Cosmology. Theory
Lecture #1

Dmitry Gorbunov

Institute for Nuclear Research of RAS, Moscow

XXIII Baikal Summer School
on Physics of Elementary Particles and Astrophysics,

JINR-ISU,
Irkutsk region, Bol'shie Koty

Standard Model: Success and Problems

Gauge fields (interactions): γ, $W^{\pm}$, Z, g

Three generations of matter: $L = \binom{\nu_L}{e_L}$, e_R; $Q = \binom{u_L}{d_L}$, d_R, u_R

- Describes
  - all experiments dealing with electroweak and strong interactions
- Does not describe
  - Neutrino oscillations
  - Dark matter (Ω_{DM})
  - Baryon asymmetry (Ω_B)
  - Inflationary stage
  - Dark energy (Ω_Λ)
  - Strong CP: ? (boundary terms, new topology, ...)
  - Gauge hierarchy: ? (No new scales!)
  - Quantum gravity

Try to explain all above

“Natural” units in particle physics

$$\hbar = c = k_{\rm B} = 1$$

measured in GeV: energy E, mass M, temperature T

$m_p = 0.938$ GeV, $\quad 1\ \text{K} = 8.6 \times 10^{-14}$ GeV

measured in GeV^{-1}: time t, length L

$1\ \text{s} = 1.5 \times 10^{24}\ \text{GeV}^{-1}, \quad 1\ \text{cm} = 5.1 \times 10^{13}\ \text{GeV}^{-1}$

Gravity (General Relativity): $V(r) = -G\frac{m_1 m_2}{r}$ $\qquad [G] = M^{-2}$

$$M_{\rm Pl} = 1.2 \times 10^{19}\ \text{GeV} = 22\ \mu\text{g} \qquad G \equiv \frac{1}{M_{\rm Pl}^2}$$

"Natural" units in cosmology

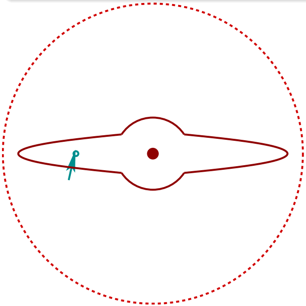
$$1 \text{ Mpc} = 3.1 \times 10^{24} \text{ cm}$$

$1 \text{ AU} = 1.5 \times 10^{13}$ cm — mean Earth-to-Sun distance

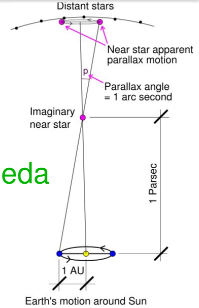
$1 \text{ ly} = 0.95 \times 10^{18}$ cm — distance light travels in one year

$1 \text{ yr} = 3.16 \times 10^{7}$ s

$1 \text{ pc} = 3.3 \text{ ly} = 3.1 \times 10^{18}$ cm — distance to object which has a parallax angle of one arcsec

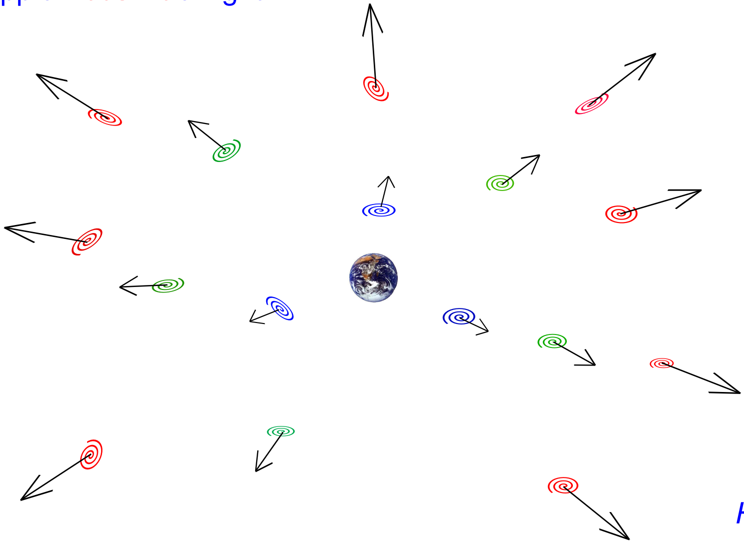


100 AU — Solar system size
1.3 pc — nearest-to-Sun stars
1 kpc — size of dwarf galaxies
50 kpc — distance to dwarves
0.8 Mpc — distance to Andromeda
1-3 Mpc — size of clusters
15 Mpc — distance to Virgo



Universe is expanding

Doppler redshift of light



$L \propto a(t)$

$n \propto a^{-3}(t)$

$H(t) = \frac{\dot{a}(t)}{a(t)}$

Hubble parameter

Hubble Law

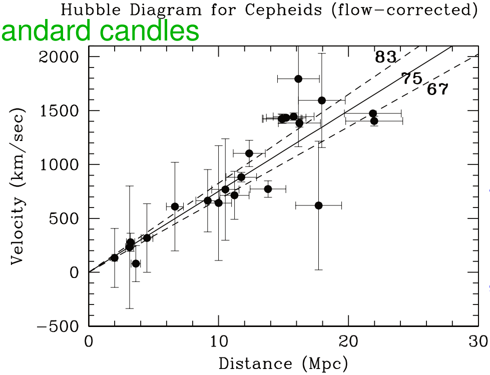
$H(t_0)\, r = v_r$

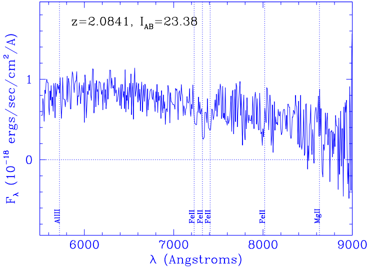
Expansion: redshift z $\qquad \lambda_{\text{abs.}}/\lambda_{\text{em.}} \equiv 1+z$

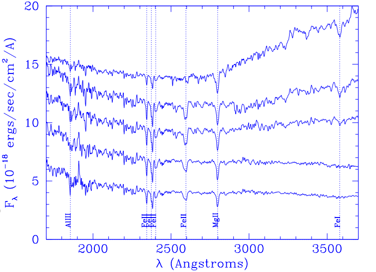
$z \ll 1$ Hubble law : $z = H_0 r$

$$H_0 = h\cdot 100\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}} \qquad h = 0.705 \pm 0.015$$

standard candles



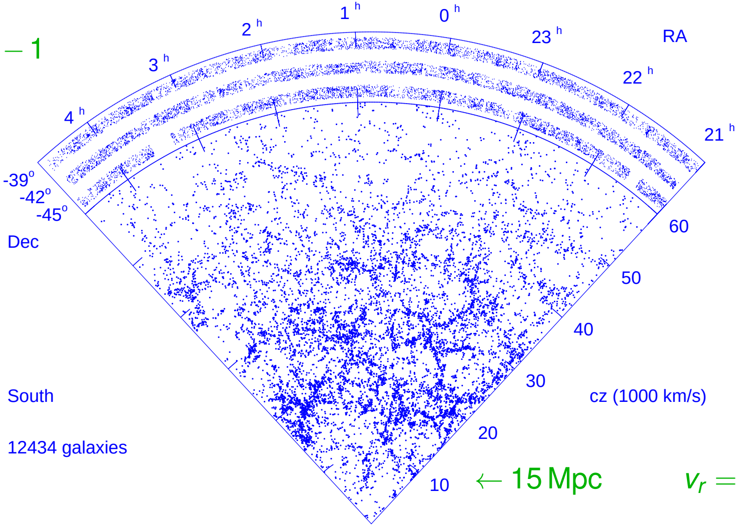




Universe is homogeneous and isotropic

redshift

$$z \equiv \frac{\lambda_{detector}}{\lambda_{source}} - 1$$



$\leftarrow$ 15 Mpc

$$v_r = c\,z$$

The Universe: age & geometry & energy density

$$[H_0] = L^{-1} = t^{-1}$$

time scale: $t_{H_0} = H_0^{-1} \approx 14 \times 10^9$ yr — age of our Universe

spatial scale: $l_{H_0} = H_0^{-1} \approx 4.3 \times 10^3$ Mpc — size of the visible Universe

