

Lectures 1&2: Linear Optics

Valeri Lebedev
JINR & Fermilab

US Particle Accelerator School,
February 2022

Lectures in JINR, spring 2023



Objectives

- Design of linear optics is a very important step in the design of an accelerator
- It determines all major parameters and properties
- In majority of cases the optics design does not require accounting of coupling between different degrees of freedom
 - ◆ And coupling can be considered in the perturbation theory
- However, in the recent years, machines, where different degrees of freedom are strongly coupled, were considered
 - ◆ Examples: Electron and Ionization cooling (including both linear and the ring-based machines), Optical stochastic cooling
- In this lecture we consider basics of linear optics for the coupled and uncoupled optics
 - ◆ We shortly refresh uncoupled optics
 - ◆ Then, having some experience, we consider x-y coupled optics

Uncoupled Betatron Motion

Equations for Uncoupled Motion

- Linearized equation of motion

$$x'' + (K_x^2 + k)x = 0$$

where: $K_x(s) \equiv K_x = eB_y(s) / Pc$, $k(s) \equiv k = eG(s) / Pc$

- In Hamiltonian form

$$\begin{cases} \frac{dx}{ds} = \frac{\partial H}{\partial p} \\ \frac{dp}{ds} = -\frac{\partial H}{\partial x} \end{cases} \quad \text{with} \quad H = \frac{p^2}{2} + (K_x^2 + k)\frac{x^2}{2}$$

- General solution of 2-nd order linear equation

$$x(s) = C(s)x(0) + S(s)\theta(0), \quad \theta(s) \equiv dx / ds$$

where $C(s)$ and $S(s)$ two linear independent solutions

We can rewrite it in matrix form

$$\begin{bmatrix} x(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{bmatrix} x(0) \\ \theta(0) \end{bmatrix} \quad \text{or} \quad \boxed{\mathbf{x}(s) = \mathbf{M}(s)\mathbf{x}(0)}$$

Conservation of the Phase Space Volume

Jacobian does not depend on time

$$\frac{d}{ds} \left(\frac{\partial(p, q)}{\partial(p_0, q_0)} \right) = \frac{d}{ds} \left(\begin{array}{cc} \frac{\partial}{\partial p_0} \left(p_0 + \frac{dp}{ds} ds \right) & \frac{\partial}{\partial p_0} \left(x_0 + \frac{dx}{ds} ds \right) \\ \frac{\partial}{\partial x_0} \left(p_0 + \frac{dp}{ds} ds \right) & \frac{\partial}{\partial x_0} \left(x_0 + \frac{dx}{ds} ds \right) \end{array} \right) = \frac{d}{ds} \left(\begin{array}{cc} 1 - \frac{\partial^2 H}{\partial s \partial p} ds & \frac{\partial^2 H}{\partial p^2} ds \\ \frac{\partial^2 H}{\partial^2 x} ds & 1 + \frac{\partial^2 H}{\partial x \partial p} ds \end{array} \right) = 0$$

where we used

$$\left\{ \begin{array}{l} \frac{dx}{ds} = \frac{\partial H}{\partial p} \\ \frac{dp}{ds} = -\frac{\partial H}{\partial x} \end{array} \right.$$

⇒ The phase space volume is conserved in the course of motion and, consequently, $|\mathbf{M}| = 1$

- The conservation of the phase space volume is also justified for multidimensional motion.
It is called **Liouville theorem**

Betatron Motion in a Ring

- Arbitrary turn-by-turn betatron motion at a given place may be presented through eigen-vectors

$$\mathbf{x}_n = \text{Re}\left(\Lambda_1^n (A_1 \mathbf{v}_1) + \Lambda_2^n (A_2 \mathbf{v}_2)\right) \text{ where } \mathbf{M}\mathbf{v}_k = \Lambda_k \mathbf{v}_k, \quad k = 1, 2$$

- ◆ Stable betatron motion requires $|\Lambda_k| = 1 \Rightarrow \Lambda_2 = \Lambda_1^*$ (since real \mathbf{M})
- ◆ Introduce betatron frequencies so that $\Lambda_{1,2} = e^{\pm i\mu}$

Corresponding betatron tune (fractional part): $Q = \mu / 2\pi$

- Description of betatron motion for the entire ring

- ◆ The eigen-vector $\mathbf{v}(s) = \mathbf{M}(0, s)\mathbf{v}$ is the eigen-vector for the total ring transfer matrix for coordinate s .
- ◆ Then we normalize the eigen-vectors so that

$$\mathbf{v}(s) = \mathbf{M}(0, s)\mathbf{v}(0)e^{-i\mu(s)}$$

and require $\text{Im}(v_1(s)) = 0$ and $\mathbf{v}^+(s)\mathbf{S}\mathbf{v}(s) = -2i$, where

$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then we can describe the entire ring betatron motion

$$\mathbf{x}(s) = \sqrt{2I} \text{Re}\left(e^{i(\psi - \mu(s))} \mathbf{v}\right)$$

where the action I and the betatron phase ψ determine initial part. pos.

