Cosmology. Theory
Lecture #2

Dmitry Gorbunov

Institute for Nuclear Research of RAS, Moscow

**XXIII Baikal Summer School
on Physics of Elementary Particles and Astrophysics,**

**JINR-ISU,
Irkutsk region, Bol'shie Koty**

Convention of GR description

metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \ , \qquad \mu,\nu = 0,1,2,3 \qquad \text{signature} \ \ (+,-,-,-)$$

$$x^\mu \to x'^\mu(x^\mu) \ : \qquad B'^{\mu}_{\nu\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}B^\sigma_{\tau\rho}(x)$$

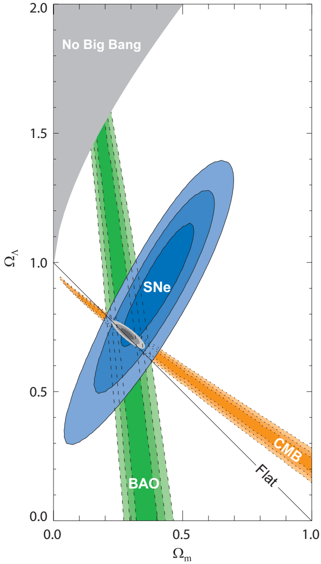
invariant volume $\int d^4x\sqrt{-g}$

$$\int d^4x \ \longrightarrow \ \int d^4x\sqrt{-g}$$

$g_{\mu\nu}(x)$ is the dynamical variable $\qquad T_{\mu\nu} \propto g_{\mu\nu}\Lambda$

$$\int d^4x\sqrt{-g} \ \Lambda$$

Dark Energy: nonclumping substence?



- estimates of Matter contribution confined in galaxies and clusters
  $\rho_c - \rho_M \neq 0$ but the Universe is flat, so
  $\rho_{curv} \simeq 0$
- corrections to the Hubble law : red shift – brightness curves for standard candles (SN Ia)
- The age of the Universe
- CMB anisotropy, large scale structures (galaxy clusters formation), etc

$$\rho_\Lambda = 0.68\rho_c$$

$$\rho_\Lambda \sim 10^{-5}\ \text{GeV/cm}^3 \sim \left(10^{-11.5}\ \text{GeV}\right)^4$$

Convention of GR description

metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \,, \qquad \mu,\nu = 0,1,2,3 \qquad \text{signature } (+,-,-,-)$$

$$x^\mu \to x'^\mu(x^\mu) \;: \qquad B'^{\mu}_{\nu\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}B^\sigma_{\tau\rho}(x)$$

Christoffel symbols: $\tilde{A}^\mu(\tilde{x}) = A^\mu(x) - \Gamma^\mu_{\nu\lambda}(x)A^\nu(x)dx^\lambda$

Covarinat derivative $A^\mu(\tilde{x}) - \tilde{A}^\mu(\tilde{x}) = \nabla_\nu A^\mu \cdot dx^\nu$

$$\nabla_\nu A^\mu(x) = \partial_\nu A^\mu(x) + \Gamma^\mu_{\nu\lambda}(x)A^\lambda$$

If $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$ **and** $\nabla_\nu g_{\mu\lambda} = 0$ $\qquad A_\mu \equiv A^\nu g_{\mu\nu}$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})$$

FLRW metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)dl^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \,,$$

Special frame: different parts look similar
it is comoving: world lines of particles at rest are geodesics,

derive it ✍

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0$$

particle trajectory $x^\mu(\tau)$, choosing $\tau = s$, we define $u^\mu = dx^\mu/ds$

$$S = -m\int ds = -m\int\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

$$\gamma_{ij} \approx \delta_{ij}$$

Photons in the expanding Universe

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}$$

$dt = ad\eta$ conformally flat metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j \longrightarrow ds^2 = a^2(\eta)[d\eta^2 - \delta_{ij}dx^i dx^j]$$

$$S = -\frac{1}{4}\int d^4x\,\eta^{\mu\nu}\eta^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}\,, \qquad A_\mu^{(\alpha)} = e_\mu^{(\alpha)}\mathrm{e}^{ik\eta - i\mathbf{kx}}\,, \quad k = |\mathbf{k}|$$

$$\Delta x = 2\pi/k\,, \quad \Delta\eta = 2\pi/k$$

$$\lambda(t) = a(t)\Delta x = 2\pi\frac{a(t)}{k}\,, \quad T = a(t)\Delta\eta = 2\pi\frac{a(t)}{k}$$

Redshift and the Hubble law $\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} \equiv \lambda_i(1+z(t_i))$

$$\mathbf{p}(t) = \frac{\mathbf{k}}{a(t)}\,, \quad \omega(t) = \frac{k}{a(t)}$$

for not very distant objects $\qquad 1\,\text{pc} \approx 3\,\text{ly}$

$$a(t_i) = a_0 - \dot{a}(t_0)(t_0 - t_i) \quad \longrightarrow \quad a(t_i) = a_0[1 - H_0(t_0 - t_i)]$$

$$z(t_i) = H_0(t_0 - t_i) = H_0 r\,, \qquad z \ll 1$$

$$H_0 = h \cdot 100\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}}\,, \qquad h = 0.675 \pm 0.005$$

similar reddening for other relativistic particles (small H, $\dot{H}$, etc.)