t_{H_0} is in agreement with various observations

homogeneity and isotropy in 3d:

flat, spherical or hyperbolic

Observations: "very" flat $l_{H_0}/R_{curv} < 0.1$

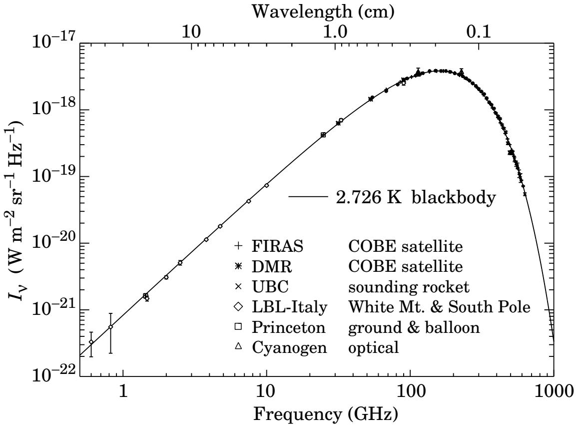
order-of-magnitude estimate: $GM_U/l_U \sim G\rho_0 l_{H_0}^3/l_{H_0} \sim 1$

flat Universe

$$\rho_c = \frac{3}{8\pi} H_0^2 M_{Pl}^2 \approx 0.53 \times 10^{-5} \frac{\text{GeV}}{\text{cm}^3}$$

$\longrightarrow$ 5 protons in each 1 m^3

Universe is occupied by “thermal” photons



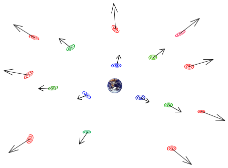
$T_0 = 2.726$ K

the spectrum (shape and normalization!) is thermal

$n_\gamma = 411$ cm^{-3}

Conclusions from observations

The Universe is homogeneous, isotropic, hot and expanding...



Conclusions

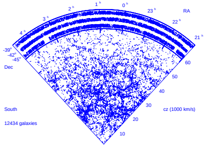
- interval between events gets modified

$$\Delta s^2 = c^2\,\Delta t^2 - a^2(t)\,\Delta \mathbf{x}^2$$

in GR expansion is described by the Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2 = H^2(t) = \frac{8\pi}{3}\,G\rho^{\text{energy}}_{\text{density}}$$

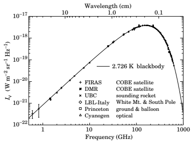
$$\rho^{\text{energy}}_{\text{density}} = \rho_{\text{radiation}} + \rho_{\text{matter}} + \dots$$



- in the past the matter density was higher, our Universe was "hotter" filled with electromagnetic plasma

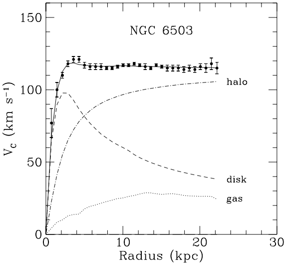
$$\rho_{\text{matter}} \propto 1/a^3(t)\,,\;\; \rho_{\text{radiation}} \propto 1/a^4(t)\,,\;\; \rho_{\text{curvature}} \propto 1/a^2(t)$$

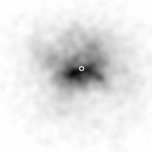
certainly known up to $T \sim 1\,\text{MeV} \sim 10^{10}\,\text{K}$



Universe content from astrophysics

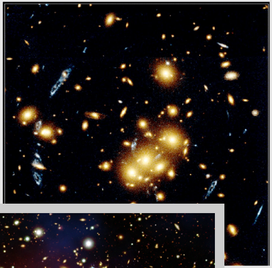
Rotational curves





X-rays from centers of galaxy clusters

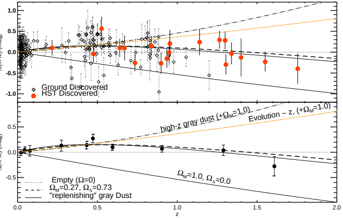
Gravitational lensing



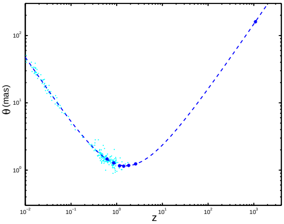
“Bullet” cluster

Universe content from cosmology

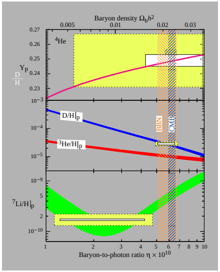
Standard candles

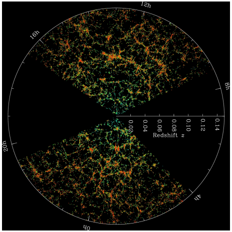


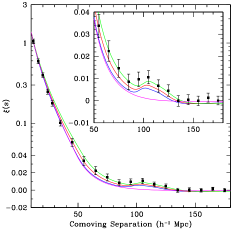
Angular distance

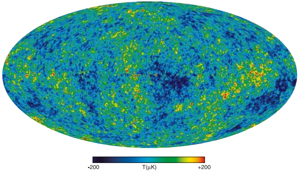


Nucleosynthesis









Large Scale Structures

Baryon acoustic oscillations

CMB anisotropy

Dark Matter Properties $p = 0$

(If) particles:

1. stable on cosmological time-scale
2. nonrelativistic long before RD/MD-transition (either Cold or Warm, $v_{RD/MD} \lesssim 10^{-3}$ at $T \sim 1$ eV)
3. (almost) collisionless
4. (almost) electrically neutral

If were in thermal equilibrium: $M_X \gtrsim 1$ keV

If not: for bosons

$\lambda = 2\pi/(M_X v_X)$, in a galaxy $v_X \sim 0.5 \cdot 10^{-3} \longrightarrow M_X \gtrsim 3 \cdot 10^{-22}$ eV

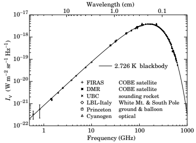
for fermions

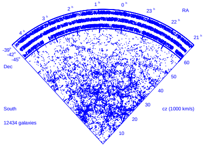
Pauli blocking: $M_X \gtrsim 750$ eV

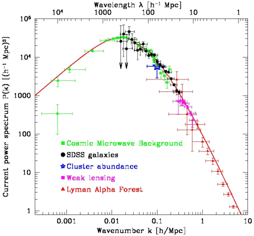
$$f(\mathbf{p},\mathbf{x}) = \frac{\rho_X(\mathbf{x})}{M_X} \cdot \frac{1}{\left(\sqrt{2\pi} M_X v_X\right)^3} \cdot e^{-\frac{\mathbf{p}^2}{2M_X^2 v_X^2}}\Big|_{\mathbf{p}=0} \leq \frac{g_X}{(2\pi)^3}$$

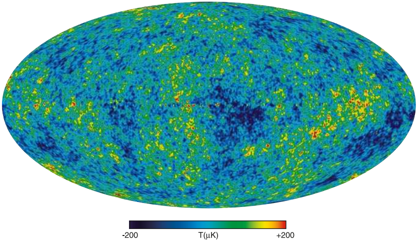
Key observable: matter perturbations

- CMB is isotropic, but "up to corrections, of course..."
  1. Earth movement with respect to CMB
  $$\frac{\Delta T_{\text{dipole}}}{T} \sim 10^{-3}$$
  2. More complex anisotropy!
  $$\frac{\Delta T}{T} \sim 10^{-4} - 10^{-5}$$
- There were matter inhomogenities $\Delta\rho/\rho \sim \Delta T/T$ at the stage of recombination ($e + p \to \gamma + H^*$)
- Jeans instability in the system of gravitating particles at rest $\Longrightarrow$ $\Delta\rho/\rho \nearrow$ $\Longrightarrow$ galaxies (CDM halos)