The Eigen-vector Parameterization

■ Parametrize the eigen-vector

$$\mathbf{v} \equiv \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \equiv \mathbf{v}(s) = \begin{bmatrix} \sqrt{\beta(s)} \\ -\frac{i + \alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix}, \quad \begin{cases} \mathbf{v}_1 = \mathbf{v} \\ \mathbf{v}_2 = \mathbf{v}^* \end{cases}$$

- ◆ we define that $\text{Im}(v_1(s)) = 0$
- ◆ The eigen-vectors are orthogonal and correctly normalized

$$\left\{ \begin{array}{l} \mathbf{v}^+ \mathbf{S} \mathbf{v} = \begin{bmatrix} \sqrt{\beta(s)} & \frac{i - \alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\beta(s)} \\ -\frac{i + \alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} = -2i \\ \mathbf{v}^T \mathbf{S} \mathbf{v} = 0 \quad \text{or} \quad \mathbf{v}_2^+ \mathbf{S} \mathbf{v}_1 = 0 \end{array} \right.$$

Courant-Snyder Invariant

The betatron amplitude (maximum particle displacement) = $\sqrt{2I\beta}$

The maximum angle = $\sqrt{\frac{2I}{\beta}(1+\alpha^2)}$

The maximum angle for $x=0$ is achieved when

$$\sqrt{2I} \operatorname{Re} \left(\left[\begin{array}{c} \sqrt{\beta(s)} \\ -\frac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{array} \right] e^{i\pi/2} \right) = \sqrt{2I} \operatorname{Re} \left(\left[\begin{array}{c} i \\ \frac{1-i\alpha(s)}{\sqrt{\beta(s)}} \end{array} \right] \right)$$

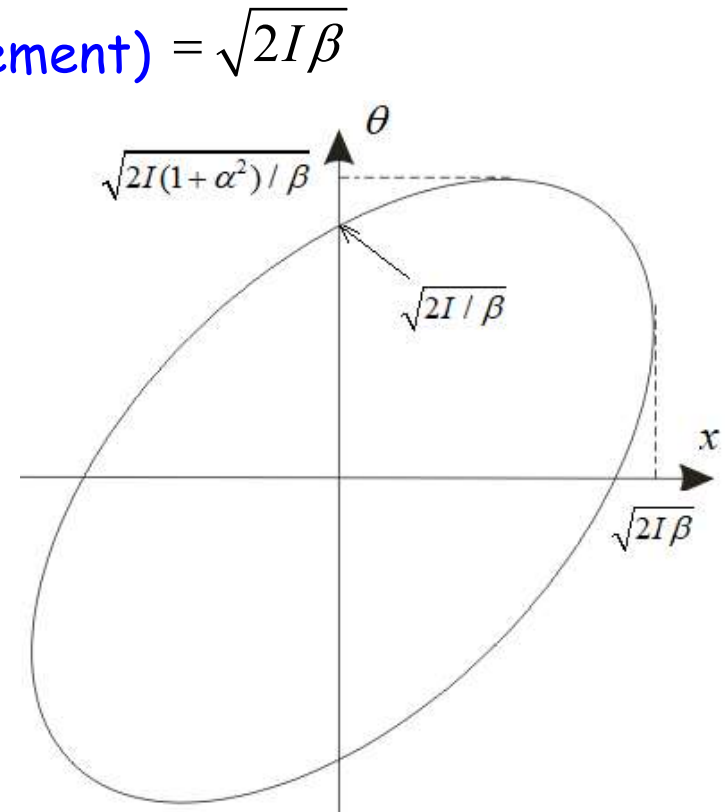
⇒ Local angular spread: $\theta_m = \sqrt{\frac{2I}{\beta}}$

■ Finding action from the known x and θ

$$\mathbf{v}^+ \mathbf{S} \left[\mathbf{x} = \sqrt{2I} \left(\frac{e^{i\psi} \mathbf{v} + CC}{2} \right) \right] \xrightarrow{\text{orthogonality condition}} \mathbf{v}^+ \mathbf{S} \mathbf{x} = -i\sqrt{2I} \rightarrow I = \frac{1}{2} |\mathbf{v}^+ \mathbf{S} \mathbf{x}|^2$$

■ Courant-Snyder invariant

$$2I = |\mathbf{v}^+ \mathbf{S} \mathbf{x}|^2 = \beta\theta^2 + 2\alpha x\theta + \frac{1+\alpha^2}{\beta} x^2$$