$\mathbf{p} = \frac{\mathbf{k}}{a(t)}$ is true for massive particles as well

Gas of free particles in the expanding Universe

homogeneous gas $\qquad dN = f(\mathbf{p},t)d^3\mathbf{X}d^3\mathbf{p} = \text{const}$

in comoving coordinates:

$$dN = f(\mathbf{k},\eta)d^3\mathbf{x}d^3\mathbf{k} = \text{const}\,, \quad d^3\mathbf{x} = \text{const}\,, \quad d^3\mathbf{k} = \text{const}$$

Hence $\qquad f(\mathbf{k},\eta) = f(\mathbf{k}) = \text{const}$

comoving volume equals physical volume

$$d^3\mathbf{x}d^3\mathbf{k} = d^3(a\mathbf{x})d^3\left(\frac{\mathbf{k}}{a}\right) = d^3\mathbf{X}d^3\mathbf{p}$$

$$f(\mathbf{p},t) = f(\mathbf{k}) = f(a(t)\cdot\mathbf{p})\,.$$

$$t = t_i\;:\quad f_i(\mathbf{p}) = f_i\left(\frac{\mathbf{k}}{a(t_i)}\right) \;\longrightarrow\; f(\mathbf{p},t) = f_i\left(\frac{\mathbf{k}}{a(t_i)}\right) = f_i\left(\frac{\mathbf{k}}{a(t_i)}\frac{a(t)}{a(t)}\right)$$

Massless bosons (photons) — fermions

$$f_i(\mathbf{p}) = f_{Pl}\left(\frac{|\mathbf{p}|}{T_i}\right) = \frac{1}{(2\pi)^3}\frac{1}{e^{|\mathbf{p}|/T_i} - 1}$$

$$f(\mathbf{p},t) = f_{Pl}\left(\frac{a(t)|\mathbf{p}|}{a_i T_i}\right) = f_{Pl}\left(\frac{|\mathbf{p}|}{T_{eff}(t)}\right)$$

$$T_{eff}(t) = \frac{a_i}{a(t)} T_i$$

decoupling at $T \gg m$: neutrinos, hot(warm) dark matter

decoupling at $T \ll m$: thermally produced cold dark matter (e.g. WIMPs)

$f(\mathbf{p}) = \frac{1}{(2\pi)^3} \exp\left(-\frac{m-\mu_i}{T_i}\right) \exp\left(-\frac{a^2(t)\mathbf{p}^2}{2ma_i^2 T_i}\right)$

calculate $f_0(p)$ for neutrinos

$$f(\mathbf{p},t) = \frac{1}{(2\pi)^3} \exp\left(-\frac{m-\mu_{eff}}{T_{eff}}\right) \exp\left(-\frac{\mathbf{p}^2}{2mT_{eff}}\right)$$

$$T_{eff}(t) = \left(\frac{a_i}{a(t)}\right)^2 T_i\,, \qquad \frac{m-\mu_{eff}(t)}{T_{eff}} = \frac{m-\mu_i}{T_i}$$

Convention of GR description

metric inv: $\int d^4x\sqrt{-g}$

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu\ , \qquad \mu,\nu = 0,1,2,3 \qquad \text{signature}\ \ (+,-,-,-)$$

$$x^\mu \to x'^\mu(x^\mu)\ : \qquad B'^{\mu}_{\nu\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}B^\sigma_{\tau\rho}(x)$$

Christoffel symbols: $\nabla_\nu A^\mu(x) = \partial_\nu A^\mu(x) + \Gamma^\mu_{\nu\lambda}(x)A^\lambda(x)$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}\right)$$

Riemann tensor

$$\nabla_\mu\nabla_\nu A^\lambda - \nabla_\nu\nabla_\mu A^\lambda = A^\sigma R^\lambda_{\sigma\mu\nu}\ , \quad R^\mu{}_{\nu\lambda\rho} = \partial_\lambda\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}$$

Ricci tensor and scalar

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}\ , \qquad R \equiv g^{\mu\nu}R_{\mu\nu} = R^{\lambda\mu}{}_{\lambda\mu}$$

Einstein equations

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \ ,$$

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R \ : \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

$$R_{00} = -3\frac{\ddot{a}}{a}\ , \quad R_{0i} = 0\ , \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2\varkappa)\gamma_{ij}\ ,$$

$$R = g^{\mu\nu}R_{\mu\nu} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\varkappa}{a^2}\right)$$

