Multichannel scattering for the Schrödinger equation on a line with different thresholds at both infinities

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Problem formulation

Matrix Shrodinger equation on the line

\[
[\partial_z g_{ij}(z)\partial_z + V_{ij}(z; \lambda)]u_j(z) = 0, \quad z \in \mathbb{R}, \quad i, j \in \{1, \ldots, N\},
\]

(1)

Where \( V_{ij}(z; \lambda) = V_{ij}(z) - \lambda g_{ij}(z) \), and \( \lambda \) is auxiliary parameter.

We also assume that there exists \( L > 0 \) such that

\[
\begin{align*}
  g_{ij}(z) \bigg|_{z>L} &= g_{ij}^+, & V_{ij}(z) \bigg|_{z>L} &= V_{ij}^+, \\
  g_{ij}(z) \bigg|_{z<-L} &= g_{ij}^-, & V_{ij}(z) \bigg|_{z<-L} &= V_{ij}^-,
\end{align*}
\]

(2)

By definition, the Jost solutions to Eq. (1) have the asymptotics

\[
(F^+_{\pm})_{is} (z; \lambda) \xrightarrow{z \to +\infty} (f_+)^{is'}(e^{\pm iK_+ z})_{s's'}, \quad (F^+_{\pm})_{is} (z; \lambda) \xrightarrow{z \to -\infty} (f_+)^{is'}(e^{\pm iK_+ z})_{s's'}.
\]

(3)
Historical review

• Single channel scattering problem for one-dimensional Shrodinger equation on the line was solved by Fadeev 1964 [1,2]
• The proof of unitarity of the $S$-matrix on the half line in the presence of closed scattering channels is given by Newton 1982 [3]
• Some scattering and analytical properties for two-Channel Hamiltonians were revealed by Melgaard 2001 [4,5]
• Multichannel scattering problem on the line for the one-dimensional Shrodinger equation on the line was investigated relatively recent, mostly by Aktosun [6-9], however the unitarity was not proved.

Nevertheless, to our knowledge, the description of properties of the $S$-matrix, of the Jost solutions, and of the bound states in the general case of multichannel scattering on a line with different thresholds at both left and right infinities is absent in the literature. Our aim is to fill this gap.
Main identities

The Jost solutions $F_\pm^+$ and $F_\pm^-$ constitute bases in the space of solutions of Eq.(1). Consequently,

$$F_+^+ = F_+^- \Phi_+ + F_-^- \Psi_+,$$
$$F_-^+ = F_+^- \Psi_+ + F_-^- \Phi_-,$$ (4)

Let us introduce the transmission matrices $t_{(1,2)}$ and the reflection matrices $r_{(1,2)}$

$$F_+^+ t_{(1)} = F_+^- + F_-^- r_{(1)}, \quad F_-^+ t_{(2)} = F_+^- + F_+^- r_{(2)},$$ (5)

Define the $S$-matrix as

$$S := \begin{bmatrix} t_{(1)} & r_{(2)} \\ r_{(1)} & t_{(2)} \end{bmatrix}.$$ (6)

Then the $S$-matrix possesses the symmetries

$$\begin{bmatrix} 0 & K_- \\ K_+ & 0 \end{bmatrix} S = S^T \begin{bmatrix} 0 & K_+ \\ K_- & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & K_- \\ K_+ & 0 \end{bmatrix} S = S^T \begin{bmatrix} 0 & K_+ \\ K_- & 0 \end{bmatrix},$$ (7)

$$\begin{bmatrix} K_+ & 0 \\ 0 & K_- \end{bmatrix} S = S^T \begin{bmatrix} K_- & 0 \\ 0 & K_+ \end{bmatrix}.$$
The case of all scattering channels are open

**Theorem 1.** If $\lambda$ belongs to none of the cuts of the functions $(K_\pm)_s$, $s \in \{1, \ldots, N\}$, i.e., when all the scattering channels are open, the $S$-matrix is unitary

$$S^+\begin{bmatrix} K_+ & 0 \\ 0 & K_- \end{bmatrix}S = \begin{bmatrix} K_- & 0 \\ 0 & K_+ \end{bmatrix}.$$  \hfill (9)