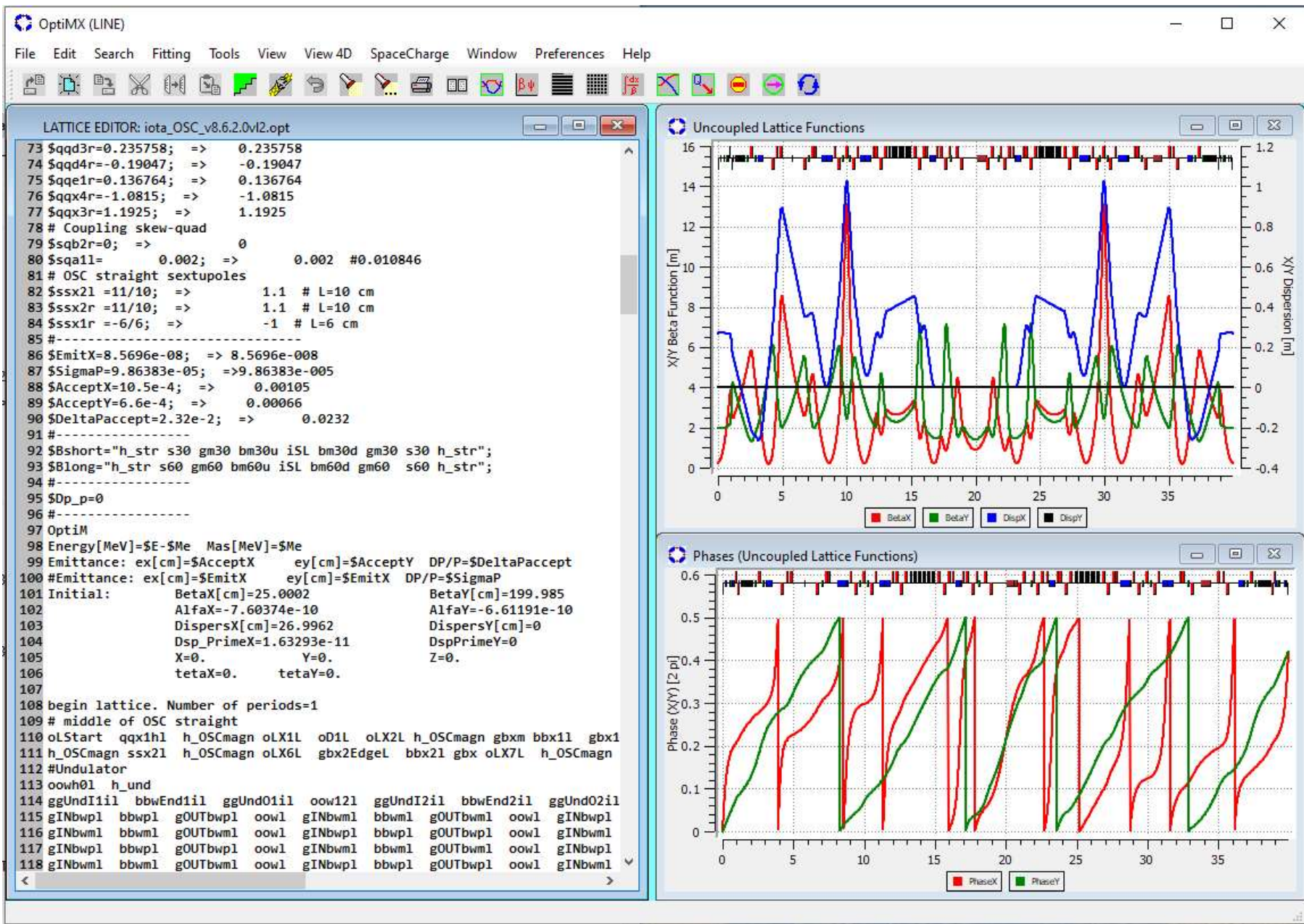


Remember that: $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Computation of Machine Optics

Software for Computation of machine optics

- There are many computer codes allowing one to compute beam optics
- I mention 3 of them
 1. MAD -> MAD-8 -> MADX - supported by CERN
<https://mad.web.cern.ch/mad/>
 2. Elegant - supported by ANL
<https://www.aps.anl.gov/Accelerator-Operations-Physics/Software#elegant>
 3. OptiMX - supported by Fermilab
<https://home.fnal.gov/~ostiguy/OptiM/> (temporary link because of Fermilab security:
<https://www.dropbox.com/s/56l4nctnwegf7w7/OptimX64-20210526-setup.exe?dl=0>)
- In this course we will be using OptiM
 - ◆ Interactive, GUI driven, easy to learn
 - ◆ Operates on major computer platforms: Windows, Unix, MAC
 - ◆ Free installation, Easy to install
 - ◆ Online help (documentation)
- Input file consists of:
 - ◆ Math header
 - ◆ Main body starting from keyword OptiM. It includes: (1) beam parameters, (2) element sequence, (3) parameters of elements, (4) service blocks



