Radiative corrections of the order $\mathcal{O}(\alpha^3 L^3)$ to unpolarized muon decay spectrum
Радиационные поправки порядка $\mathcal{O}(\alpha^3 L^3)$ к спектру распада неполяризованного мюона

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Обсуждается расчет радиационных поправок высших порядков к спектру распада неполяризованного мюона. Приведены результаты для порядков $\mathcal{O}(\alpha^2 L)$, $\mathcal{O}(\alpha^3 L^3)$ и $\mathcal{O}(\alpha^3 L^2)$.

Calculation of higher-order radiative corrections to unpolarized muon decay spectrum is discussed. Results for the orders $\mathcal{O}(\alpha^2 L)$, $\mathcal{O}(\alpha^3 L^3)$ и $\mathcal{O}(\alpha^3 L^2)$ are presented.

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Introduction

The process of muon decay is almost a pure weak-interaction process with small QED, QCD, and possibly new physics additions. In high-precision and high-sensitivity experiments with muons small deviations from the Standard Model pointing to new physics can be seen.

To predict the results of such experiments accurate theoretical predictions are needed. In this work corrections to unpolarized muon decay spectrum up to the order $\mathcal{O}(\alpha^3 L^2)$ are presented.

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Parton distribution functons approach

Parton distribution functions approach is used. This approach came from QCD to QED and allows calculating radiative corrections without direct loop calculation [1].

In parton distribution functions (PDF) approach expansion not only in powers of coupling constant but also in powers of the large logarithm is made. Large logarithm is where μ_F is factorisation scale and μ_R is renormalizaton scale. For muon decay μ_F is taken equal to muon mass, and μ_R is taken equal to electron mass, and $L = \ln \frac{\mu_F^2}{\mu_R^2} = \ln \frac{m_\mu^2}{m_e^2} \approx 10.66$.

In PDF approach parton distribution functions and splitting functions are used. Both types of functions are independent of the process. To calculate radiative corrections to a particular process we have to make a convolution of them with the functions containing information of the process. For muon decay such functions are [2]:

$$
\begin{aligned}
&f_e^{(0)}(z) = z^2(3-2z), \quad f_\gamma^{(0)}(z) = 0\\
&f_e^{(1)}(z) = 2z^2(2z-3)\big(4\zeta(2) - 4\mathrm{Li}_2(z) + 2\ln z^2 - 3\ln z \ln(1-z)\\
&-\ln(1-z)^2\big) + \left(\frac{5}{3} - 2z - 13z^2 + \frac{34}{3}z^3\right)\ln(1-z)\\
&+\left(\frac{5}{3} + 4z - 2z^2 - 6z^3\right)\ln z + \frac{5}{6} - \frac{23}{3}z - \frac{3}{2}z^2 + \frac{7}{3}z^3\\
&f_\gamma^{(1)}(z) = \ln z\left(-\frac{10}{3} + \frac{2}{z} + 4z\right) + \ln(1-z)\left(-\frac{5}{3} + \frac{1}{z} + 2z - 2z^2 + \frac{2}{3}z^3\right)\\
&+\frac{1}{3} - \frac{1}{z} + \frac{35}{12}z - 2z^2 - \frac{1}{4}z^3
\end{aligned}
$$

Evolution equation

Process-independent fragmentation functions were calculated solving the QED parton distribution functions evolution equation:

$$
D_{ba}(x, \frac{\mu_R^2}{\mu_F^2}) = \delta(1-x)\delta_{ba} + \sum_{i=e,\bar{e},\gamma} \int\limits_{\mu_R^2}^{\mu_F^2} \frac{dt\alpha(t)}{2\pi t} \int\limits_x^1 \frac{dy}{y} D_{ia}(y, \frac{\mu_R^2}{t}) P_{bi}\left(\frac{x}{y}, t\right), \quad (1)
$$

by iterations. Details of the calculation of PDFs can be found in [3]. We use the complete expression for running coupling in the $\overline{\mathrm{MS}}$ scheme that can be found in [4].