$T_{\mu\nu}$: macroscopic description

$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - g_{\mu\nu}p$

$\frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}$

liquid with $\rho(t)$ and $p(t)$

in the comoving frame $u^0 = 1$, $\mathbf{u} = 0$

$$T_{\mu\nu} = diag(\rho,\ a^2(t)\gamma_{ij}p)$$

Friedmann equation for the present Universe

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\left(\rho_{\rm M} + \rho_{rad} + \rho_\Lambda + \rho_{\rm curv}\right)$$

$$\frac{8\pi}{3} G\rho_{\rm curv} = -\frac{\varkappa}{a^2}\,, \quad \rho_c \equiv \frac{3}{8\pi G} H_0^2$$

$$\rho_c = \rho_{{\rm M},0} + \rho_{rad,0} + \rho_{\Lambda,0} = \rho_c = 0.53 \cdot 10^{-5} \frac{\rm GeV}{\rm cm^3}\,,$$

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_c}\,, \qquad\qquad \frac{a_0}{a} = 1 + z$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda + \Omega_{curv}\left(\frac{a_0}{a}\right)^2\right]$$

Obs: Equality ($\Omega_M = \Omega_{rad}$, $z \approx 3000$), Acceleration ($\ddot{a} = 0$); Age (14 Gy)

Homogeneous and isotropic 3d manifolds

$$dl^2 = d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$r(\rho) = \begin{cases} R\sin(\rho/R)\,, & \text{3-sphere} \\ \rho\,, & \text{3-plane} \\ R\sinh(\rho/R)\,, & \text{3-hyperboloid} \end{cases}$$

ρ is a geodesic distance; $S = 4\pi r^2(\rho)$; $\Delta\theta = \frac{l}{r(\rho)}$

Brightness–redshift dependence in the Universe

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

coordinate distance $\chi = \int_{t_i}^{t_0} \frac{dt}{a(t)}$ $\qquad z(t) = \frac{a_0}{a(t)} - 1$

$$\chi(z) = \int_0^z \frac{dz'}{a_0 H_0} \frac{1}{\sqrt{\Omega_{\rm M}(z'+1)^3 + \Omega_\Lambda + \Omega_{curv}(z'+1)^2}}$$

$$a_0^2 H_0^2 \Omega_{curv} = 1\,, \quad \Omega_{\rm M} + \Omega_\Lambda + \Omega_{curv} = 1$$

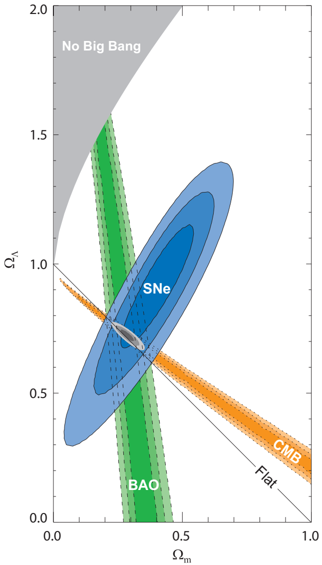
$$S(z) = 4\pi r^2(z)\,, \qquad r(z) = a_0 \sinh\chi(z)$$

detector: $N_\gamma \propto S^{-1}$, $\omega = \omega_i/(1+z)$, $dt_0 = (1+z)dt_i$

brightness (energy flux measured by a detector) is

$$J = \frac{L}{(1+z)^2 S(z)} \equiv \frac{L}{4\pi r_{ph}^2}\,, \quad r_{ph} = (1+z)\cdot r(z)$$

Astrophysical and cosmological data are in agreement



$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi}{3} G \rho_{\text{density}}^{\text{energy}}$$

$$\rho_{\text{density}}^{\text{energy}} = \rho_{\text{radiation}} + \rho_{\text{matter}}^{\text{ordinary}} + \rho_{\text{matter}}^{\text{dark}} + \rho_\Lambda$$

$$\rho_{\text{radiation}} \propto 1/a^4(t) \propto T^4(t)\,, \quad \rho_{\text{matter}} \propto 1/a^3(t)$$

$$\rho_\Lambda = \text{const}$$

$$\frac{3H_0^2}{8\pi G} = \rho_{\text{density}}^{\text{energy}}(t_0) \equiv \rho_c \approx 0.53 \times 10^{-5}\,\frac{\text{GeV}}{\text{cm}^3}$$

radiation: $\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_c} = 0.5 \times 10^{-4}$

Baryons (H, He): $\Omega_B \equiv \frac{\rho_B}{\rho_c} = 0.046$

Neutrino: $\Omega_\nu \equiv \frac{\sum \rho_{\nu_i}}{\rho_c} < 0.01$

Dark matter: $\Omega_{DM} \equiv \frac{\rho_{DM}}{\rho_c} = 0.28$

Dark energy: $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = 0.68$

Initial singularity (Bang!)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho\,, \qquad p = w\rho\,, \qquad \textit{repeat}$$

dust: $p = 0$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^3}\,, \quad a(t) = const\cdot(t-t_s)^{2/3}\,, \quad \rho(t) = \frac{\text{const}}{(t-t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{2}{3t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{1}{6\pi G}\frac{1}{t^2}$$

radiation: $p = \frac{1}{3}\rho$ singular at $t = t_s$

$$\rho = \frac{\text{const}}{a^4}\,, \quad a(t) = const\cdot(t-t_s)^{1/2}\,, \quad \rho(t) = \frac{\text{const}}{(t-t_s)^2}$$

$$t_s = 0\,, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{3}{32\pi G}\frac{1}{t^2}$$