Computations can be done in a ring and beam line modes

X-Y Coupled Betatron Motion

Equations for X-Y Coupled Motion

■ Linearized equations of motion

$$\begin{cases} x'' + (K_x^2 + k)x + \left(N - \frac{1}{2}R'\right)y - Ry' = 0 \\ y'' + (K_y^2 - k)y + \left(N + \frac{1}{2}R'\right)x + Rx' = 0 \end{cases}$$

where: $K_{x,y}(s) \equiv K_{x,y} = eB_{y,x}(s) / Pc$, $k(s) \equiv k = eG(s) / Pc$, $N = eG_s / Pc$, $R = eB_s / Pc$

■ In Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \left(K_x^2 + k + \frac{R^2}{4}\right)\frac{x^2}{2} + \left(K_y^2 - k + \frac{R^2}{4}\right)\frac{y^2}{2} + Nxy + \frac{R}{2}(yp_x - xp_y)$$

the corresponding canonical momenta are:

$$\begin{cases} p_x = x' - \frac{R}{2}y \\ p_y = y' + \frac{R}{2}x \end{cases}$$

In matrix form: $\hat{\mathbf{x}} = \mathbf{R}\mathbf{x}$

$$\hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Form of Equations for X-Y Coupled Motion

$$H = \frac{1}{2} \hat{\mathbf{x}}^T \mathbf{H} \hat{\mathbf{x}} \quad \text{where}$$

$$\mathbf{H} = \begin{bmatrix} K_x^2 + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^2 - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix}$$

Then the motion equations are

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{U} \mathbf{H} \hat{\mathbf{x}}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Properties of matrix \mathbf{U} (called unit symplectic matrix)
 - $\mathbf{U}^T \mathbf{U} = \mathbf{I}$ and $\mathbf{U} \mathbf{U} = -\mathbf{I}$, where \mathbf{I} is the identity matrix
- Similar to the single dimensional motion we introduce 4-dimensional transfer matrix, $\hat{\mathbf{x}} = \hat{\mathbf{M}}(0, s) \hat{\mathbf{x}}_0$, for the 2-dimensional motion
 The cap here and below denotes that we consider the transfer matrix which uses canonical momenta instead of angles

Motion Symplecticity

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{U}\mathbf{H}\hat{\mathbf{x}}$$

■ Lagrange invariant

$$\frac{d}{ds}(\hat{\mathbf{x}}_1^T \mathbf{U} \hat{\mathbf{x}}_2) = \frac{d\hat{\mathbf{x}}_1^T}{ds} \mathbf{U} \hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1^T \mathbf{U} \frac{d\hat{\mathbf{x}}_2}{ds} = \hat{\mathbf{x}}_1^T \mathbf{H}^T \mathbf{U}^T \mathbf{U} \hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1^T \mathbf{U} \mathbf{U} \mathbf{H} \hat{\mathbf{x}}_2 = 0$$

$$\Rightarrow \hat{\mathbf{x}}_1^T \mathbf{U} \hat{\mathbf{x}}_2 = \text{const}$$

■ Motion symplecticity

Substituting $\hat{\mathbf{x}} = \hat{\mathbf{M}}\hat{\mathbf{x}}_0$ into above equation one obtains

$$\hat{\mathbf{x}}_1^T \mathbf{U} \hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1^T \hat{\mathbf{M}}(0, s)^T \mathbf{U} \hat{\mathbf{M}}(0, s) \hat{\mathbf{x}}_2 = \text{const}$$

As the above equation is satisfied for any $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ it yields

$$\hat{\mathbf{M}}^T \mathbf{U} \hat{\mathbf{M}} = \mathbf{U}$$

This property is called symplecticity and matrix $\hat{\mathbf{M}}$ symplectic

■ $\hat{\mathbf{M}}(0, s)^T \mathbf{U} \hat{\mathbf{M}}(0, s)$ is antisymmetric

\Rightarrow Only six $((n^2-n)/2 = 6)$ of these equations are independent (4 diagonal ones are identities). Thus, out of 16 matrix elements of matrix \mathbf{M} the motion symplecticity leaves only 10 elements linearly independent

Symplecticity of Eigen-Vectors

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_k = \lambda_k \hat{\mathbf{v}}_k, \quad k = 1, \dots, 4$$