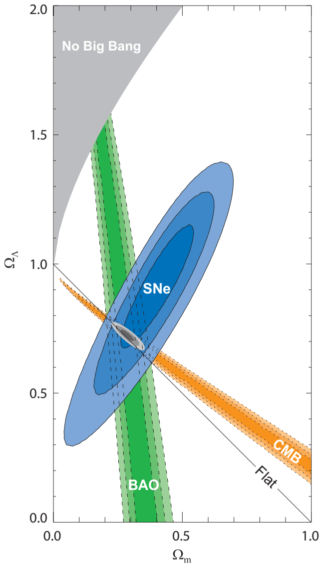






-

Astrophysical and cosmological data are in agreement



$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi}{3} G \rho_{\text{density}}^{\text{energy}}$$

$$\rho_{\text{density}}^{\text{energy}} = \rho_{\text{radiation}} + \rho_{\text{matter}}^{\text{ordinary}} + \rho_{\text{matter}}^{\text{dark}} + \rho_\Lambda$$

$$\rho_{\text{radiation}} \propto 1/a^4(t) \propto T^4(t)\,, \quad \rho_{\text{matter}} \propto 1/a^3(t)$$

$$\rho_\Lambda = \text{const}$$

$$\frac{3H_0^2}{8\pi G} = \rho_{\text{density}}^{\text{energy}}(t_0) \equiv \rho_c \approx 0.53 \times 10^{-5}\,\frac{\text{GeV}}{\text{cm}^3}$$

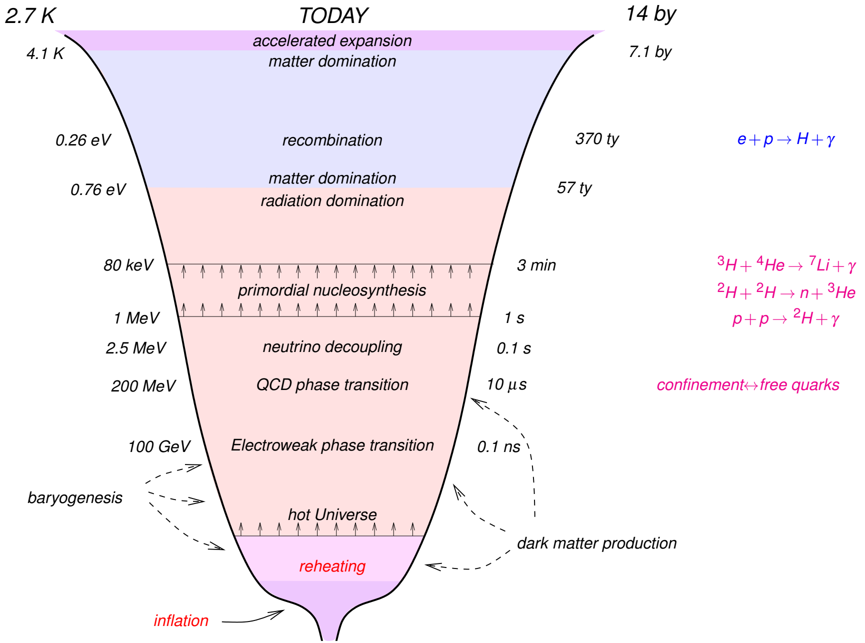
radiation: $\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_c} = 0.5 \times 10^{-4}$

Baryons (H, He): $\Omega_B \equiv \frac{\rho_B}{\rho_c} = 0.046$

Neutrino: $\Omega_\nu \equiv \frac{\sum \rho_{\nu_i}}{\rho_c} < 0.01$

Dark matter: $\Omega_{DM} \equiv \frac{\rho_{DM}}{\rho_c} = 0.28$

Dark energy: $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = 0.68$


2.7 K
TODAY
14 by
accelerated expansion
4.1 K
matter domination
7.1 by
0.26 eV
recombination
370 ty
e + p → H + γ
matter domination
0.76 eV
57 ty
radiation domination
80 keV
3 min
³H + ⁴He → ⁷Li + γ
primordial nucleosynthesis
²H + ²H → n + ³He
1 MeV
1 s
p + p → ²H + γ
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 µs
confinement↔free quarks
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter production
reheating
inflation

Friedmann equation for the present Universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\left(\rho_{\mathrm{M}} + \rho_{rad} + \rho_{\Lambda} + \rho_{\mathrm{curv}}\right)$$

$$\frac{8\pi}{3} G\rho_{\mathrm{curv}} = -\frac{\varkappa}{a^2}\,, \quad \rho_c \equiv \frac{3}{8\pi G} H_0^2$$

$$\rho_c = \rho_{\mathrm{M},0} + \rho_{rad,0} + \rho_{\Lambda,0} = \rho_c = 0.53 \cdot 10^{-5} \frac{\mathrm{GeV}}{\mathrm{cm}^3}\,,$$

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_c}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{\mathrm{M}} \left(\frac{a_0}{a}\right)^3 + \Omega_{rad} \left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda}\right]$$

Simple tasks to be solved ✍

1. Refine the estimate of the age of our Universe

$$t_0 = \frac{1}{H_0} = 14 \times 10^9 \, \text{years}$$

2. When (z_{acc}, t_{acc}, T_γ – ?) did deceleration-acceleration transition happen?
3. When (z_{EQ}, t_{EQ}, T_γ – ?) did matter-radiation transition (Equality) happen?

   Hint: neutrino contribution to radiation energy density is 70% of photon contribution
4. Find the time of Electroweak phase transition, $T \sim 100\,\text{GeV}$

   The early Universe to be tested at LHC . . .

   Hint: all SM particles contribute to energy density as about 50 photon species

Convention of GR description

metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu\ , \qquad \mu,\nu = 0,1,2,3 \qquad \text{signature}\ \ (+,-,-,-)$$

$$x^\mu \to x'^\mu(x^\mu)\ : \qquad B'^\mu_{\nu\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}B^\sigma_{\tau\rho}(x)$$

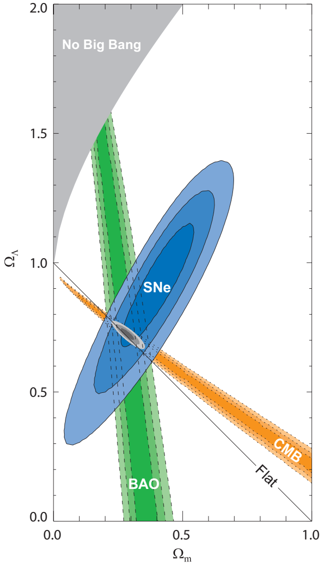
invariant volume $\int d^4x\sqrt{-g}$

$$\int d^4x \ \longrightarrow\ \int d^4x\sqrt{-g}$$

$g_{\mu\nu}(x)$ is the dynamical variable $T_{\mu\nu} \propto g_{\mu\nu}\Lambda$

$$\int d^4x\sqrt{-g}\ \Lambda$$

Dark Energy: nonclumping matter?



- estimates of Matter contribution confined in galaxies and clusters
  $\rho_c - \rho_M \neq 0$ but the Universe is flat, so
  $\rho_{curv} \simeq 0$
- corrections to the Hubble law : red shift – brightness curves for standard candles (SN Ia)
- The age of the Universe
- CMB anisotropy, large scale structures (galaxy clusters formation), etc

$$\rho_\Lambda = 0.73\rho_c$$

$$\rho_\Lambda \sim 10^{-5}\ \text{GeV/cm}^3 \sim \left(10^{-11.5}\ \text{GeV}\right)^4$$

Convention of GR description

metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \,, \qquad \mu,\nu = 0,1,2,3 \qquad \text{signature } (+,-,-,-)$$

$$x^\mu \to x'^\mu(x^\mu) \;: \qquad B'^{\mu}_{\nu\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}B^\sigma_{\tau\rho}(x)$$

Christoffel symbols:

$$\tilde{A}^\mu(\tilde{x}) = A^\mu(x) - \Gamma^\mu_{\nu\lambda}(x)A^\nu(x)dx^\lambda$$

Covarinat derivative

$$A^\mu(\tilde{x}) - \tilde{A}^\mu(\tilde{x}) = \nabla_\nu A^\mu \cdot dx^\nu$$

$$\nabla_\nu A^\mu(x) = \partial_\nu A^\mu(x) + \Gamma^\mu_{\nu\lambda}(x)A^\lambda$$

If $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$ and $\nabla_\nu g_{\mu\lambda} = 0$ $\qquad A_\mu \equiv A^\nu g_{\mu\nu}$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})$$

