Multichannel scattering for the Schrödinger equation on a line with different thresholds at both infinities

P.O. Kazinski and <u>P.S. Korolev</u>

Physics Faculty, Tomsk State University, Russia

Problem formulation

Matrix Shrodinger equation on the line

$$[\partial_z g_{ij}(z)\partial_z + V_{ij}(z;\lambda)]u_j(z) = 0, \quad z \in \mathbb{R}, \quad i,j \in \{1,\dots,N\}, \tag{1}$$

Where $V_{ij}(z;\lambda) = V_{ij}(z) - \lambda g_{ij}(z)$, and λ is auxiliary parameter.

We also assume that there exists $L > 0$ such that

$$\begin{cases} g_{ij}(z)\Big|_{z>L} = g_{ij}^{+}, & V_{ij}(z)\Big|_{z>L} = V_{ij}^{+}, \\ g_{ij}(z)\Big|_{z<-L} = g_{ij}^{-}, & V_{ij}(z)\Big|_{z<-L} = V_{ij}^{-}, \end{cases} \tag{2}$$

By definition, the Jost solutions to Eq. (1) have the asymptotics

$$(F_{\pm}^{+})_{is}\,(z;\lambda) \xrightarrow[z\to+\infty]{} (f_{+})_{is'}\left(e^{\pm iK_{+}z}\right)_{s's}, \qquad (F_{\pm}^{+})_{is}\,(z;\lambda) \xrightarrow[z\to-\infty]{} (f_{+})_{is'}\left(e^{\pm iK_{+}z}\right)_{s's}. \tag{3}$$

Historical review

- Single channel scattering problem for one-dimensional Shrodinger equation on the line was solved by Fadeev 1964 [1,2]
- The proof of unitarity of the S-matrix on the half line in the presence of closed scattering channels is given by Newton 1982 [3]
- Some scattering and analytical properties for two-Channel Hamiltonians were revealed by Melgaard 2001 [4,5]
- Multichannel scattering problem on the line for the one-dimensional Shrodinger equation on the line was investigated relatively recent, mostly by Aktosun [6-9], however the unitarity was not proved.

Nevertheless, to our knowledge, the description of properties of the S-matrix, of the Jost solutions, and of the bound states in the general case of multichannel scattering on a line with different thresholds at both left and right infinities is absent in the literature. Our aim is to fill this gap.

Main identities

The Jost solutions $F_\pm^+$ and $F_\pm^-$ constitute bases in the space of solutions of Eq.(1). Consequently,

$$F_+^+ = F_+^- \Phi_+ + F_-^- \Psi_+, \qquad F_-^+ = F_+^- \Psi_- + F_-^- \Phi_-, \tag{4}$$

Let us introduce the transmission matrices $t_{(1,2)}$ and the reflection matrices $r_{(1,2)}$

$$F_+^+ t_{(1)} = F_+^- + F_-^- r_{(1)}, \qquad F_-^- t_{(2)} = F_-^+ + F_+^+ r_{(2)}, \tag{5}$$

Define the S-matrix as

$$S \coloneqq \begin{bmatrix} t_{(1)} & r_{(2)} \\ r_{(1)} & t_{(2)} \end{bmatrix}. \tag{6}$$

Then the S-matrix possesses the symmetries

$$\begin{bmatrix} 0 & K_- \\ K_+ & 0 \end{bmatrix} S = S^T \begin{bmatrix} 0 & K_+ \\ K_- & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & K_- \\ K_+ & 0 \end{bmatrix} \overline{S} = \overline{S}^T \begin{bmatrix} 0 & K_+ \\ K_- & 0 \end{bmatrix},$$
$$\overline{S}^T \begin{bmatrix} K_+ & 0 \\ 0 & K_- \end{bmatrix} S = \begin{bmatrix} K_- & 0 \\ 0 & K_+ \end{bmatrix}. \tag{7}$$

$$\overline{\Phi}_\pm \coloneqq \Phi_\mp, \qquad \overline{\Psi}_\pm \coloneqq \Psi_\mp.$$