- For any two eigen-vectors the symplecticity yields the identity

$$0 = \lambda_j \hat{\mathbf{v}}_j^T \mathbf{U} (\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_i \hat{\mathbf{v}}_i) = (\hat{\mathbf{M}}\hat{\mathbf{v}}_j)^T \mathbf{U} \hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_j \hat{\mathbf{v}}_j^T \mathbf{U} \lambda_i \hat{\mathbf{v}}_i = (1 - \lambda_j \lambda_i) \hat{\mathbf{v}}_j^T \mathbf{U} \hat{\mathbf{v}}_i$$

where we substituted: $(\hat{\mathbf{M}}\hat{\mathbf{v}}_j)^T \mathbf{U} \hat{\mathbf{M}}\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_j^T \hat{\mathbf{M}}^T \mathbf{U} \hat{\mathbf{M}}\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_j^T \mathbf{U} \hat{\mathbf{v}}_i$

It determines that the eigen-values of stable motion always appear in two reciprocal pairs, and, consequently, the four eigen-values split into two complex conjugate pairs: λ_1, λ_1^* and λ_2, λ_2^*

For $\lambda_1 \neq \lambda_2$ (non-degenerate case) we obtain the orthogonality condition

$$\hat{\mathbf{v}}_1^+ \mathbf{U} \hat{\mathbf{v}}_1 \neq 0,$$

$$\hat{\mathbf{v}}_2^+ \mathbf{U} \hat{\mathbf{v}}_2 \neq 0,$$

$$\hat{\mathbf{v}}_i^+ \mathbf{U} \hat{\mathbf{v}}_j = 0 \quad \text{if } i \neq j,$$

$$\hat{\mathbf{v}}_i^T \mathbf{U} \hat{\mathbf{v}}_j = 0,$$

Normalizing
eigen-vectors
we obtain:

$$\hat{\mathbf{v}}_1^+ \mathbf{U} \hat{\mathbf{v}}_1 = -2i, \quad \hat{\mathbf{v}}_2^+ \mathbf{U} \hat{\mathbf{v}}_2 = -2i,$$

$$\hat{\mathbf{v}}_1^T \mathbf{U} \hat{\mathbf{v}}_1 = 0, \quad \hat{\mathbf{v}}_2^T \mathbf{U} \hat{\mathbf{v}}_2 = 0,$$

$$\hat{\mathbf{v}}_2^T \mathbf{U} \hat{\mathbf{v}}_1 = 0, \quad \hat{\mathbf{v}}_2^+ \mathbf{U} \hat{\mathbf{v}}_1 = 0.$$

Out of 2 complex conjugated vectors we choose one which satisfies the normalization condition. Normalization of CC vector has different sign.

Parameterization of Eigen-vectors

- Betatron motion is described similar to 1D case:

$$\hat{\mathbf{x}}(s) == \text{Re}\left(\sqrt{2I_1}\hat{\mathbf{v}}_1(s)e^{-i(\psi_1+\mu_1(s))} + \sqrt{2I_2}\hat{\mathbf{v}}_2(s)e^{-i(\psi_2+\mu_2(s))}\right)$$

- There are 2 popular parameterizations: Edwards-Teng and Mais-Ripken
 - ◆ Here we shortly consider the extended Mais-Ripken

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \frac{\sqrt{\beta_{1x}}}{i(1-u) + \alpha_{1x}} \\ \frac{\sqrt{\beta_{1x}}}{\sqrt{\beta_{1y}} e^{iv_1}} \\ \frac{i u + \alpha_{1y}}{\sqrt{\beta_{1y}}} e^{iv_1} \end{bmatrix} \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} \frac{\sqrt{\beta_{2x}} e^{iv_2}}{i u + \alpha_{2x}} \\ \frac{\sqrt{\beta_{2x}}}{\sqrt{\beta_{2y}}} \\ \frac{i(1-u) + \alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

- The betatron motion is described by 10 linearly independent functions:
 - 4 β -functions, 4 α -functions, and 2 betatron phase advances
- Symplecticity allows one to compute functions u , v_1 & v_2 from known α 's & β 's. However, there are 4 solutions for their values and additional information is required to choose α 's and β 's.

In practice, first, we find the eigen-vectors from known transfer matrix, and, then unique solutions for all 4D-Twiss functions

4D Ellipsoid in the Phase Space

$$\hat{\mathbf{x}} = \text{Re}\left(A_1 e^{-i\psi_1} \hat{\mathbf{v}}_1 + A_2 e^{-i\psi_2} \hat{\mathbf{v}}_2\right) = A_1 \left(\hat{\mathbf{v}}_1' \cos \psi_1 + \hat{\mathbf{v}}_1'' \sin \psi_1\right) + A_2 \left(\hat{\mathbf{v}}_2' \cos \psi_2 + \hat{\mathbf{v}}_2'' \sin \psi_2\right)$$