FLRW metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) dl^2 = dt^2 - a^2(t)\gamma_{ij} dx^i dx^j \,,$$

Special frame: different parts look similar

it is comoving: world lines of particles at rest are geodesics,

derive it ✍

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0$$

particle trajectory $x^\mu(\tau)$, choosing $\tau = s$, we define $u^\mu = dx^\mu/ds$

$$S = -m\int ds = -m\int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} d\tau$$

$$\gamma_{ij} \approx \delta_{ij}$$

Photons in the expanding Universe

$$S = -\frac{1}{4}\int d^4x \sqrt{-g} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}$$

$dt = a d\eta$ conformally flat metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij} dx^i dx^j \longrightarrow ds^2 = a^2(\eta)[d\eta^2 - \delta_{ij} dx^i dx^j]$$

$$S = -\frac{1}{4}\int d^4x\, \eta^{\mu\nu} \eta^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}\,, \qquad A_\mu^{(\alpha)} = e_\mu^{(\alpha)} \mathrm{e}^{ik\eta - i\mathbf{kx}}\,, \quad k = |\mathbf{k}|$$

$$\Delta x = 2\pi/k\,, \quad \Delta\eta = 2\pi/k$$

$$\lambda(t) = a(t)\Delta x = 2\pi\frac{a(t)}{k}\,, \quad T = a(t)\Delta\eta = 2\pi\frac{a(t)}{k}$$

Redshift and the Hubble law $\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} \equiv \lambda_i(1+z(t_i))$

$$\mathbf{p}(t) = \frac{\mathbf{k}}{a(t)}\;,\;\; \omega(t) = \frac{k}{a(t)}$$

for not very distant objects $\quad 1\ \text{pc} \approx 3\ \text{ly}$

$$a(t_i) = a_0 - \dot{a}(t_0)(t_0 - t_i) \;\;\longrightarrow\;\; a(t_i) = a_0[1 - H_0(t_0 - t_i)]$$

$$z(t_i) = H_0(t_0 - t_i) = H_0 r\;, \quad z \ll 1$$

$$H_0 = h \cdot 100\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}}\;, \quad h = 0.675 \pm 0.005$$

similar reddening for other relativistic particles (small H, $\dot{H}$, etc.)

$\mathbf{p} = \frac{\mathbf{k}}{a(t)}$ is true for massive particles as well

Gas of free particles in the expanding Universe

homogeneous gas $dN = f(\mathbf{p}, t) d^3\mathbf{X} d^3\mathbf{p} = \text{const}$

in comoving coordinates:

$$dN = f(\mathbf{k}, \eta) d^3\mathbf{x} d^3\mathbf{k} = \text{const}\,, \quad d^3\mathbf{x} = \text{const}\,, \quad d^3\mathbf{k} = \text{const}$$

Hence $f(\mathbf{k}, \eta) = f(\mathbf{k}) = \text{const}$

comoving volume equals physical volume

$$d^3\mathbf{x} d^3\mathbf{k} = d^3(a\mathbf{x}) d^3\left(\frac{\mathbf{k}}{a}\right) = d^3\mathbf{X} d^3\mathbf{p}$$

$$f(\mathbf{p}, t) = f(\mathbf{k}) = f(a(t) \cdot \mathbf{p})\,.$$

$$t = t_i\ : \quad f_i(\mathbf{p}) = f_i\left(\frac{\mathbf{k}}{a(t_i)}\right) \ \longrightarrow \ f(\mathbf{p}, t) = f_i\left(\frac{\mathbf{k}}{a(t_i)}\right) = f_i\left(\frac{\mathbf{k}}{a(t_i)} \frac{a(t)}{a(t)}\right)$$

Massless bosons (photons) fermions

$$f_i(\mathbf{p}) = f_{\rm Pl}\left(\frac{|\mathbf{p}|}{T_i}\right) = \frac{1}{(2\pi)^3}\frac{1}{e^{|\mathbf{p}|/T_i} - 1}$$

$$f(\mathbf{p},t) = f_{\rm Pl}\left(\frac{a(t)|\mathbf{p}|}{a_i T_i}\right) = f_{\rm Pl}\left(\frac{|\mathbf{p}|}{T_{eff}(t)}\right)$$

$$T_{eff}(t) = \frac{a_i}{a(t)}T_i$$

decoupling at $T \gg m$: neutrinos, hot(warm) dark matter

decoupling at $T \ll m$: thermally produced cold dark matter (e.g. WIMPs)

$f(\mathbf{p}) = \frac{1}{(2\pi)^3}\exp\left(-\frac{m-\mu_i}{T_i}\right)\exp\left(-\frac{a^2(t)\mathbf{p}^2}{2ma_i^2 T_i}\right)$ calculate $f_0(p)$ for neutrinos

$$f(\mathbf{p},t) = \frac{1}{(2\pi)^3}\exp\left(-\frac{m-\mu_{eff}}{T_{eff}}\right)\exp\left(-\frac{\mathbf{p}^2}{2mT_{eff}}\right)$$

$$T_{eff}(t) = \left(\frac{a_i}{a(t)}\right)^2 T_i\,, \qquad \frac{m-\mu_{eff}(t)}{T_{eff}} = \frac{m-\mu_i}{T_i}$$

Convention of GR description

metric inv: $\int d^4x\sqrt{-g}$

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu\ , \qquad \mu,\nu = 0,1,2,3 \qquad \text{signature}\ \ (+,-,-,-)$$

$$x^\mu \to x'^\mu(x^\mu)\ : \qquad B'^{\mu}_{\nu\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}B^\sigma_{\tau\rho}(x)$$

Christoffel symbols: $\nabla_\nu A^\mu(x) = \partial_\nu A^\mu(x) + \Gamma^\mu_{\nu\lambda}(x)A^\lambda(x)$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})$$

Riemann tensor

$$\nabla_\mu\nabla_\nu A^\lambda - \nabla_\nu\nabla_\mu A^\lambda = A^\sigma R^\lambda_{\sigma\mu\nu}\ , \quad R^\mu{}_{\nu\lambda\rho} = \partial_\lambda\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}$$

Ricci tensor and scalar

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}\ , \qquad R \equiv g^{\mu\nu}R_{\mu\nu} = R^{\lambda\mu}{}_{\lambda\mu}$$

Einstein equations

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \ ,$$

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R \ : \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

$$R_{00} = -3\frac{\ddot{a}}{a}\,, \quad R_{0i} = 0\,, \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2\varkappa)\gamma_{ij}\,,$$

$$R = g^{\mu\nu}R_{\mu\nu} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\varkappa}{a^2}\right)$$

$T_{\mu\nu}$: macroscopic description

$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - g_{\mu\nu}p$

$\frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}$

liquid with $\rho(t)$ and $p(t)$

in the comoving frame $u^0 = 1$, $\mathbf{u} = 0$

$$T_{\mu\nu} = diag(\rho,\ a^2(t)\gamma_{ij}p)$$

Friedmann equation for the present Universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\left(\rho_{\rm M} + \rho_{rad} + \rho_\Lambda + \rho_{\rm curv}\right)$$

$$\frac{8\pi}{3} G\rho_{\rm curv} = -\frac{\varkappa}{a^2}\,, \quad \rho_c \equiv \frac{3}{8\pi G} H_0^2$$

$$\rho_c = \rho_{\rm M,0} + \rho_{rad,0} + \rho_{\Lambda,0} = \rho_c = 0.53 \cdot 10^{-5} \frac{\rm GeV}{\rm cm^3}\,,$$

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_c}\,, \qquad \frac{a_0}{a} = 1 + z$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda + \Omega_{curv}\left(\frac{a_0}{a}\right)^2\right]$$

Obs: Equality ($\Omega_M = \Omega_{rad}$, $z \approx 3000$), Acceleration ($\ddot{a} = 0$); Age (14 Gy)

Homogeneous and isotropic 3d manifolds

$$dl^2 = d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$r(\rho) = \begin{cases} R\sin(\rho/R)\,, & \text{3-sphere} \\ \rho\,, & \text{3-plane} \\ R\sinh(\rho/R)\,, & \text{3-hyperboloid} \end{cases}$$

