \ni v}}$$
$$= T_\lambda^{*,m}{}_{:m} \sup d_0^4\left(B\overline{I_O^\pm}(x^*(A,\alpha_c),\varepsilon_0)\right)_{IntI^-{}_{U \ni v}}$$
$$= sup\bar{B}\ \left(J_{U^+ \cap U^-}^O\left(x^*\left(\frac{\partial}{\partial\tau} g\left(\frac{\partial}{\partial\tau}.\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial\tau} g\left(\frac{\partial}{\partial\tau}.\frac{\partial}{\partial t}\right)_j = \frac{\partial}{\partial\tau} g_j\big(J(u)\big)\partial s\partial t = \frac{1}{2}\frac{\partial}{\partial t_j}\ g\big(J(u)\big)\partial s\partial\tau\right.\right.\right.$$
$$= g\left(J(u)\frac{\partial}{\partial\tau}\frac{D}{\partial s}\frac{\partial}{\partial t_j}\right) = g(u,v)expx(q)\ \partial s\partial u\frac{D}{\partial s}\frac{\partial}{\partial t_j} = g(u-v)\frac{D}{\partial s}\frac{\partial}{\partial t_j} g^{ba}{}_{ab} = g_j\frac{D}{\partial t}\left(\frac{\partial t}{\partial t_j}\right)\partial s\partial,\alpha_c\Bigg),\varepsilon_0\Bigg)\Bigg)_{u_{1\ldots}u_n}$$
$$= T_\lambda^{*,m}{}_{:m}\sup\left(B\overline{I_O^\pm}\left(x^*\left(\left(\frac{1}{2}u\left(v_{.b}{}^a - \frac{D\partial}{\partial t}_j\right)g_{ab}^{ba}\right),\alpha_c\right),\varepsilon_0\right)\right)_{IntI^-{}_{U \ni v}}$$
$$= T_{[\omega \ldots, \acute{\omega}]_n^s}\sup\left(B\overline{I_O^\pm}\left(x^*\left(\left(\frac{1}{2}u\left(v_{.b}{}^a - \frac{D\partial}{\partial t}_j\right)g_{ab}^{ba}\right)g_{\sigma,j}\right),\varepsilon_0\right)\right)$$

For closed time-like circuit propagator

$$\partial\frac{J_O^\pm}{\partial\tau} g\left(\frac{\partial}{\partial\tau}.\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial\tau} g\left(\frac{\partial}{\partial\tau}.\frac{\partial}{\partial t}\right)_j = \frac{\partial}{\partial\tau} g_j\big(J(u)\big)\partial s\partial t = \frac{1}{2}\frac{\partial}{\partial t_j}\ g\big(J(u)\big)\partial s\partial\tau$$
$$= g\left(J(u)\frac{\partial}{\partial\tau}\frac{D}{\partial s}\frac{\partial}{\partial t_j}\right) = g(u,v)\sup T_{\omega\acute{\lambda}}^* B(x^*,)\ \partial s\partial u\frac{D}{\partial s}\frac{\partial}{\partial t_j}$$
$$= g(u-v)\frac{D}{\partial s}\frac{\partial}{\partial t_j} g^{ba}{}_{ab} = g_j\frac{D}{\partial t}\left(\frac{\partial t}{\partial t_j}\right)\partial s\partial\tau$$

**Is it possible to build a weak *- topology on "open light cones"?**

$$C^{1-y} \to M^0: \sum I^+{}_O \in \left(\overline{Int^-}_p p\acute{q}_n(2^n)\#\langle\langle\pi_L^c:\tau_c^l\rangle\rangle\left[x_p{}^*\ x_{p+1}\right] - q_0\right)\left[q^{*:i,s}\right]^{2p+1^-}$$
$$\to Int\{I_i[x_i, x_{i+1}]\}_{i=0}{}^{p-1}$$
$$= IntI_{U \ni v} \subset I^-(q) \subset IntI_j \quad \cup\ TInt\left[x_p{}^*\ x_{p+1}\right] \subset IntI^+(q)I^+(p) \subset I^-(q) \subset I^-{}_i \subset IntI_j$$