- Rewrite it in matrix form

$$\hat{\mathbf{x}} = \hat{\mathbf{V}} \mathbf{A} \xi_A \quad \text{where} \quad \hat{\mathbf{V}} = \begin{bmatrix} \hat{\mathbf{v}}_1' & -\hat{\mathbf{v}}_1'' & \hat{\mathbf{v}}_2' & -\hat{\mathbf{v}}_2'' \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \quad \xi_A = \begin{bmatrix} \cos \psi_1 \\ -\sin \psi_1 \\ \cos \psi_2 \\ -\sin \psi_2 \end{bmatrix}$$

- To obtain a 4D ellipsoid which

includes all particles we need to

account that the mode amplitudes are interdependent. To account it

we put:
$$\xi = \begin{bmatrix} \cos \psi_1 \cos \psi_3 \\ -\sin \psi_1 \cos \psi_3 \\ \cos \psi_2 \sin \psi_3 \\ -\sin \psi_2 \sin \psi_3 \end{bmatrix}$$

so that vector ξ stays at 3D surface with unit radius, i. e. $(\xi, \xi) = 1$

- ◆ Substituting \mathbf{x} in this equation we obtain: $\hat{\mathbf{x}}^T \left((\hat{\mathbf{V}} \mathbf{A})^{-1} \right)^T (\hat{\mathbf{V}} \mathbf{A})^{-1} \hat{\mathbf{x}} = 1$

- ◆ Matrix symplecticity yields $\hat{\mathbf{V}}^{-1} = \mathbf{U}^T \hat{\mathbf{V}}^T \mathbf{U}$ using this equation we

finally obtain:
$$\hat{\mathbf{x}}^T \hat{\mathbf{E}} \hat{\mathbf{x}} = 1, \quad \hat{\mathbf{E}} = \mathbf{U} \hat{\mathbf{V}} \hat{\mathbf{E}}' \hat{\mathbf{V}}^T \mathbf{U}^T, \quad \hat{\mathbf{E}}' = \mathbf{A}^{-1} \mathbf{A}^{-1}$$

1D and 2D Emittances

- We define the beam emittance as a product of the ellipsoid semi-axes (omitting the factor $\pi^2/2$ correcting for the real 4D volume of the ellipsoid):

$$\varepsilon_{4D} = \frac{1}{\sqrt{\hat{\mathbf{E}}'_{11}\hat{\mathbf{E}}'_{22}\hat{\mathbf{E}}'_{33}\hat{\mathbf{E}}'_{44}}} = \frac{1}{\sqrt{\det(\hat{\mathbf{E}}')}}$$

- Consequently:

$$\varepsilon_1\varepsilon_2 = \varepsilon_{4D}, \quad \hat{\mathbf{E}}' = \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}$$

- Gaussian distribution: $f(\hat{\mathbf{x}}) = \frac{1}{4\pi^2\varepsilon_1\varepsilon_2} \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{E}}\hat{\mathbf{x}}\right)$

- Second order moments

$$\hat{\Sigma}_{ij} \equiv \overline{\hat{x}_i\hat{x}_j} = \int \hat{x}_i\hat{x}_j f(\hat{\mathbf{x}})d\hat{x}^4 = \frac{1}{4\pi^2\varepsilon_1\varepsilon_2} \int \hat{x}_i\hat{x}_j \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{E}}\hat{\mathbf{x}}\right)d\hat{x}^4$$