ρ is a geodesic distance; $S = 4\pi r^2(\rho)$; $\Delta\theta = \frac{l}{r(\rho)}$

Brightness–redshift dependence in the Universe

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

coordinate distance $\chi = \int_{t_i}^{t_0} \frac{dt}{a(t)}$ $\qquad z(t) = \frac{a_0}{a(t)} - 1$

$$\chi(z) = \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{\Omega_M (z'+1)^3 + \Omega_\Lambda + \Omega_{curv}(z'+1)^2}}$$

$$a_0^2 H_0^2 \Omega_{curv} = 1\,, \quad \Omega_M + \Omega_\Lambda + \Omega_{curv} = 1$$

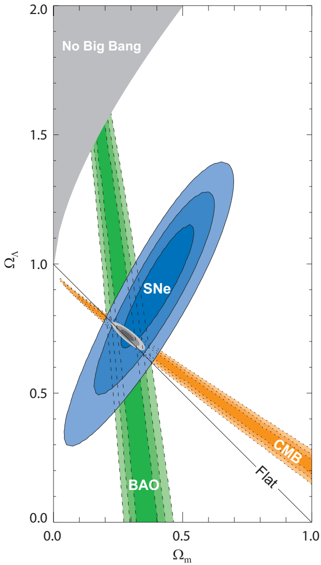
$$S(z) = 4\pi r^2(z)\,, \qquad r(z) = a_0 \sinh\chi(z)$$

detector: $N_\gamma \propto S^{-1}$, $\omega = \omega_i/(1+z)$, $dt_0 = (1+z)dt_i$

brightness (energy flux measured by a detector) is

$$J = \frac{L}{(1+z)^2 S(z)} \equiv \frac{L}{4\pi r_{ph}^2}\,, \quad r_{ph} = (1+z)\cdot r(z)$$

Astrophysical and cosmological data are in agreement



$$\left(\frac{\dot a}{a}\right)^2 = H^2(t) = \frac{8\pi}{3} G\rho^{\text{energy}}_{\text{density}}$$

$$\rho^{\text{energy}}_{\text{density}} = \rho_{\text{radiation}} + \rho^{\text{ordinary}}_{\text{matter}} + \rho^{\text{dark}}_{\text{matter}} + \rho_\Lambda$$

$$\rho_{\text{radiation}} \propto 1/a^4(t) \propto T^4(t)\,, \quad \rho_{\text{matter}} \propto 1/a^3(t)$$

$$\rho_\Lambda = \text{const}$$

$$\frac{3H_0^2}{8\pi G} = \rho^{\text{energy}}_{\text{density}}(t_0) \equiv \rho_c \approx 0.53 \times 10^{-5}\,\frac{\text{GeV}}{\text{cm}^3}$$

radiation: $\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_c} = 0.5 \times 10^{-4}$

Baryons (H, He): $\Omega_B \equiv \frac{\rho_B}{\rho_c} = 0.046$

Neutrino: $\Omega_\nu \equiv \frac{\sum \rho_{\nu_i}}{\rho_c} < 0.01$

Dark matter: $\Omega_{DM} \equiv \frac{\rho_{DM}}{\rho_c} = 0.28$

Dark energy: $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = 0.68$

Initial singularity (Bang!)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho\,, \qquad p = w\rho\,, \qquad \textit{repeat}$$

dust: $p = 0$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^3}\,, \quad a(t) = const\cdot(t-t_s)^{2/3}\,, \quad \rho(t) = \frac{\text{const}}{(t-t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{2}{3t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{1}{6\pi G}\frac{1}{t^2}$$

radiation: $p = \frac{1}{3}\rho$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^4}\,, \quad a(t) = const\cdot(t-t_s)^{1/2}\,, \quad \rho(t) = \frac{\text{const}}{(t-t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{3}{32\pi G}\frac{1}{t^2}$$

Entropy $T^{\nu}_{\mu} = (p+\rho)u_{\mu}u^{\nu} - \delta^{\nu}_{\mu}p$

$$\nabla_{\mu} T^{\mu}_{0} = 0 \quad \longrightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

for equation of state

$$p = p(\rho)$$

of the primordial plasma we obtain

$$-3d(\ln a) = \frac{d\rho}{p+\rho} = d(\ln s)$$

entropy is conserved in a comoving volume

$$sa^3 = \text{const}$$

For the visible part of the Universe: $S \sim s_{\gamma,0} \cdot l_H^3 \sim 10^{88}$

Horizon problem $l_H(t)$

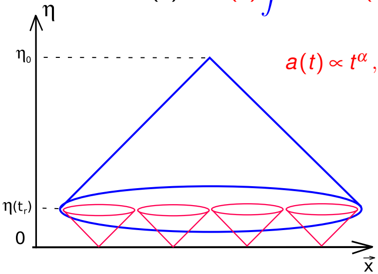
a distance covered by photon emitted at $t = 0$

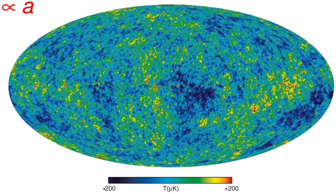
size of the causally connected part,
i.e. the visible part of the Universe ("inside horison")

$$ds^2 = dt^2 - a^2(t)\,dx^2 = a^2(\eta)\left(d\eta^2 - dx^2\right) \qquad ds^2 = 0$$

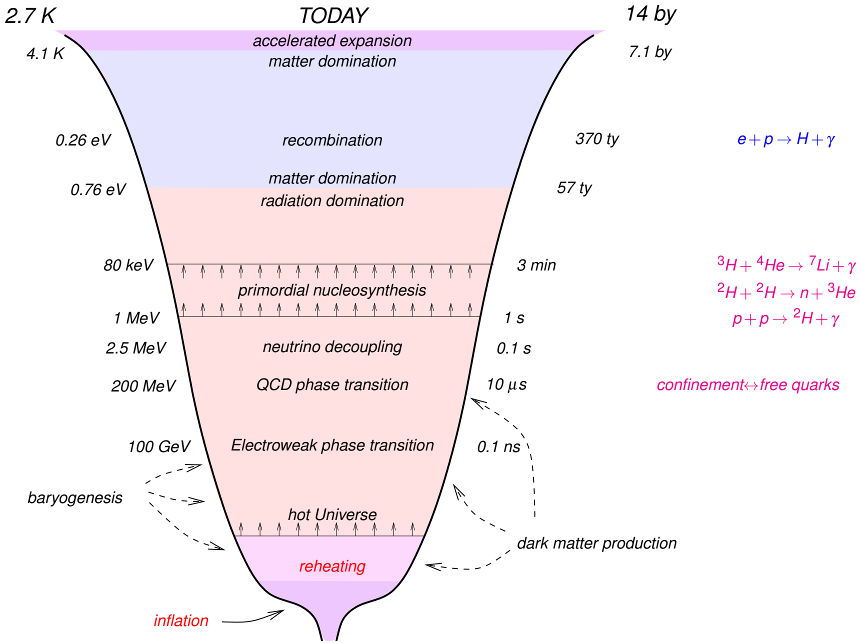
$$l_H(t) = a(t)\int dx = a(t)\int d\eta = a(t)\int_0^t \frac{c\,dt'}{a(t')} \propto t \quad \propto 1/H(t)$$

$$a(t) \propto t^\alpha\,,\ 0<\alpha<1\,,\ L_{phys} \propto a$$





$$l_{H_0}/l_{H,r}(t_0) \sim l_{H_0}/l_{H,r}(t_r)\,a(t_r)/a_0 \sim H_r/H_0\,a(t_r)/a_0 \sim \sqrt{1+z_r} \simeq 30$$



2.7 K
TODAY
14 by
accelerated expansion
4.1 K
matter domination
7.1 by
0.26 eV
recombination
370 ty
$e+p \to H+\gamma$
matter domination
0.76 eV
57 ty
radiation domination
80 keV
3 min
$^3H + {}^4He \to {}^7Li + \gamma$
primordial nucleosynthesis
$^2H + {}^2H \to n + {}^3He$
1 MeV
1 s
$p+p \to {}^2H + \gamma$
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 μs
confinement$\leftrightarrow$free quarks
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter production
reheating
inflation