Entropy $T_\mu^\nu = (p+\rho)u_\mu u^\nu - \delta_\mu^\nu p$

$$\nabla_\mu T_0^\mu = 0 \quad \longrightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

for equation of state

$$p = p(\rho)$$

of the primordial plasma we obtain

$$-3d(\ln a) = \frac{d\rho}{p+\rho} = d(\ln s)$$

entropy is conserved in a comoving volume

$$sa^3 = \text{const}$$

For the visible part of the Universe: $S \sim s_{\gamma,0} \cdot l_H^3 \sim 10^{88}$

Horizon problem $l_H(t)$

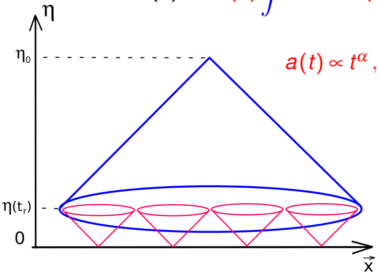
a distance covered by photon emitted at $t = 0$

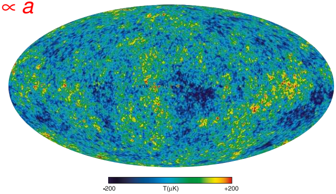
size of the causally connected part,
i.e. the visible part of the Universe (“inside horison”)

$$ds^2 = dt^2 - a^2(t)\,dx^2 = a^2(\eta)\left(d\eta^2 - dx^2\right) \qquad ds^2 = 0$$

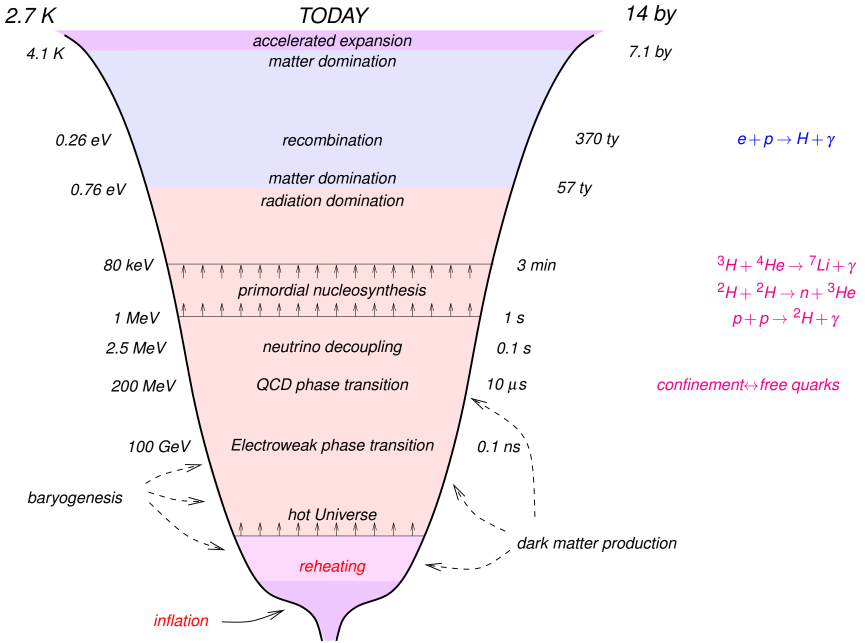
$$l_H(t) = a(t)\int dx = a(t)\int d\eta = a(t)\int_0^t \frac{c\,dt'}{a(t')} \propto t \quad \propto 1/H(t)$$

$a(t) \propto t^\alpha\,,\ 0 < \alpha < 1\,,\ L_{phys} \propto a$





$$l_{H_0}/l_{H,r}(t_0) \sim l_{H_0}/l_{H,r}(t_r)\,a(t_r)/a_0 \sim H_r/H_0\,a(t_r)/a_0 \sim \sqrt{1+z_r} \simeq 30$$


2.7 K
TODAY
14 by
accelerated expansion
4.1 K
matter domination
7.1 by
0.26 eV
recombination
370 ty
$e + p \to H + \gamma$
matter domination
0.76 eV
57 ty
radiation domination
80 keV
3 min
$^3H + {}^4He \to {}^7Li + \gamma$
primordial nucleosynthesis
$^2H + {}^2H \to n + {}^3He$
1 MeV
1 s
$p + p \to {}^2H + \gamma$
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 μs
confinement↔free quarks
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter production
reheating
inflation