From here, we get

$$C^\gamma \to M\left[\left(Int\{I_i \coloneqq [x_i \ldots, x_{i+1}] \subset \acute{O}\}\left(\overline{Int^-}_p p\acute{q}_n(2^n)\#\langle\langle\pi_L^c: T_{\omega \ldots, ijkl}^{v\ \ *}\rangle\rangle\left[x_p{}^*\ x_{p+1}\right] - q_0\right)_{i=0}^{p-1}\left[q^{*:i,s}\right]^{2p+1}\right)\right]$$
$$= C^p\left(\overline{Int^-}_p p\acute{q}_n(2^n)\#\langle\langle\pi_L^c: T_{\omega \ldots, ijkl}^{v\ \ *}\rangle\rangle\left[x_p{}^*\ x_{p+1}\right] - q_0\right)_{i=0}^{p-1}\left[q^{*:i,s}\right]^{2p+1}\Bigg)\Bigg]$$

$$U \ni N \subset \acute{N} \subset \bar{N}(p) \ni C^p \to = \overline{N^*} \bigcap V$$

Captured of past-geodesic

$$\cap\ \bar{I}_{\omega_0 \ldots \omega_n} = \overline{TInt^-{}_p \subset IntI_j \cup IntI_{U \ni v} \cap_{k=0}^n Tx(\bar{\omega})} \subset \overline{IntI^-(q)} =$$
$$TInt^-{}_p \subset IntI_j \cup IntI_{U \ni v} T_p^* \overline{N}(\lambda(p)), \cap_{k=0}^n,$$

$$(3) there, TInt\left[x_p{}^*\ x_{p+1}\right] \subset IntI^+(q)I^+(p) \subset I^-(q) \subset I^-{}_i \subset IntI_j$$
$$= T\overline{Int_p}^- IntI^+{}_i \subset IntI_j \bigcup IntI_{U \ni v} \cap \overline{IntI^-(q)} \cap I^+(p) \subset IntI_j\ (\quad)$$
$$= T\overline{Int_p}^- IntI^+{}_i \subset IntI_j \bigcup IntI_{U \ni v} \bigcap \overline{IntI^-(q)} \subset IntI_j$$
$$= T\overline{Int_p}^- IntI^+{}_i \subset IntI_j \bigcup IntI_{U \ni v} \bigcap \bar{I}_{\omega_0 \ldots \omega_n}$$
$$= \overline{TInt^-{}_p \subset IntI_j \bigcup IntI_{U \ni v} \cap_{k=0}^n Tx(\bar{\omega})} \subset \overline{IntI^-(q)}$$
$$= T\overline{Int\left[x_p{}^*\ x_{p+1}\right]} I_U^+(p) \bigcup \overline{T^{-k} \cap_{k=0}^n IntI_{U \ni v} \subset I_U^-}(q)$$

**Proof of the equivalence of the formalism of the area-preserving plane transformation with infinite hierarchical structure and causal sets**

For transition between different level of dynamic system

1. $IntI^+(q)IntI_j \subset \overline{IntI^-(p) \subset IntI^+(p)} \sim I_O^\pm$
2. $I^-(q) \subset \overline{IntI_p} \cap I(q)^+ IntI^i \qquad IntI_j \subset I^+(p)$
3. $IntI_j \subset Int\{I_i[x_i, x_{i+1}]\}_{i=0}$
4. $IntI_i \coloneqq [x\ldots, x_i]^p \subset IntI_j \quad \cup IInt\left[x_p{}^*, x_{p+1}\right] =$ $IntI_{U \ni v} \subset I^-(q) IntI^-(p) IntI^+(q) T \cap IntI_p = IntI_{\omega_{..}\acute{\omega}_0} \cup IntI_{\acute{\omega}_{0\ \ldots}\bar{\acute{\omega}}}$
5. $\left[x_p{}^*, x_{p+1}\right] = T_\omega{}^* \subset \overline{IntI^-(p) \subset IntI^+(p)}$