Ultra-relativistic particles

radiation: $p = \frac{1}{3}\rho$ singular at $t = 0$

$$\rho = \frac{\text{const}}{a^4}\,, \quad a(t) \propto t^{1/2}\,, \quad \rho(t) = \frac{\text{const}}{t^2}$$

$$H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{3}{32\pi G}\frac{1}{t^2}$$

$$l_H(t) = a(t)\int_0^t \frac{dt'}{a(t')} = 2t = \frac{1}{H(t)}\,.$$

In case of thermal equilibrium $T = \text{const}/(a g_*^{1/3}(T)) \approx \text{const}/a$

$$\rho_b = \frac{\pi^2}{30} g_b T^4\,, \quad \rho_f = \frac{7}{8}\frac{\pi^2}{30} g_f T^4$$

find $\frac{7}{8}$

$$\rho = \frac{\pi^2}{30} g_* T^4\,, \quad g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f = g_*(T)\,, \quad s = \frac{\pi^2}{45} g_* T^3$$

Last scattering: $\gamma e \to \gamma e$

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \approx 0.67 \cdot 10^{-24}\,\mathrm{cm}^2\,, \qquad \tau_\gamma = \frac{1}{\sigma_{\mathrm{T}} \cdot n_e(T)}$$

last scattering: $\tau_\gamma(T_f) \simeq H^{-1}(T_f) \simeq t_f$

$$T_f = 0.26\ \mathrm{eV}\,, \quad z = 1100\,, \quad t_f = 370\,000\ \mathrm{yr}$$

for general processes one should solve kinetic equations

$$\frac{dn_{X_i}}{dt} + 3Hn_{X_i} = \int (production - destruction)$$

Boltzmann equation in a comoving volume: $\frac{d}{dt}\left(n\,a^3\right) = a^3 \int \dots$

Recombination: horizon

matter domination: $l_{\mathrm{H},r} = 2H_r^{-1}$

$$H_r^2 = \frac{8\pi}{3} G\rho_{\mathrm{M}}(t_r) = \frac{8\pi}{3} G\rho_{\mathrm{M},0}\left(\frac{a_0}{a_r}\right)^3 = \frac{8\pi}{3} G\rho_c \Omega_{\mathrm{M},0}(1+z_r)^3 \, .$$

at recombination: $l_{\mathrm{H},r} = \frac{2}{H_0\sqrt{\Omega_{\mathrm{M}}}} \frac{1}{(1+z_r)^{3/2}}$

today: $l_{\mathrm{H},r}(t_0) = \frac{2}{H_0\sqrt{\Omega_{\mathrm{M}}}} \frac{1}{\sqrt{1+z_r}}$

$$\frac{l_{H_0}}{l_{\mathrm{H},r}(t_0)} \sim \sqrt{1+z_r} \simeq 30$$

Recombination: angle

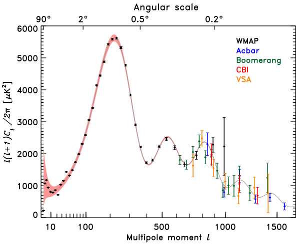
$$\chi_r = \int_{t_r}^{t_0} \frac{dt}{a(t)}\,, \quad \Delta\theta_r = \frac{l_{H,r}}{r_a(z_r)}$$

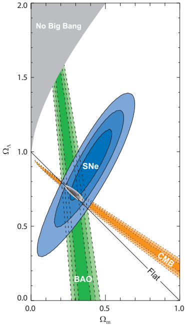
$$r_a(z_r) = (1+z_r)^{-1} \cdot a_0 \cdot \sinh\chi_r$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\,, \quad \Omega_{curv} = \Omega_\Lambda = 0\,.$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}} \frac{2\sqrt{\Omega_{curv}/\Omega_M}}{\sinh\left(2\sqrt{\Omega_{curv}/\Omega_M}\, I\right)}\,.$$

$$I = \int_0^1 \frac{dy}{\sqrt{1+\frac{\Omega_\Lambda}{\Omega_M}y^6}}$$





Acoustic oscillations in relativistic plasma:
What matters is the sound horizon:

$$l_{s,r} = l_{H,r} \cdot v_s \approx$$

Recombination: angle

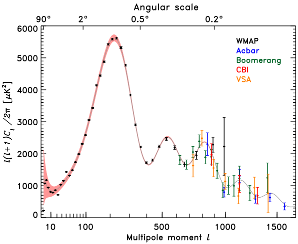
$$\chi_r = \int_{t_r}^{t_0} \frac{dt}{a(t)}\,, \quad \Delta\theta_r = \frac{l_{H,r}}{r_a(z_r)}$$

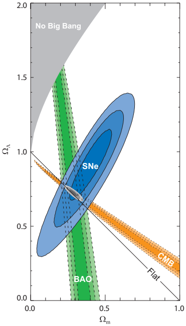
$$r_a(z_r) = (1+z_r)^{-1}\cdot a_0 \cdot \sinh\chi_r$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\,, \quad \Omega_{curv} = \Omega_\Lambda = 0\,.$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\frac{2\sqrt{\Omega_{curv}/\Omega_M}}{\sinh\left(2\sqrt{\Omega_{curv}/\Omega_M}\,I\right)}\,.$$

$$I = \int_0^1 \frac{dy}{\sqrt{1+\frac{\Omega_\Lambda}{\Omega_M}y^6}}$$



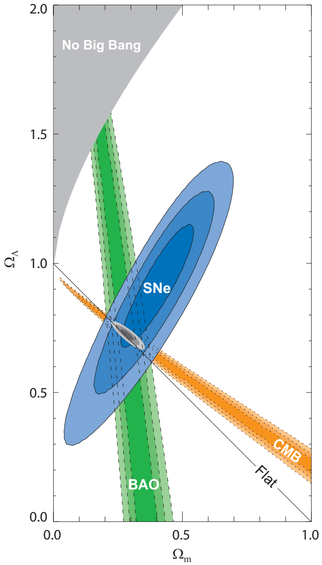


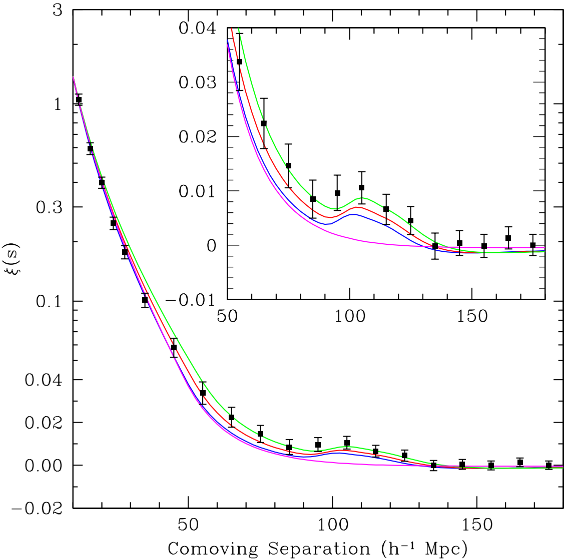
Acoustic oscillations in relativistic plasma:
What matters is the sound horizon:

$$l_{s,r} = l_{H,r}\cdot v_s \approx l_{H,r}/\sqrt{3}$$

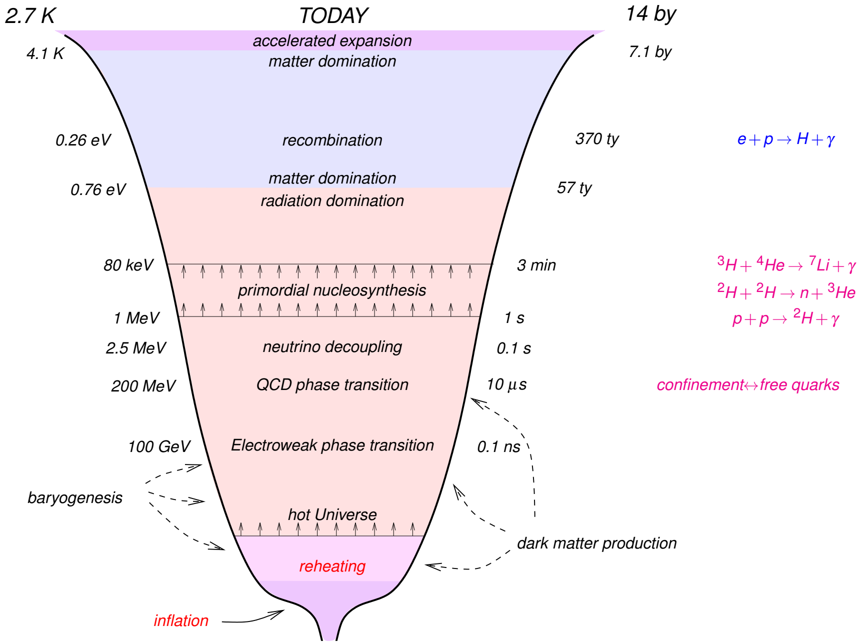
Then $\Delta\theta_{r,s} =$

$$\frac{1}{\sqrt{3}}\frac{1}{\sqrt{1+z}}\times\frac{180^\circ}{\pi} \simeq 1^\circ$$





$$110/0.7\,\mathrm{Mpc} \simeq l_{H,r}(t_0) \times \sqrt{v_s^2} \simeq l_{H_0}/\sqrt{3}/\sqrt{1+z_r}$$