**Remark. Introducing the notation**

$$\tilde{\Phi}_\pm := \frac{1}{2}K_-^{\pm}\Phi_\pm K_+^{-\frac{1}{2}}, \quad \tilde{\Psi}_\pm := \frac{1}{2}K_-^{\pm}\Psi_\pm K_+^{-\frac{1}{2}}.$$ \hfill (10)

One can reduce (9) to the standard form $\tilde{S}^+\tilde{S} = 1.$

**Proposition 1.** If all scattering channels are open, there are no bound states.
Identities in the subspace of open channels

We split the relations (5) into blocks with respect to the indices $s, s'$ in accordance with splitting into open and closed channels,

\[
(F_+^+)_ot_{(1)oo} + (F_+^-)_ct_{(1)co} = (F_+^-)_o + (F_-^-)_or_{(1)oo} + (F_-^-)_cr_{(1)co},
\]
\[
(F_+^+)_ot_{(1)oc} + (F_+^-)_ct_{(1)cc} = (F_+^-)_c + (F_-^-)_or_{(1)oc} + (F_-^-)_cr_{(1)cc},
\]
\[
(F_-^-)_ot_{(2)oo} + (F_-^-)_ct_{(2)co} = (F_-^-)_o + (F_+^+)_or_{(2)oo} + (F_+^+)_cr_{(2)co},
\]
\[
(F_-^-)_ot_{(2)oc} + (F_-^-)_ct_{(2)cc} = (F_-^-)_c + (F_+^+)_or_{(2)oc} + (F_+^+)_cr_{(2)cc}.
\]

(11)

**Theorem 2.** The *S*-matrix in the subspace of open channels is unitary.

\[
t^*_{(2)oo}t_{(1)oo} + r^*_{(1)oo}r_{(1)oo} = 1,
\]
\[
t_{(1)oo}(K_+)o_t_{(1)oo} + r_{(1)oo}(K_-)o_r_{(1)oo} = (K_-)_0.
\]

(12)

**Theorem 3.** The following condition

\[
\det \Phi_+(\lambda) = 0, \quad \lambda \in \mathbb{R},
\]

is a necessary and sufficient condition for the existence of bound states of Eq. (1).
An electrodynamic example: wire metamaterial

An non-local effective permittivity tensor in a wire metamaterial reads as

\[ \hat{\varepsilon}_{ij} = \varepsilon_0 \delta_{ij} - z \frac{\varepsilon_0 \omega_p^2}{\varepsilon_0 k_0^2 - (\hat{k}_z)^2} z, \quad z = (0,0,1). \]  

(13)

The standard Maxwell equation in media

\( (\text{rot}^2_{ij} - k_0^2 \hat{\varepsilon}_{ij}) A_j = 0, \quad \hat{k}_i (\hat{\varepsilon}_{ij} A_j) = 0. \)  

(14)

Introducing the scalar plasmon field \( \Psi \), the non-locality of equations (1) may be resolved

\( (\varepsilon_0 k_0^2 - (\hat{k}_z)^2) \Psi + \omega_p^2 A_3 = 0, \)

\( (\varepsilon_0 k_0^2 \delta_{ij} - \text{rot}^2_{ij}) A_j + \varepsilon_0 k_0^2 z_i \Psi = 0. \)  

(15)

Unitarity relation holds

\[ |r_1|^2 + |r_2|^2 + |t_1|^2 + |t_2|^2 = 1 \]
Summary

• It has been shown that the Jost solutions are the different branches of the same vector-valued analytic function on a double-sheeted Riemann surface.
• The key result of this work is Theorem 2 that proves unitarity of the scattering matrix in the subspace of open channels. The proof of this fact appears to be obtained for the first time.
• The weak version of the Theorem 2, the statement of about unitarity in the case when all scattering channels are open is also proved.
• The condition determining the bound states has been obtained.
• The results of the work are applicable in electrodynamic of continuous media, in particular, for the theories with large spatial dispersion. Corresponding wire-metamaterial example is presented.
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References.