Muon decay spectrum corrections

The expression for the differential distribution of electrons for unpolarized muon decay reads [5]

$$\frac{d\Gamma}{dz} = \Gamma_0 F(z), \quad \Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3},\ z = \frac{2m_\mu E_e}{m_\mu^2 + m_e^2}, \quad z_0 \leq z \leq 1, \quad z_0 = \frac{2m_\mu m_e}{m_\mu^2 + m_e^2}, \tag{2}$$

where G_F is Fermi coupling constant, E_e and z are energy and energy fraction of the electron, and

$$F(z) = f_e^{(0)}(z) + \sum_{i,j} \alpha^i L^j F_{ij}(z). \tag{3}$$

Unpolarized muon decay spectrum corrections of the lower orders were calculated in the works [5] and [6]. We recalculated the corrections of the orders $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^3 L^3)$. In $\mathcal{O}(\alpha^2 L)$ we have a new term $d_{\gamma e}^{(1)}(x) \otimes P_{e\gamma}^{(0)}$ in D_{ee}, which comes from the iterative solution of the evolution equation. In $\mathcal{O}(\alpha^3 L^3)$ we have corrected the coefficient of $P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}$ in D_{ee}, it should be $\frac{2}{9}$, and not $-\frac{1}{9}$. The expressions for D_{ee} can be found in [3].

To get the NLL contribution of the order $\mathcal{O}(\alpha^3 L^2)$ we have to make convolutions of the fragmentation functions with functions $f_e^i(z)$ and $f_\gamma^i(z)$:

$$\left(f_e^{(0)}(z) + \frac{\alpha}{2\pi} f_e^{(1)}(z)\right) \otimes \big[D_{ee}\big]_T + \left(f_\gamma^{(0)}(z) + \frac{\alpha}{2\pi} f_\gamma^{(1)}(z)\right) \otimes \big[D_{e\gamma}\big]_T, \tag{4}$$

and take only the terms proportional to $\alpha^3 L^2$ from the result. Index T denotes fragmentation function. The results are given below.

$$\begin{aligned}
F_{21}(z) &= -\frac{4405}{216} + \frac{2\zeta_2 z^3}{3} - 9\zeta_2 z^2 + \left(8\zeta_2 z^3 - 12\zeta_2 z^2 - \frac{32z^3}{9} - 19z^2 - 13z - \frac{97}{12}\right)\ln(z) \\
&+12\zeta_2 z + \left(8z^3 - 12z^2\right)\left(-\mathrm{Li}_3(z) + \mathrm{Li}_2(z)\ln(z) + \frac{1}{2}\ln(1-z)\ln^2(z) + \zeta_3\right) \\
&+\left(-\frac{16z^3}{3} + 6z^2 - 6z\right)\mathrm{Li}_2(1-z) + \left(24z^2 - 16z^3\right)\mathrm{Li}_3(1-z) \\
&+\left(16z^3 - 24z^2\right)\mathrm{Li}_2(1-z)\ln(1-z) + \left(8z^3 - 12z^2\right)\mathrm{Li}_2(1-z)\ln(z) - 12z^3\zeta_3 \\
&-\frac{167z^3}{54} + \left(\frac{16z^3}{3} - 12z\right)\ln^2(1-z) + 18z^2\zeta_3 + \frac{449z^2}{9} + \left(12z^2 - 8z^3\right)\ln^3(z) \\
&+\left(-\frac{32z^3}{3} + 11z^2 - 3z - \frac{5}{4}\right)\ln^2(z) + \left(24z^3 - 36z^2\right)\ln(1-z)\ln^2(z) \\
&+\left(12z^2 - 8z^3\right)\ln^2(1-z)\ln(z) + \left(-\frac{8z^3}{9} + \frac{4z^2}{3} - 16z + \frac{2}{3z} - \frac{8}{3}\right)\ln(1-z) \\
&+\left(\frac{8z^3}{3} - 14z^2 + 22z + \frac{20}{3}\right)\ln(1-z)\ln(z) - \frac{1195z}{36} - \frac{3}{z},
\end{aligned} \tag{5}$$