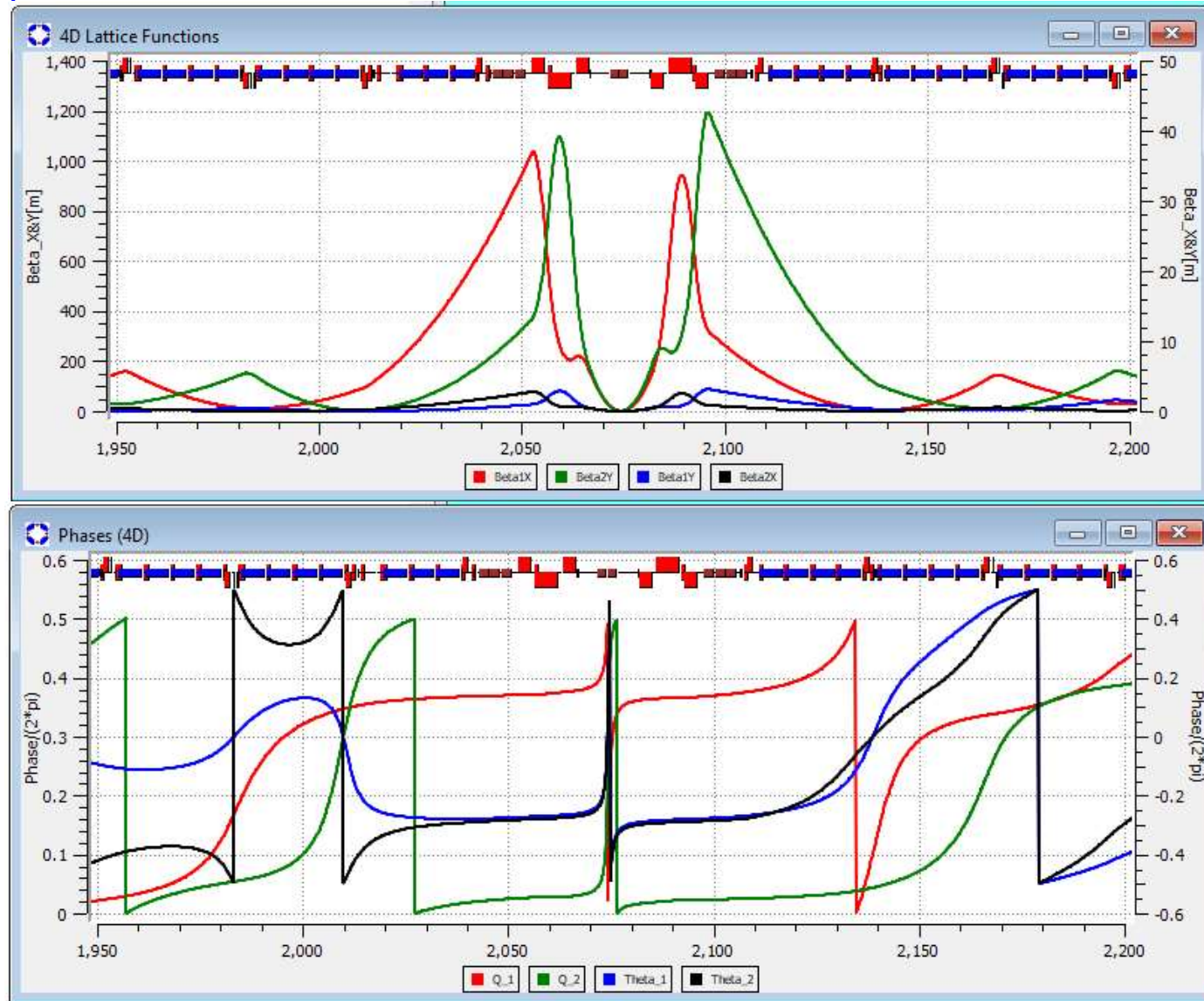
To carry out the integration we use a transform $\hat{\mathbf{y}} = \hat{\mathbf{V}}^{-1}\hat{\mathbf{x}}$. It reduces matrix to the diagonal form. $\Rightarrow \hat{\Sigma} = \hat{\mathbf{V}}\hat{\mathbf{E}}'^{-1}\hat{\mathbf{V}}^T = \hat{\mathbf{E}}^{-1}$

General Remarks

- ε_1 and ε_2 are the motion invariants - they are conserved
- In practical applications the longitudinal magnetic field at boundaries of elements is zero. Consequently, the difference between variables with and without caps disappears.

OptiMX 4D Calculations

- 4D Twiss parameters for Tevatron near B0 (CDF detector)



Perturbations of Uncoupled Betatron Motion

Perturbed Betatron Motion in Uncoupled Case

- To simplify equations, we transit to new variables

$$x = \frac{X}{\sqrt{\beta}}, \quad p = \beta \frac{d}{ds} \frac{X}{\sqrt{\beta}} = \beta \left(\frac{1}{\sqrt{\beta}} \frac{dX}{ds} - \frac{X}{2\beta^{3/2}} \frac{d\beta}{ds} \right) = \sqrt{\beta} \theta + \alpha \frac{X}{\sqrt{\beta}}$$

- In the new variables the motion description is greatly simplified.

Accounting that $d\mu/ds = 1/\beta$ we obtain

$$\frac{d^2 X}{d\mu^2} = -X, \quad \mathbf{M} = \begin{bmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{bmatrix}$$

Consequently, the unperturbed solution is characterized by $\beta = 1, \alpha = 1$

- Choose perturbed initial particle coordinates as following:

$$\mathbf{X}_0 = \sqrt{\varepsilon} \operatorname{Re} \left(\begin{bmatrix} \sqrt{\hat{\beta}} \\ i + \hat{\alpha} \\ -\frac{1}{\sqrt{\hat{\beta}}} \end{bmatrix} e^{i\psi} \right)$$