Ultra-relativistic particles

radiation: $p = \frac{1}{3}\rho$ singular at $t = 0$

$$\rho = \frac{\text{const}}{a^4}\,, \quad a(t) \propto t^{1/2}\,, \quad \rho(t) = \frac{\text{const}}{t^2}$$

$$H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}\,, \quad \rho = \frac{3}{8\pi G}H^2 = \frac{3}{32\pi G}\frac{1}{t^2}$$

$$l_H(t) = a(t)\int_0^t \frac{dt'}{a(t')} = 2t = \frac{1}{H(t)}\,.$$

In case of thermal equilibrium $T = \text{const}/(a g_*^{1/3}(T)) \approx \text{const}/a$

$$\rho_b = \frac{\pi^2}{30} g_b T^4\,, \quad \rho_f = \frac{7}{8}\frac{\pi^2}{30} g_f T^4$$

find $\frac{7}{8}$

$$\rho = \frac{\pi^2}{30} g_* T^4\,, \quad g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f = g_*(T)\,, \quad s = \frac{\pi^2}{45} g_* T^3$$

Last scattering: $\gamma e \to \gamma e$

$$\sigma_{\rm T} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \approx 0.67\cdot 10^{-24}\ \text{cm}^2\,, \qquad \tau_\gamma = \frac{1}{\sigma_{\rm T}\cdot n_e(T)}$$

last scattering: $\tau_\gamma(T_f) \simeq H^{-1}(T_f) \simeq t_f$

$$T_f = 0.26\ \text{eV}\,, \quad z = 1100\,, \quad t_f = 370\,000\ \text{yr}$$

for general processes one should solve kinetic equations

$$\frac{dn_{X_i}}{dt} + 3Hn_{X_i} = \int (production - destruction)$$

Boltzmann equation in a comoving volume: $\frac{d}{dt}\left(n a^3\right) = a^3 \int \dots$

Recombination: horizon

matter domination: $l_{\mathrm{H},r} = 2H_r^{-1}$

$$H_r^2 = \frac{8\pi}{3}G\rho_{\mathrm{M}}(t_r) = \frac{8\pi}{3}G\rho_{\mathrm{M},0}\left(\frac{a_0}{a_r}\right)^3 = \frac{8\pi}{3}G\rho_c\Omega_{\mathrm{M},0}(1+z_r)^3 \,.$$

at recombination: $l_{\mathrm{H},r} = \frac{2}{H_0\sqrt{\Omega_{\mathrm{M}}}}\frac{1}{(1+z_r)^{3/2}}$

today: $l_{\mathrm{H},r}(t_0) = \frac{2}{H_0\sqrt{\Omega_{\mathrm{M}}}}\frac{1}{\sqrt{1+z_r}}$

$$\frac{l_{H_0}}{l_{\mathrm{H},r}(t_0)} \sim \sqrt{1+z_r} \simeq 30$$

Recombination: angle

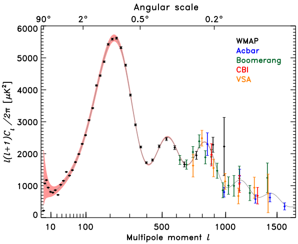
$$\chi_r = \int_{t_r}^{t_0} \frac{dt}{a(t)}\,, \quad \Delta\theta_r = \frac{l_{H,r}}{r_a(z_r)}$$

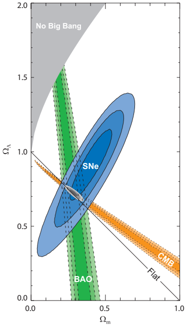
$$r_a(z_r) = (1+z_r)^{-1} \cdot a_0 \cdot \sinh\chi_r$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\,, \quad \Omega_{curv} = \Omega_\Lambda = 0\,.$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}} \frac{2\sqrt{\Omega_{curv}/\Omega_M}}{\sinh\left(2\sqrt{\Omega_{curv}/\Omega_M}\, I\right)}\,.$$

$$I = \int_0^1 \frac{dy}{\sqrt{1+\frac{\Omega_\Lambda}{\Omega_M} y^6}}$$





Acoustic oscillations in relativistic plasma:
What matters is the sound horizon:

$$l_{s,r} = l_{H,r} \cdot v_s \approx$$

Recombination: angle

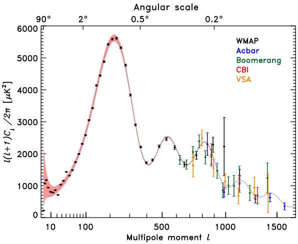
$$\chi_r = \int_{t_r}^{t_0} \frac{dt}{a(t)}\,, \quad \Delta\theta_r = \frac{l_{H,r}}{r_a(z_r)}$$

