Differential equations method for expansion of hypergeometric functions

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In quantum field theory an important role is played by various hypergeometric functions. Of particular interest is their close relationship with Feynman loop integrals. The latter are used to calculate higher corrections in perturbation theory to the measurable physical processes. Which becomes especially important now that the accuracy of measurements is increasing. There are many ways to solve Feynman loop integrals using hypergeometric functions. These solutions have the common property that the indices of the hypergeometric function linearly depends on a small parameter. And for practical calculations, it is necessary to obtain a Laurent expansion in this small parameter. In this case, it is desirable that the expansion elements be expressed in terms of well-defined functions that can be calculated with arbitrary precision. In this work we study the expansion of various hypergeometric functions in a Laurent series with respect to a small parameter in terms of multiple-polylogarithms. For this purpose, we mainly use the differential equation method and the Lee algorithm. Specifically, we will be interested in the generalized hypergeometric functions, the Appell and Lauricella functions. In these calculations, a particularly important role is played by the replacement of the variable: rational in one direction and irrational in the other. This issue is discussed with special attention.

PACS: 44.25.+f; 44.90.+c

Introduction

The problem of expansion of hypergeometric functions often arises in quantum field theory when calculating scalar Feynman integrals. It is important that the expansion be expressed in a class of well-defined functions. For such functions we will use so called Goncharov multiple polylogarithms [1,2]. The later can be defined recursively:

$$G(a_1, ..., a_n; x) = \int\limits_0^x \frac{G(a_2, ..., a_n; x')}{x' - a_1} dx', \qquad n > 0 \tag{1}$$

and the recursion starts with $G(;x) = 1$. We also define the regularization rule $G(\underbrace{0, \ldots, 0}_{n}; x) = \frac{\log^n x}{n!}$. For this purpose we will use methods originally developed in quantum field theory in applications for Feynman loop integrals. Such methods include the method of differential equations (DE) [3–7] and

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the Lee algorithm [8]. This possibility for generalized hypergeometric series was mentioned in the work [9].

The paper is organized as follows. First, we describe the main hypergeometric functions under consideration. Next, we give a brief schematic description of how such functions can be expanded. Finally, we come to our conclusion.

Hypergeometric series

Here we will give a brief description of the functions under consideration and the corresponding systems of DE. The simplest example is the generalized hypergeometric function defined as

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n,\ldots,(a_p)_n}{(b_1)_n,\ldots,(b_q)_n}\frac{z^n}{n!} \tag{2}$$

where $(\ )_n$ denotes the Pochhammer symbol. For practical applications, we will be interested in the case when $p = q+1$ and the indices linearly depend on the small parameter ε. We will look for a series expansion with respect to this ε parameter. We will be most interested in the differential equation for ${}_pF_q$ which, as well known, can be written as

$$\left[z(\theta+a_1)(\theta+a_2)\ldots(\theta+a_p)-\theta(\theta+b_1-1)(\theta+b_2-1)\ldots(\theta+b_q-1)\right]{}_pF_q = 0 \tag{3}$$

where $\theta = z\frac{d}{dz}$. This single DE can be naturally rewritten as a system of DE. To do this, we will choose a basis as $J = \{f_0, f_1, \ldots, f_q\}$ with

$$f_0 = {}_pF_q, \qquad f_n = \theta(\theta-1)\ldots(\theta-n+1)f_0 = z^n\frac{d^n f_0}{dz^n}. \tag{4}$$

For $n < q$, one can easily write the differential relations as $\frac{d}{dz}f_n = \frac{n}{z}f_n + \frac{1}{z}f_{n+1}$, for $n = q$ the differential relation can be obtained directly from the equation (3). As a result, we get a system of DE in the form

$$\frac{d}{dz}J = \left(\frac{\mathbf{A}}{z} + \frac{\mathbf{B}}{z-1}\right)J \tag{5}$$

where $\mathbf{A}$ and $\mathbf{B}$ are some $p\times p$ matrices. Note that this system is immediately in Fuchsian form due to the basis choice. The boundary conditions can be easily obtained directly from the definition (2) and we get $J\Big|_{z\to 0} \sim \{1, 0, \ldots, 0\}$. A generalization of hypergeometric functions to the case of many variables are the Lauricella functions [10]

$$F_A^{(n)}(\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}m_1!\ldots m_n!}x_1^{m_1}\ldots x_n^{m_n}, \tag{6}$$

$$F_B^{(n)}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\ldots(\alpha_n)_{m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n}, \quad (7)$$

$$F_D^{(n)}(\alpha;\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n}. \quad (8)$$