$$
\begin{aligned}
F_{32} ={}& \frac{53623}{1296} + \frac{1}{108z} + \frac{1201z^3}{162} - \frac{2131z^2}{72} - \frac{49z}{2} + \left(8z^3 - 4z^2 - 12z\right)\ln^3(1-z) \\
&+ \left(\frac{92z^3}{9} - \frac{41z^2}{3} - \frac{7z}{3} - \frac{35}{36}\right)\ln^3(z) + \Bigg[\frac{142z^3}{9} + \frac{152z^2}{3} + \frac{161z}{12} + \zeta_2\left(60z^2 - 40z^3\right) \\
&+ \left(4z^3 - 6z^2\right)\ln^2(1-z) + \left(-\frac{56z^3}{3} + 58z^2 + 44z + \frac{125}{6}\right)\ln(1-z) + \frac{37}{8}\Bigg]\ln^2(z) \\
&+ \left(4z^3 - 6z^2\right)\mathrm{Li}_2(1-z)^2 + \zeta_2\left(-6z^2 + 16z + \frac{139}{18}\right) + \zeta_4\left(60z^2 - 40z^3\right) \\
&+\zeta_2^2\left(36z^3 - 54z^2\right) + \left(104z^3 - 156z^2\right)\mathrm{H}(3,0,z) + \left(80z^3 - 120z^2\right)\mathrm{H}(1,2,0,z) \\
&+ \left(104z^3 - 156z^2\right)\mathrm{H}(2,0,0,z) + \left(104z^3 - 156z^2\right)\mathrm{H}(2,1,0,z) \\
&+ \left(64z^3 - 96z^2\right)\mathrm{H}(0,0,0,0,z) + \left(56z^3 - 84z^2\right)\mathrm{H}(1,0,0,0,z) \\
&+ \left(96z^3 - 144z^2\right)\mathrm{H}(1,1,0,0,z) + \left(80z^3 - 120z^2\right)\mathrm{H}(1,1,1,0,z) \\
&+ \left(\frac{136z^3}{9} + \frac{185z^2}{3} - \frac{247z}{3} + \zeta_2\left(60z^2 - 40z^3\right) - \frac{160}{3} - \frac{6}{z}\right)\mathrm{Li}_2(1-z) \\
&+ \ln^2(1-z)\Bigg(-\frac{62z^3}{9} + \frac{37z^2}{6} - \frac{62z}{3} + \zeta_2\left(40z^3 - 60z^2\right) + \left(16z^3 - 24z^2\right)\mathrm{Li}_2(1-z) \\
&- \frac{121}{18} + \frac{1}{z}\Bigg) + \left(32z^3 - 40z^2 + 4z - \frac{10}{3}\right)\mathrm{Li}_3(1-z) + \Bigg(\frac{64z^3}{3} + 72z^2 + 97z \\
&+ \frac{140}{3}\Bigg)\mathrm{Li}_3(z) + \left(72z^2 - 48z^3\right)\mathrm{Li}_4(1-z) + \left(120z^3 - 180z^2\right)\mathrm{Li}_4(z) \\
&+ \left(120z^2 - 80z^3\right)\mathrm{S}_{2,2}(z) + \ln(z)\Bigg[\frac{283z^3}{27} - \frac{799z^2}{12} + \frac{539z}{36} + \left(12z^2 - 8z^3\right)\ln^3(1-z) \\
&+ \left(-\frac{16z^3}{3} - 8z^2 + 36z + 10\right)\ln^2(1-z) + \zeta_2\left(-\frac{100z^3}{3} - 32z^2 - 125z - \frac{160}{3}\right) \\
&+ \left(\frac{8z^3}{3} + 22z^2 + 74z + \frac{185}{6}\right)\mathrm{Li}_2(1-z) + \ln(1-z)\Bigg(-\frac{4z^3}{3} - 11z^2 - \frac{z}{2} \\