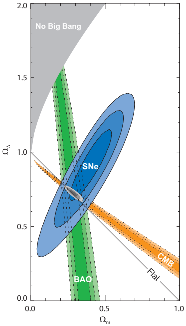
$$r_a(z_r) = (1+z_r)^{-1}\cdot a_0 \cdot \sinh\chi_r$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\,, \quad \Omega_{curv} = \Omega_\Lambda = 0\,.$$

$$\Delta\theta_r = \frac{1}{\sqrt{z_r+1}}\,\frac{2\sqrt{\Omega_{curv}/\Omega_M}}{\sinh\left(2\sqrt{\Omega_{curv}/\Omega_M}\,I\right)}\,.$$

$$I = \int_0^1 \frac{dy}{\sqrt{1+\frac{\Omega_\Lambda}{\Omega_M}y^6}}$$



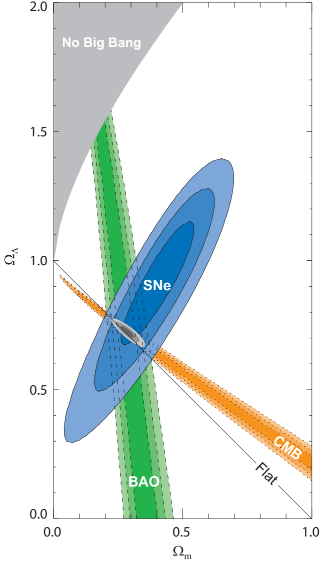


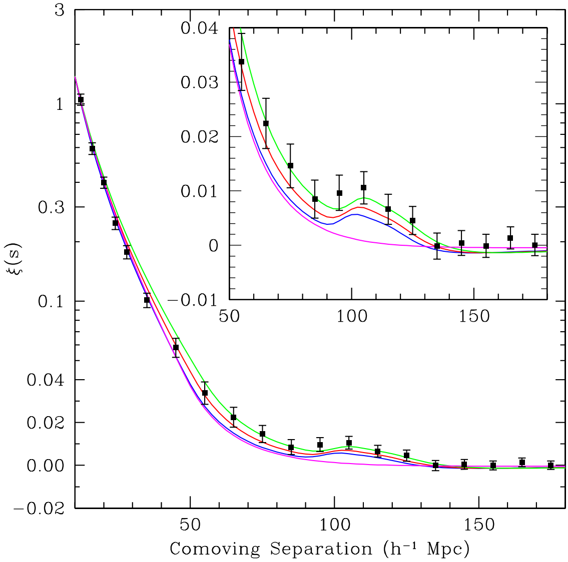
Acoustic oscillations in relativistic plasma:
What matters is the sound horizon:

$$l_{s,r} = l_{H,r}\cdot v_s \approx l_{H,r}/\sqrt{3}$$

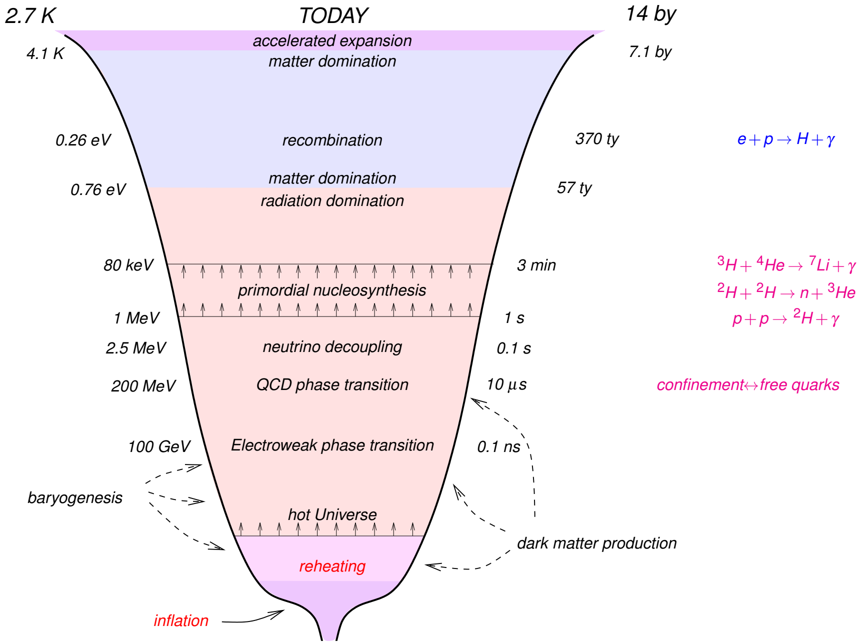
Then $\Delta\theta_{r,s} =$

$$\frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{1+z}} \times \frac{180^\circ}{\pi} \simeq 1^\circ$$





$$110/0.7\,\mathrm{Mpc} \simeq l_{H,r}(t_0) \times \sqrt{v_s^2} \simeq l_{H_0}/\sqrt{3}/\sqrt{1+z_r}$$