In a particular case $n = 2$, these functions are reduced to the more well-known Appell functions $F_A^{(2)} = F_2, F_B^{(2)} = F_3, F_D^{(2)} = F_1$. In order to obtain a system of differential equations for the Lauricella functions, we write the basis as

$$\left\{\theta_{x_{j_1}}\ldots\theta_{x_{j_k}}F_i \;\middle|\; 0 \le k \le n,\ j_1 < j_2 < \cdots < j_k\right\}, \qquad i = A, B, \quad (9)$$

and

$$\left\{F_D,\ \theta_{x_j}F_D \;\middle|\; j = 1,\ldots,n\right\}, \quad (10)$$

with $\theta_a = \partial/\partial a$. As a simple example, consider the Appell function F_1. The basis will be $J_1 = \left\{F_1, x\frac{\partial}{\partial x}F_1, y\frac{\partial}{\partial y}F_1\right\}$, then the DE system will have the form

$$\frac{\partial}{\partial x}J_1 = \left(\frac{\mathbf{A}_0}{x} + \frac{\mathbf{A}_1}{x-1} + \frac{\mathbf{A}_y}{x-y}\right)J_1. \quad (11)$$

The boundary conditions for the Lauricella functions can be obtained recursively. In practical calculations we consider functions up to $n = 4$, with more complex functions computational difficulties arise.

Expansion of hypergeometric series

To obtain the decomposition, we use the Lee algorithm [8] to reduce the resulting system of DE to ε-form. It can then be easily integrated in terms of polylogarithms, taking into account the boundary conditions, which gives the final answer. Lee algorithm only works if all eigenvalues of matrix residues are integers (ignoring the linear part in ε). If this condition is not met, then one need to use a special variable replacement that will convert all eigenvalues to integers. If the eigenvalues are half-integer at points z_1 and z_2, then one can use the variable replacement $z = (x^n z_2 - z_1)/(x^n - 1)$, where n - the least common denominator of eigenvalues at these points and x is the new variable. If there are more than two "non-integer" points, then the replacement may not exist and each such case should be considered individually. In the case of generalized hypergeometric functions of one variable, the system has only three singular points, so we can classify all the variable changes that we used. Define n - the least common denominator for all eigenvalues. We consider the following cases

- Case A: $n = 1$ which means all eigenvalues are integers and no variable replacement is required.
- Case B: $n > 1$ and eigenvalues are non-integer at points $z = 0, 1$, then we use variable change $z \to \frac{z_1^n}{1+z_1^n}$.
- Case C: $n > 1$ and eigenvalues are non-integer at points $z = 0, \infty$, then we use variable change $z \to z_2^n$.
- Case D: $n > 1$ and eigenvalues are non-integer at points $z = 1, \infty$, then we use variable change $z \to 1 - z_3^n$.
- Case E: $n = 2$ and eigenvalues are non-integer at points $z = 0, 1, \infty$, then we use variable change $z \to -\frac{4z_4^2}{(z_4^2-1)^2}$.
- Case F: $n > 2$ and eigenvalues are non-integer at points $z = 0, 1, \infty$, then we use transformation matrix $T = z^{-1/n}\mathcal{I}$ and variable change $z \to \frac{z_1^n}{1+z_1^n}$.