The case of all scattering channels are open

Theorem 1. *If λ belongs to none of the cuts of the functions $(K_{\pm})_s, s \in \{1, \dots, N\}$, i.e., when all the scattering channels are open, the S-matrix is unitary*

$$S^{\dagger} \begin{bmatrix} K_{+} & 0 \\ 0 & K_{-} \end{bmatrix} S = \begin{bmatrix} K_{-} & 0 \\ 0 & K_{+} \end{bmatrix}, \qquad (9)$$

Remark. *Introducing the notation*

$$\widetilde{\Phi}_{\pm} := K_{-}^{\frac{1}{2}} \Phi_{\pm} K_{+}^{-\frac{1}{2}}, \quad \tilde{\psi}_{\pm} := K_{-}^{\frac{1}{2}} \Psi_{\pm} K_{+}^{-\frac{1}{2}}, \qquad (10)$$

One can reduce (9) to the standard form $\tilde{S}^{\dagger}\tilde{S} = 1$.

Proposition 1. *If all scattering channels are open, there are no bound states.*

Identities in the subspace of open channels

We split the relations (5) into blocks with respect to the indices s, s' in accordance with splitting into open and closed channels,

$$\begin{aligned}
(F_+^+)_o t_{(1)oo} + (F_+^+)_c t_{(1)co} &= (F_+^-)_o + (F_-^-)_o r_{(1)oo} + (F_-^-)_c r_{(1)co}, \\
(F_+^+)_o t_{(1)oc} + (F_+^+)_c t_{(1)cc} &= (F_+^-)_c + (F_-^-)_o r_{(1)oc} + (F_-^-)_c r_{(1)cc}, \\
(F_-^-)_o t_{(2)oo} + (F_-^-)_c t_{(2)co} &= (F_-^+)_o + (F_+^+)_o r_{(2)oo} + (F_+^+)_c r_{(2)co}, \\
(F_-^-)_o t_{(2)oc} + (F_-^-)_c t_{(2)cc} &= (F_-^+)_c + (F_+^+)_o r_{(2)oc} + (F_+^+)_c r_{(2)cc}.
\end{aligned} \tag{11}$$

Theorem 2. The *S-matrix in the subspace of open channels is unitary.*

$$\begin{aligned}
t^*_{(2)oo} t_{(1)oo} + r^*_{(1)oo} r_{(1)oo} &= 1, \\
t^\dagger_{(1)oo} (K_+)_0 t_{(1)oo} + r^\dagger_{(1)oo} (K_-)_0 r_{(1)oo} &= (K_-)_0.
\end{aligned} \tag{12}$$

Theorem 3. The following condition $\det \Phi_+(\lambda) = 0, \quad \lambda \in \mathbb{R},$

is a necessary and sufficient condition for the existence of bound states of Eq. (1).

An electrodynamic example: wire metamaterial

An non-local effective permittivity tensor in a wire metamaterial reads as

$$\hat{\varepsilon}_{ij} = \varepsilon_0 \delta_{ij} - \boldsymbol{z} \frac{\varepsilon_0 \omega_p^2}{\varepsilon_0 k_0^2 - (\hat{k}_z)^2} \boldsymbol{z}, \quad \boldsymbol{z} = (0,0,1). \tag{13}$$

The standard Maxwell equation in media

$$\left(rot_{ij}^2 - k_0^2 \hat{\varepsilon}_{ij}\right) A_j = 0, \qquad \hat{k}_i\left(\hat{\varepsilon}_{ij} A_j\right) = 0. \tag{14}$$

Introducing the scalar plasmon field Ψ, the non-locality of equations (1) may be resolved

$$(\varepsilon_0 k_0^2 - (\hat{k}_z)^2)\Psi + \omega_p^2 A_3 = 0,$$
$$(\varepsilon_0 k_0^2 \delta_{ij} - rot_{ij}^2)\, A_j + \varepsilon_0 k_0^2 z_i \Psi = 0. \tag{15}$$