2.7 K
TODAY
14 by
accelerated expansion
4.1 K
matter domination
7.1 by
0.26 eV
recombination
370 ty
$e + p \to H + \gamma$
matter domination
0.76 eV
57 ty
radiation domination
80 keV
3 min
$^3H + {}^4He \to {}^7Li + \gamma$
primordial nucleosynthesis
$^2H + {}^2H \to n + {}^3He$
1 MeV
1 s
$p + p \to {}^2H + \gamma$
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 μs
confinement↔free quarks
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter production
reheating
inflation

Neutrino freeze-out

$T > m_e$ $e^+e^- \leftrightarrow \nu\bar{\nu},\ e\nu \leftrightarrow e\nu$

$$\sigma_\nu \sim G_F^2 E^2$$

neutrino interaction rate

$$\tau_\nu = \frac{1}{\langle \sigma_\nu n v \rangle} \sim \frac{1}{G_F^2 T^5}$$

$$\tau_\nu(T) \sim H^{-1}(T) = \frac{M_{Pl}^*}{T^2}$$

$$T_{\nu,f} \sim \left(\frac{1}{G_F^2 M_{Pl}^*} \right)^{1/3} \sim 2 \div 3\ \text{MeV}$$

Neutron decoupling $p + e \longleftrightarrow n + \nu_e$

typical energy scales

$$T \gtrsim \Delta m = 1.3\ \text{MeV}, \qquad T \gtrsim m_e = 0.5\ \text{MeV}$$

neutron interaction rate

$$\tau_{n\leftrightarrow p} = \frac{1}{\Gamma_{n\leftrightarrow p}} = \frac{1}{C_n G_F^2 T^5}$$

neutron decoupling $\Gamma_{n\leftrightarrow p}(T) \sim H(T) = T^2/M_{Pl}^*$

$$T_n = \frac{1}{\left(C_n M_{Pl}^* G_F^2\right)^{1/3}} \approx 1.4\ \text{MeV}$$

$g_* = 2 + \frac{7}{8}\cdot 4 + \frac{7}{8}\cdot 2\cdot N_\nu$ $\qquad t = \frac{1}{2H(T_n)} = \frac{M_{Pl}^*}{2T_n^2} = 1.2\ \text{s}$ $\qquad T_n \approx 0.8\ \text{MeV}$

Neutron density after decoupling

$$n_n = g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} e^{\frac{\mu_n - m_n}{T}} \qquad \mu_n + \mu_\nu = \mu_p + \mu_e$$

$$\frac{n_n}{n_p} = e^{-\frac{m_n - m_p}{T}} e^{\frac{\mu_n - \mu_p}{T}}$$

for relativistic e^+ and e^- Why ■

$$n_{e^-} - n_{e^+} \sim \mu_e T^2 \longrightarrow \frac{\mu_e}{T} \sim \frac{n_{e^-} - n_{e^+}}{T^3} = \frac{n_p}{T^3} \sim \eta_B \sim 10^{-9}$$

✍ find coefficient ⇑

$$\frac{n_n}{n_p} = e^{-\frac{m_n - m_p}{T_n}} \equiv e^{-\frac{\Delta m}{T_n}} \approx \frac{1}{5} e^{-\frac{\mu_\nu}{T_n}}$$

$p+n \longrightarrow \mathrm{D}+\gamma$ Saha equation

$$n_n = 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \mathrm{e}^{\frac{\mu_n - m_n}{T}}\,, \qquad n_p = 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \mathrm{e}^{\frac{\mu_p - m_p}{T}}\,,$$

Chemical equilibrium: $\mu_D = \mu_p + \mu_n$

✍ prove that: $n_D = n_p n_n g_D 2^{-1/2}\left(\frac{2\pi}{m_p T}\right)^{\frac{3}{2}} \mathrm{e}^{\frac{\Delta_D}{T}}$

Then $n_D \sim n_n$ at $T \simeq 70\,\mathrm{keV}$

Interesting quantity

$$n_{\mathrm{B}} = \eta_{\mathrm{B}} \cdot n_\gamma = 0.24 \eta_{\mathrm{B}} T^3$$

Helium abundance (NO chemical equilibrium)

Neutrons remain mostly in helium

$$n_{^4\mathrm{He}}(T_{NS}) = \frac{1}{2} n_n(T_{NS}) \,,$$

neutron-to-proton ratio $\tau_n \approx 886$ s

$$\frac{n_n(T_{NS})}{n_p(T_{NS})} \approx \frac{1}{5} \cdot \mathrm{e}^{-\frac{t_{NS}}{\tau_n}} \cdot \mathrm{e}^{-\frac{\mu_\nu}{T_n}} \approx \frac{1}{7} \,,$$

$$Y_p \equiv X_{^4\mathrm{He}} = \frac{m_{^4\mathrm{He}} \cdot n_{^4\mathrm{He}}(T_{NS})}{m_p\,(n_p(T_{NS}) + n_n(T_{NS}))} = \frac{2}{\frac{n_p(T_{NS})}{n_n(T_{NS})} + 1} \approx 25\% \,.$$

from observations of relic helium abundance:

$$\Delta N_{\nu,eff} \leq 0.3 \,, \qquad \left|\frac{\mu_\nu}{T_n}\right| \lesssim 0.01$$

Main nuclear reactions

1. $p(n,\gamma)$D — deuterium production, BBN starts.
2. D$(p,\gamma)^3$He, D(D,n)^{3}He, D(D,p)T, ^{3}He(n,p)T — intermediate stage.
3. T(D,n)^{4}He, ^{3}He(D,p)^{4}He — production of ^{4}He.
4. T$(\alpha,\gamma)^7$Li, ^{3}He$(\alpha,\gamma)^7$Be, ^{7}Be$(n,p)^7$Li — production of the heaviest baryonic relics.
5. ^{7}Li$(p,\alpha)^4$He — ^{7}Li burning.