Traceless tensor of momentum- energy- matter

$$\sup T_{\omega\lambda}^* T_{\omega\lambda}^* \overline{B^{Z(r) \times Z(r)}}\left(x^*, dimU^{\Psi i}\left(J_{U_{\psi i}{}^+ \cap U^-{}_{\psi i}}^+\right)\max: \#\langle\langle\pi_L^c:\tau_c^l\rangle\rangle\left(C_p^*(A) \ni N^* \ \ni M\right.\right.$$
$$\left.\ni \acute{U} + 2\sum_{n=1}^{\infty}(N+C)^{1-y} - q_0\right), \varepsilon_0\Big)$$
$$= \sup T_{\omega\lambda}^* T_{\omega\lambda}^* \langle\langle\pi_L^c\rangle\rangle N \subset \acute{N} C_p^*(A) \ni M \ni \acute{U} : \overline{B^{Z(r) \times Z(r)}}\left(x^* \cdot dimU^{\Psi i}\left(J_{U_{\psi i}{}^+ \cap U^-{}_{\psi i}}^+\right)\right)$$
$$= supT_{\omega\lambda}^*\left(J\left(w_G(U.\omega)\right)\right)$$

This proves the symplectic nesting

$$T\bar{N}(p)_{M_{\omega\grave{\lambda}} \ni V} there M_{\omega\grave{\lambda}} \ni V$$

$$(4) T_\lambda^*\bar{N}(p)\ \cap V = \overline{Int_p^-} Tx(\bar{\omega}) = \overline{Int_p^-} T_{\omega \ldots, \acute{\omega}} x^q \subset IntI_j \cup\ IntI_{U \ni V} \coloneqq$$
$$T_n^{-k} \cap_{k=0}^n \tilde{T}_{[[\omega]]_s}\frac{1}{\lambda} \subset IntI_{U \ni v} \subset IntI_j \subset I_{U_{p \ni P}}^+ \cup T^*{}_{\omega\grave{\lambda}} \subset Int_{V \ni N} \subset U \ni N^* \subset \tilde{N} =$$
$$\tilde{T}_{[[\omega]]_s}\frac{1}{\lambda} \subset IntI_{U \ni V} \subset IntI_j \subset I_{U_{p \ni P}}^+ \subset Int_{V \ni N} \cup T^*{}_{\omega\grave{\lambda}} \subset U \ni N^* \subset \tilde{N} \ni M = IntT^p{}_{M_{\omega\grave{\lambda}}} \ni$$
$$T_p^*\overline{N}(\lambda(p)), \cap_{k=0}^n \acute{N} \subset IntI_{U \ni v} \cap Int^p I_{-i} \subset I_U{}^- = T_{\lambda p}{}^*\bar{N} \cap_{k=0}^n \acute{N} \subset Int^p I_{-i} \subset I_U{}^- IntI_{U \ni v} =$$
$$T_{\lambda p}{}^*\bar{N} \cap_{k=0}^n \acute{N} \subset Int^p I_{-i} \subset I_U{}^- IntI_{U \ni v} \subset TInt\left[x_p{}^*\ x_{p+1}\right] \subset IntI^+(q) \subset IntI_j =$$
$$T_\lambda{}^*\ \bar{N}(p) \cap_{k=0}^n \acute{N} \subset Int^p I_{-i} \subset I_U{}^- \cap \subset TInt\left[x_p{}^*\ x_{p+1}\right] TIntI_{U \ni v} \cap Int_{V \ni N} \in IntI_i^- \in \subset I^-(q) \subset IntI_j =$$
$$\cap\ TInt\left[x_p{}^*\ x_{p+1}\right] TIntI_{U \ni v} \cap_{k=0}^n Tx_{[[\omega]]_n^s}{}^{q'} \ni Int_{V \ni N} = T_\lambda{}^*\ \bar{N}(p) \cap_{k=0}^n \acute{N} \subset Int^p I_{-i} = IntI_{U \ni v} T_{[\omega \ldots, \acute{\omega}]_n^s}^* x^q \ni Int_{V \ni N} =$$
$$IntI_{U \ni v} T^*{}_{[\omega \ldots, \acute{\omega}]_n^s} I_O^\pm = T_\lambda^*\bar{N}(p)\ \cap V.$$