2.7 K
TODAY
14 by
accelerated expansion
4.1 K
matter domination
7.1 by
0.26 eV
recombination
370 ty
$e+p \to H+\gamma$
matter domination
0.76 eV
radiation domination
57 ty
80 keV
3 min
$^3H+{}^4He \to {}^7Li+\gamma$
primordial nucleosynthesis
$^2H+{}^2H \to n+{}^3He$
1 MeV
1 s
$p+p \to {}^2H+\gamma$
2.5 MeV
neutrino decoupling
0.1 s
200 MeV
QCD phase transition
10 μs
confinement↔free quarks
100 GeV
Electroweak phase transition
0.1 ns
baryogenesis
hot Universe
dark matter production
reheating
inflation

Neutrino freeze-out

$T > m_e$ $e^+e^- \leftrightarrow \nu\bar{\nu}$, $e\nu \leftrightarrow e\nu$

$$\sigma_\nu \sim G_F^2 E^2$$

neutrino interaction rate

$$\tau_\nu = \frac{1}{\langle \sigma_\nu n v \rangle} \sim \frac{1}{G_F^2 T^5}$$

$$\tau_\nu(T) \sim H^{-1}(T) = \frac{M_{Pl}^*}{T^2}$$

$$T_{\nu,f} \sim \left(\frac{1}{G_F^2 M_{Pl}^*} \right)^{1/3} \sim 2 \div 3 \text{ MeV}$$

Neutron decoupling $p+e \longleftrightarrow n+\nu_e$

typical energy scales

$$T \gtrsim \Delta m = 1.3 \text{ MeV}, \qquad T \gtrsim m_e = 0.5 \text{ MeV}$$

neutron interaction rate

$$\tau_{n\leftrightarrow p} = \frac{1}{\Gamma_{n\leftrightarrow p}} = \frac{1}{C_n G_F^2 T^5}$$

neutron decoupling $\Gamma_{n\leftrightarrow p}(T) \sim H(T) = T^2/M_{Pl}^*$

$$T_n = \frac{1}{\left(C_n M_{Pl}^* G_F^2\right)^{1/3}} \approx 1.4 \text{ MeV}$$

$g_* = 2 + \frac{7}{8}\cdot 4 + \frac{7}{8}\cdot 2\cdot N_\nu$ $\qquad t = \frac{1}{2H(T_n)} = \frac{M_{Pl}^*}{2T_n^2} = 1.2 \text{ s}$ $\qquad T_n \approx 0.8 \text{ MeV}$

Neutron density after decoupling

$$n_n = g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} e^{\frac{\mu_n - m_n}{T}} \qquad \mu_n + \mu_\nu = \mu_p + \mu_e$$

$$\frac{n_n}{n_p} = e^{-\frac{m_n - m_p}{T}} e^{\frac{\mu_n - \mu_p}{T}}$$

for relativistic e^+ and e^- Why ■

$$n_{e^-} - n_{e^+} \sim \mu_e T^2 \longrightarrow \frac{\mu_e}{T} \sim \frac{n_{e^-} - n_{e^+}}{T^3} = \frac{n_p}{T^3} \sim \eta_B \sim 10^{-9}$$

✍ find coefficient ⇑

$$\frac{n_n}{n_p} = e^{-\frac{m_n - m_p}{T_n}} \equiv e^{-\frac{\Delta m}{T_n}} \approx \frac{1}{5} e^{-\frac{\mu_\nu}{T_n}}$$

$p+n \longrightarrow \mathrm{D}+\gamma$ Saha equation

$$n_n = 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \mathrm{e}^{\frac{\mu_n - m_n}{T}}\,, \qquad n_p = 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \mathrm{e}^{\frac{\mu_p - m_p}{T}}\,,$$

Chemical equilibrium: $\mu_D = \mu_p + \mu_n$

✍ prove that: $$n_D = n_p n_n g_D 2^{-1/2}\left(\frac{2\pi}{m_p T}\right)^{\frac{3}{2}} \mathrm{e}^{\frac{\Delta_D}{T}}$$

Then $n_D \sim n_n$ at $T \simeq 70\,\mathrm{keV}$

Interesting quantity

$$n_{\mathrm{B}} = \eta_{\mathrm{B}} \cdot n_\gamma = 0.24\eta_{\mathrm{B}} T^3$$

Helium abundance (NO chemical equilibrium)

Neutrons remain mostly in helium

$$n_{^4\mathrm{He}}(T_{NS}) = \frac{1}{2} n_n(T_{NS}) \,,$$

neutron-to-proton ratio $\tau_n \approx 886$ s

$$\frac{n_n(T_{NS})}{n_p(T_{NS})} \approx \frac{1}{5} \cdot \mathrm{e}^{-\frac{t_{NS}}{\tau_n}} \cdot \mathrm{e}^{-\frac{\mu_\nu}{T_n}} \approx \frac{1}{7} \,,$$

$$Y_p \equiv X_{^4\mathrm{He}} = \frac{m_{^4\mathrm{He}} \cdot n_{^4\mathrm{He}}(T_{NS})}{m_p\,(n_p(T_{NS}) + n_n(T_{NS}))} = \frac{2}{\frac{n_p(T_{NS})}{n_n(T_{NS})} + 1} \approx 25\% \,.$$

from observations of relic helium abundance:

$$\Delta N_{\nu,eff} \leq 0.3 \,, \qquad \left|\frac{\mu_\nu}{T_n}\right| \lesssim 0.01$$