⇒ Dependence of beam size on μ is

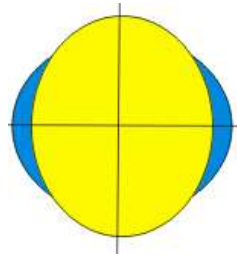
$$A(s) = \sqrt{\varepsilon} \operatorname{Re} \left(\left(\sqrt{\hat{\beta}} \cos \mu - \frac{i + \hat{\alpha}}{\sqrt{\hat{\beta}}} \sin \mu \right) e^{i\psi} \right)_{\max} = \sqrt{\varepsilon \left(\left(\sqrt{\hat{\beta}} \cos \mu - \frac{\hat{\alpha}}{\sqrt{\hat{\beta}}} \sin \mu \right)^2 + \left(\frac{\sin \mu}{\sqrt{\hat{\beta}}} \right)^2 \right)}$$

$$A(s) = \sqrt{\varepsilon \left(c^2 \hat{\beta} + s^2 \left(\frac{1 + \hat{\alpha}^2}{\hat{\beta}} \right) - 2\hat{\alpha}cs \right)} = \sqrt{\varepsilon \left(\frac{1 + \cos(2\mu)}{2} \hat{\beta} + \frac{1 - \cos(2\mu)}{2} \left(\frac{1 + \hat{\alpha}^2}{\hat{\beta}} \right) - \hat{\alpha} \sin(2\mu) \right)}$$

Perturbed Betatron Motion (2)

- The beam size oscillates at the double betatron frequency

$$A(s) = \sqrt{\frac{\varepsilon}{2} \left(\left(\hat{\beta} + \frac{1 + \hat{\alpha}^2}{\hat{\beta}} \right) + \left(\hat{\beta} - \frac{1 + \hat{\alpha}^2}{\hat{\beta}} \right) \cos(2\mu) - 2\hat{\alpha} \sin(2\mu) \right)}$$



Consequently, the perturbed beta-function oscillates at the double betatron frequency as well. Here $\hat{\beta} = 1 + \Delta\beta / \beta$.

What is missed in the Lecture?

- Not all calculations are shown in detail
- Edwards-Teng parameterization
- How to find eigen-vectors from matrices Σ and Ξ and vice versa
- How to express a transfer matrix from known Twiss parameters or eigen vectors and betatron phase advances

- These details are not required to follow other lectures

- Look for details in:
 - ◆ V. A. Lebedev (Fermilab), S. A. Bogacz (Jefferson Lab), “Betatron motion with coupling of horizontal and vertical degrees of freedom”, <https://arxiv.org/abs/1207.5526>
 - ◆ or “Accelerator Physics at the Tevatron Collider”, edited by V. Lebedev and V. Shiltsev, Springer, 2014.

Problems

1. For uncoupled betatron motion prove that the normalization of eigen-vectors, $\hat{\mathbf{v}}_k^+ \mathbf{S} \hat{\mathbf{v}}_k = -2i$, yields that $d\mu / ds = 1 / \beta$. (For the proof use top Eq. of page 7)
2. Prove that if \mathbf{v} is the eigen-vector for matrix \mathbf{M} corresponding to the one turn matrix starting at $s=0$ (point 1) then the vector $\mathbf{M}_{12} \mathbf{v}$ will be the eigen vector of the transfer matrix corresponding to the point 2. Here \mathbf{M}_{12} is the transfer matrix from point 1 to point 2.
3. Find 2D analog of Courant-Snyder invariant
4. Prove that matrix $\hat{\mathbf{V}} = \begin{bmatrix} \hat{v}_1', -\hat{v}_1'', \hat{v}_2', -\hat{v}_2'' \end{bmatrix}$ is symplectic
5. Fill missed calculations in computation $\hat{\Sigma} = \hat{\mathbf{V}} \hat{\mathbf{E}}'^{-1} \hat{\mathbf{V}}^T = \hat{\mathbf{E}}^{-1}$
6. Prove that for a symplectic matrix, defined by the following equation $\hat{\mathbf{M}}^T \mathbf{U} \hat{\mathbf{M}} = \mathbf{U}$, its determinant is $|\hat{\mathbf{M}}| = 1$, $\hat{\mathbf{M}}^{-1} = \mathbf{U}^T \hat{\mathbf{M}}^T \mathbf{U}$ and the matrix also satisfies to $\hat{\mathbf{M}} \mathbf{U} \hat{\mathbf{M}}^T = \mathbf{U}$.

7. Assuming that the motion after exit from KRION ion source is uncoupled and described uncoupled Twiss-parameters find equations describing the horizontal and vertical rms sizes in the downstream beam transport for two below cases.
- (1) Ions exit at the axis of magnetic field. Beam parameters at the ion source center: magnetic field - B_0 , ion rms beam size - σ , transverse temperature - T .
- (2) Now add that the ions exit solenoid with offset r_0 directed at angle θ from the horizontal plane.