If a case falls into this classification, this does not mean that it can be calculated in terms of G-functions. Let k be the number of non-integer upper indices $k = \#\{a_i | i = 1, \ldots, q; a_i \notin \mathbb{Z}\}$, l the number of non-integer lower indices $l = \#\{b_j | j = 1, \ldots, p; b_j \notin \mathbb{Z}\}$. Empirically, we have established that the generalized hypergeometric function cannot be expanded in terms of polylogarithms **by our method** if one of the following conditions is satisfied

1. $n = 2$ and $|k - l| \geq 2$.
2. $n > 2$ and $(k \geq 2$ or $l \geq 2)$ and $\{a_i - a_1, b_j - a_1 | i = 1, \ldots, q, j = 1, \ldots, p\} \not\subset \mathbb{Z}$.
3. $n > 2$, $k = l = 1$ and $a_i - b_j \notin \mathbb{Z}$ where a_i and b_j are two non-integer indices.

For Lauricella functions the situation is similar. Only in this case, the singularities will depend on additional parameters.

Conclusion

In this short note, we briefly describe the method of expanding hypergeometric functions in terms of multiple polylogarithms using the method of differential equations. A classification was given of possible variable changes which can be used in these calculations. A more detailed description of the algorithm as well as a description of the package that automates this algorithm in the Wolfram Mathematica language will be given in future publications. Even though similar packages already exist [11–13] we believe that this work will still be useful to the community. The reason for this is that we consider many cases here that have not been considered previously. They are mainly associated with non-trivial changes of variable.

Acknowledgments

I would like to thank A.I.Onishchenko, R.N.Lee and V.V.Bytev for interesting and stimulating discussions. This work was supported by the Russian Science Foundation, grant number 20-12-00205.

REFERENCES

1. *Goncharov A.B.* Multiple polylogarithms, cyclotomy and modular complexes // Mathematical Research Letters. — 1998. — V. 5. — P. 497–516.

2. *Goncharov A.B.* Multiple polylogarithms and mixed Tate motives // arXiv preprint math/0103059. — 2001.

3. *Kotikov A.* Differential equations method. New technique for massive Feynman diagram calculation // Physics Letters B. — 1991. — V. 254, no. 1. — P. 158 – 164. — URL: http://www.sciencedirect.com/science/article/pii/037026939190413K.

4. *Kotikov A.* Differential equation method. The calculation of N-point Feynman diagrams // Physics Letters B. — 1991. — V. 267, no. 1. — P. 123–127.

5. *Kotikov A.* Differential equations method: the calculation of vertex-type Feynman diagrams // Physics Letters B. — 1991. — V. 259, no. 3. — P. 314–322.

6. *Remiddi E.* Differential equations for Feynman graph amplitudes // Il Nuovo Cimento A (1971-1996). — 1997. — V. 110, no. 12. — P. 1435–1452.

7. *Gehrmann T., Remiddi E.* Differential equations for two-loop four-point functions // Nuclear Physics B. — 2000. — V. 580, no. 1-2. — P. 485–518.

8. *Lee R.N.* Reducing differential equations for multiloop master integrals // JHEP. — 2015. — V. 04. — P. 108. — arXiv:1411.0911 [hep-ph].

9. *Bezuglov M.A., Kotikov A.V., Onishchenko A.I.* On Series and Integral Representations of Some NRQCD Master Integrals // JETP Lett. — 2022. — V. 116, no. 1. — P. 61–69. — arXiv:2205.14115.

10. *Schlosser M.J.* Multiple Hypergeometric Series: Appell Series and Beyond // LHCPhenoNet School: Integration, Summation and Special Functions in Quantum Field Theory. — 2013. — P. 305–324. — arXiv:1305.1966 [math.CA].

11. *Bera S.* `MultiHypExp`: A Mathematica Package For Expanding Multivariate Hypergeometric Functions In Terms Of Multiple Polylogarithms. — 2023. — 6. — arXiv:2306.11718.

12. *Huber T., Maitre D.* HypExp: A Mathematica package for expanding hypergeometric functions around integer-valued parameters // Comput. Phys. Commun. — 2006. — V. 175. — P. 122–144. — arXiv:hep-ph/0507094.

13. *Huber T., Maitre D.* HypExp 2, Expanding Hypergeometric Functions about Half-Integer Parameters // Comput. Phys. Commun. — 2008. — V. 178. — P. 755–776. — arXiv:0708.2443 [hep-ph].