One has to compare reaction rates to the expansion rate

$$H(T_{NS} = 70\ \text{keV}) = 4 \cdot 10^{-3}\ \text{s}^{-1}$$

to obtain nonequilibrium concentrations

Neutron burning $p + n \longrightarrow \mathrm{D} + \gamma$

@ $T = T_{NS} = 65$ keV

$$(\sigma v)_{p(n,\gamma)D} \approx 6 \cdot 10^{-20} \ \frac{\mathrm{cm}^3}{\mathrm{s}} \ .$$

for the rate of neutron disappearance (it meets proton!)

$$\Gamma_{p(n,\gamma)D} = n_p \cdot (\sigma v)_{p(n,\gamma)D} = \eta_B \cdot 2\frac{\zeta(3)}{\pi^2} T^3 \cdot (\sigma v)_{p(n,\gamma)D} = 0.31 \ \mathrm{s}^{-1}$$

for $\eta_B = 6.15 \cdot 10^{-10}$ and $T = T_{NS}$

So, neutrons disappear very rapidly

$$\Gamma_{p(n,\gamma)D} \gg H(T_{NS}) = 4 \cdot 10^{-3} \ \mathrm{s}^{-1}$$

Deuterium burning $D(D,n)^3He$, $D(D,p)T$

Coloumb barier: tunneling $T_9 \equiv T/(10^9\ \mathrm{K}) = T/(86\ \mathrm{keV})$

$$\langle \sigma v \rangle_{DD} = 3 \cdot 10^{-15} \frac{\mathrm{cm}^3}{\mathrm{s}} \cdot T_9^{-2/3} \cdot e^{-4.26 \cdot T_9^{-1/3}} .$$

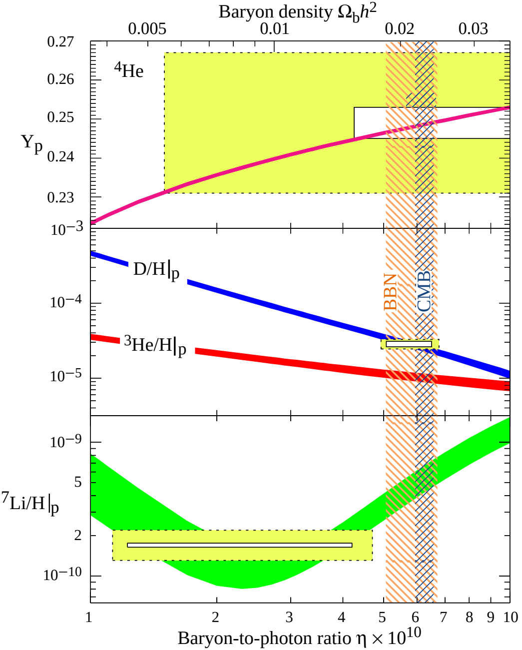
deuterium stops burning when $T = T_{NS} (T_9 = 0.75)$

$$\Gamma_{DD} = n_D(T) \cdot \langle \sigma v \rangle_{DD}(T) \sim H(T) .$$

Then relic deuterium abundance is estimated as

$$\frac{n_D}{n_p} = \frac{1}{0.75 \eta_B} \cdot \frac{n_D}{n_\gamma(T_{NS})} = 0.3 \cdot 10^{-4}$$

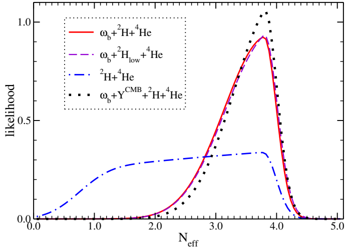
for $\eta_B = 6.15 \cdot 10^{-10}$



Lack of Lithium. . .

$$Y_p = 0.2581 \pm 0.025\,,$$
$$D/H|_p = (2.87 \pm 0.21) \times 10^{-5}$$

1103.1261



similar results from other recent studies including structure formation

1001.4440, 1001.5218, 1202.2889

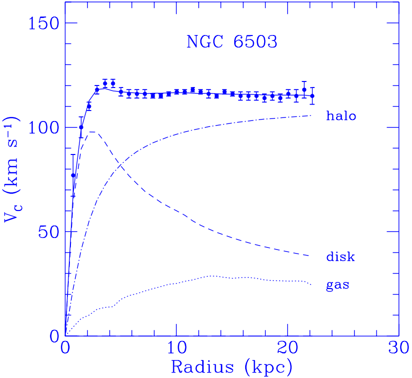
$N_\nu < 4.2$ @ 95%CL

$N_\nu < 4.3$ with shorter neutron's life...

BACKUP SLIDES

$$v(R) = \sqrt{G\frac{M(R)}{R}}$$

$$M(R) = 4\pi\int_0^R \rho(r)r^2 dr$$



observations: $v(R) \simeq \text{const}$

visible matter:

internal regions $v(R) \propto \sqrt{R}$

external ("empty") regions $v(R) \propto 1/\sqrt{R}$

Dark Matter in clusters

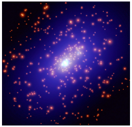
X-rays from hot gas in clusters

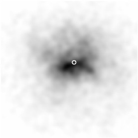
$$\frac{dP}{dR} = -\mu n_e(R) m_p \frac{GM(R)}{R^2}\,, \quad M(R) = 4\pi \int_0^R \rho(r) r^2 dr\,, \quad P(R) = n_e(R) T_e(R)$$

galaxies in clusters — virial theorem

$$U + 2E_k = 0$$

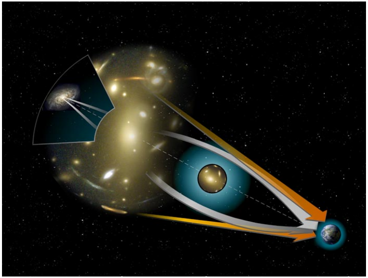
$$3M\langle v_r^2 \rangle = G\frac{M^2}{R}$$

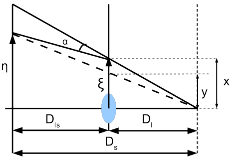




Milky Way: Virgo infall

Gravitational lensing in GR: $\alpha = 4GM/(c^2 b)$



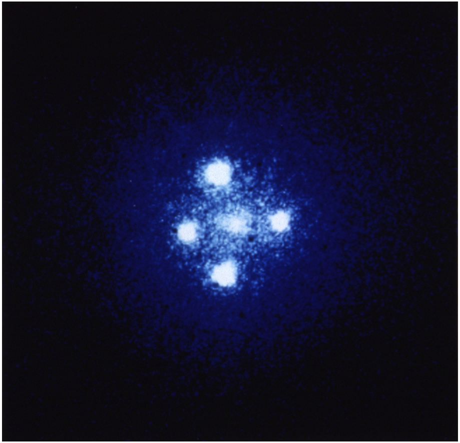


$$\vec{\eta} = \frac{D_s}{D_l}\vec{\xi} - D_{ls}\vec{\alpha}\left(\vec{\xi}\right)$$

common lens with specific refraction coefficient

$$\vec{\alpha}\left(\vec{\xi}\right) = \frac{4G}{c}\int \frac{\vec{\xi}-\vec{\xi}'}{\left|\xi-\vec{\xi}'\right|^2} d^2\xi' \int \rho\left(\vec{\xi}',z\right) dz$$

Einstein Cross

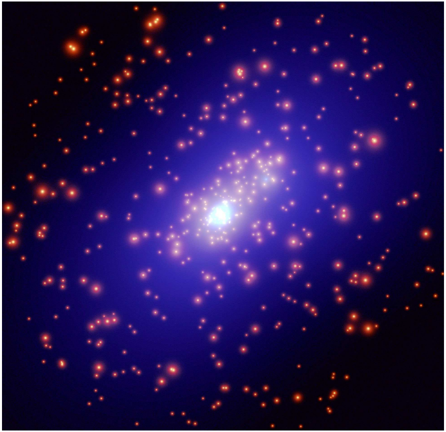


source: quasar $D_s = 2.4$ Gpc

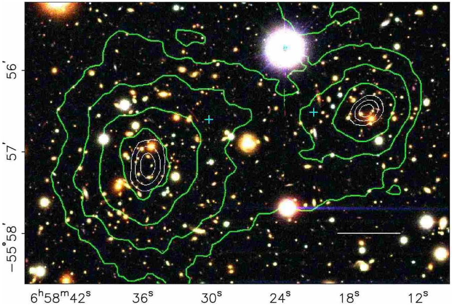
lens: galaxy $D_l = 120$ Mpc

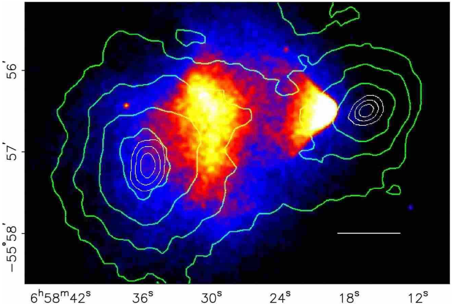
Dark Matter in clusters

gravitational lensing $\rho_B \approx 0.25\rho_{DM}$



Colliding clusters (Bullet clusters 1E0657-558)





gravitational lensing

Observations in X-rays

$M \simeq 10 \times m$

scale is 200 kpc

clusters are at 1.5 Gpc