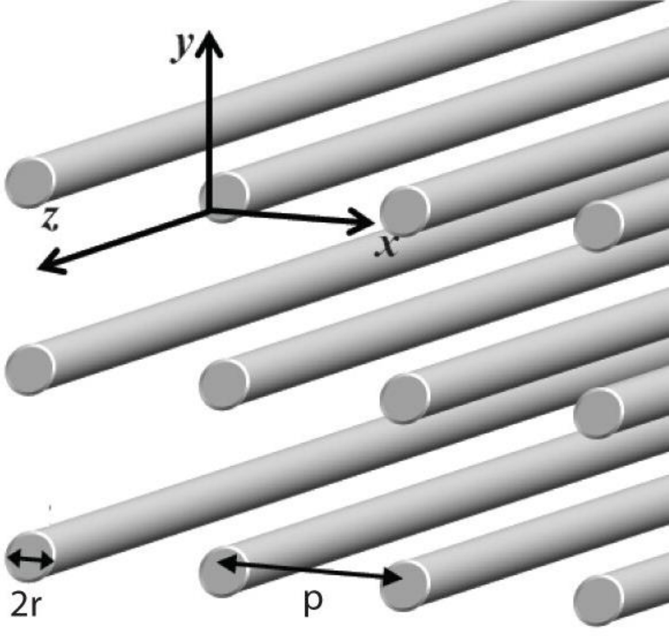

Fig. 1

Unitarity relation holds

$$|r_1|^2 + |r_2|^2 + |t_1|^2 + |t_2|^2 = 1$$

$\boldsymbol{A}_r = (r_1 \boldsymbol{f}_1 + r_2 \boldsymbol{f}_2) e^{i\boldsymbol{k}_\perp \boldsymbol{x}_\perp - ik_3 z}$

$\boldsymbol{A}_t = (t_1 \boldsymbol{f}_1 + t_2 \boldsymbol{f}_2) e^{i\boldsymbol{k}_\perp \boldsymbol{x}_\perp + ik_3 z}$

$\boldsymbol{A}_{in} = \boldsymbol{f}_{in} e^{i\boldsymbol{k}\boldsymbol{x}}$

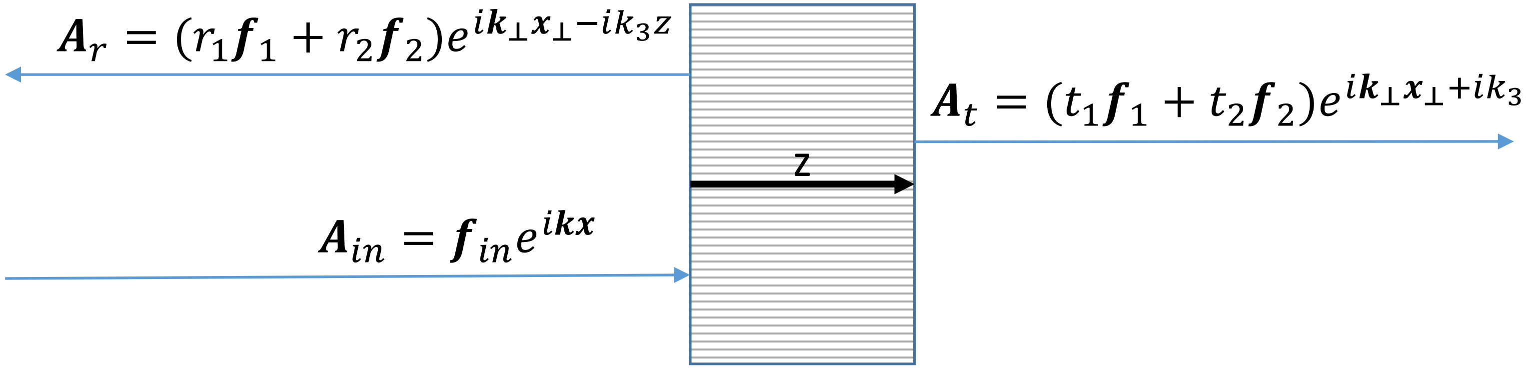

Fig. 2

Summary

- It has been shown that the Jost solutions are the different branches of the same vector-valued analytic function on a double-sheeted Riemann surface.
- The key result of this work is Theorem 2 that proves unitarity of the scattering matrix in the subspace of open channels. The proof of this fact appears to be obtained for the first time.
- The weak version of the Theorem 2, the statement of about unitarity in the case when all scattering channels are open is also proved.
- The condition determining the bound states has been obtained.
- The results of the work are applicable in electrodynamic of continuous media, in particular, for the theories with large spatial dispersion. Corresponding wire-metamaterial example is presented.

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