&+\zeta_2\left(16z^3 - 24z^2\right) + \left(8z^3 - 12z^2\right)\mathrm{Li}_2(1-z) - \frac{57}{4} + \frac{8}{3z}\Bigg) + \left(48z^2 - 32z^3\right)\zeta_3 + \frac{1261}{108}\Bigg] \\
&+ \ln(1-z)\Bigg[-\frac{155z^3}{27} + \frac{2221z^2}{36} - \frac{677z}{36} + \zeta_2\left(-\frac{20z^3}{3} - 14z^2 + 36z\right) + \Bigg(-32z^3 + \\
&40z^2 - 4z + \frac{10}{3}\Bigg)\mathrm{Li}_2(1-z) + \left(48z^2 - 32z^3\right)\mathrm{Li}_3(1-z) + \left(144z^2 - 96z^3\right)\mathrm{Li}_3(z) \\
&+ \left(88z^3 - 132z^2\right)\zeta_3 - \frac{6281}{108} - \frac{32}{9z}\Bigg] + \left(\frac{8z^3}{3} - 92z^2 - 109z - \frac{125}{3}\right)\zeta_3, \qquad (6) \\
F_{33}(z) ={}& -\frac{619}{1296} + \frac{20z}{9} + \frac{4}{27z} + \frac{16z^3}{81} - \frac{5z^2}{9} + \zeta_2\left(-\frac{16z^3}{3} + \frac{8z^2}{3} + \frac{2z}{3} + \frac{5}{18}\right) \\
&+ \ln(z)\Bigg[\zeta_2\left(12z^2 - 8z^3\right) + \left(\frac{16z^3}{3} - 8z^2\right)\mathrm{Li}_2(z) + \frac{32z^3}{27} + \frac{52z^2}{9} - \frac{67z}{36} - \frac{41}{108} \\
&+ \left(8z^3 - 12z^2\right)\ln^2(1-z) + \left(-\frac{8z^3}{3} + 4z^2 - 4z - \frac{5}{3}\right)\ln(1-z)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\ln(1-z)\Bigg[\zeta_2\left(8z^3-12z^2\right)+\left(8z^3-12z^2\right)\mathrm{Li}_2(1-z)-\frac{32z^3}{27}-\frac{44z^2}{9}+\frac{15z}{2}+\frac{4}{9z}\\
&+\frac{289}{108}\Bigg]+\left(\frac{8z^3}{3}+\frac{4z^2}{3}-\frac{14z}{3}-\frac{35}{18}\right)\mathrm{Li}_2(z)+\left(12z^2-8z^3\right)\mathrm{Li}_3(1-z)\\
&+\left(\frac{8z^3}{3}-4z^2+4z+\frac{5}{3}\right)\ln^2(1-z)+\left(\frac{4z^3}{3}-\frac{2z^2}{3}+\left(2z^2-\frac{4z^3}{3}\right)\ln(1-z)-\frac{z}{2}-\frac{5}{24}\right)\ln^2(z)\\
&+\left(8z^2-\frac{16z^3}{3}\right)\mathrm{Li}_3(z)+\left(4z^2-\frac{8z^3}{3}\right)\ln^3(1-z)+\left(\frac{4z^3}{9}-\frac{2z^2}{3}\right)\ln^3(z). \qquad (7)
\end{aligned}
$$

Conclusion

We have received new results for $\mathcal{O}(\alpha^3L^2)$ order and recalculated results for $\mathcal{O}(\alpha^2L)$ and $\mathcal{O}(\alpha^3L^3)$. Our results have some difference from the works [5, 6] because of correction of some mistakes. Results will be useful for new experiments including searching for new physics [7].

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