Main nuclear reactions

1. $p(n,\gamma)$D — deuterium production, BBN starts.
2. D$(p,\gamma)^3$He, D(D,n)^{3}He, D(D,p)T, ^{3}He(n,p)T — intermediate stage.
3. T(D,n)^{4}He, ^{3}He(D,p)^{4}He — production of ^{4}He.
4. T$(\alpha,\gamma)^7$Li, ^{3}He$(\alpha,\gamma)^7$Be, ^{7}Be$(n,p)^7$Li — production of the heaviest baryonic relics.
5. ^{7}Li$(p,\alpha)^4$He — ^{7}Li burning.

One has to compare reaction rates to the expansion rate

$$H(T_{NS} = 70\ \text{keV}) = 4 \cdot 10^{-3}\ \text{s}^{-1}$$

to obtain nonequilibrium concentrations

Neutron burning $p + n \longrightarrow \mathrm{D} + \gamma$

@ $T = T_{NS} = 65$ keV

$$(\sigma v)_{p(n,\gamma)D} \approx 6 \cdot 10^{-20}\ \frac{\mathrm{cm}^3}{\mathrm{s}}\ .$$

for the rate of neutron disappearance (it meets proton!)

$$\Gamma_{p(n,\gamma)D} = n_p \cdot (\sigma v)_{p(n,\gamma)D} = \eta_B \cdot 2\frac{\zeta(3)}{\pi^2} T^3 \cdot (\sigma v)_{p(n,\gamma)D} = 0.31\ \mathrm{s}^{-1}$$

for $\eta_B = 6.15 \cdot 10^{-10}$ and $T = T_{NS}$

So, neutrons disappear very rapidly

$$\Gamma_{p(n,\gamma)D} \gg H(T_{NS}) = 4 \cdot 10^{-3}\ \mathrm{s}^{-1}$$

Deuterium burning $D(D,n)^3He$, $D(D,p)T$

Coloumb barier: tunneling $T_9 \equiv T/(10^9\ \mathrm{K}) = T/(86\ \mathrm{keV})$

$$\langle\sigma v\rangle_{DD} = 3\cdot 10^{-15}\frac{\mathrm{cm}^3}{\mathrm{s}}\cdot T_9^{-2/3}\cdot \mathrm{e}^{-4.26\cdot T_9^{-1/3}}\ .$$

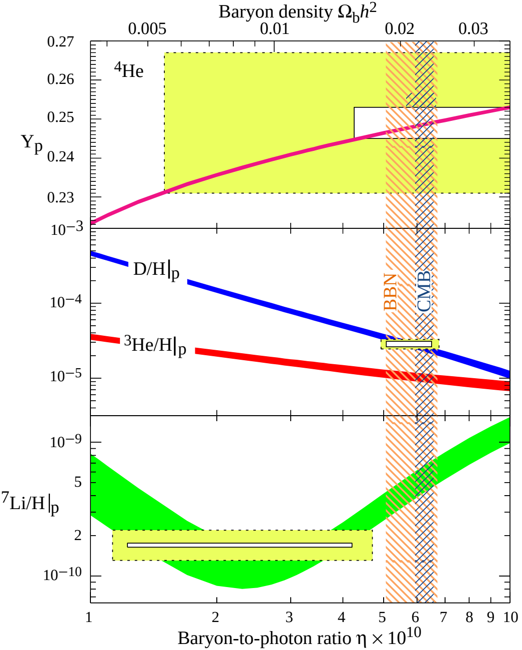
deuterium stops burning when $T = T_{NS}(T_9 = 0.75)$

$$\Gamma_{DD} = n_D(T)\cdot\langle\sigma v\rangle_{DD}(T) \sim H(T)\ .$$

Then relic deuterium abundance is estimated as

$$\frac{n_D}{n_p} = \frac{1}{0.75\eta_B}\cdot\frac{n_D}{n_\gamma(T_{NS})} = 0.3\cdot 10^{-4}$$

for $\eta_B = 6.15\cdot 10^{-10}$

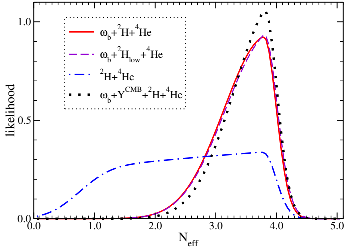


Lack of Lithium...

$$Y_p = 0.2581 \pm 0.025\,,$$

$$D/H\big|_p = (2.87 \pm 0.21) \times 10^{-5}$$

1103.1261



similar results from other recent studies including structure formation

1001.4440, 1001.5218, 1202.2889

$N_\nu < 4.2$ @ 95%CL

$N_\nu < 4.3$ with shorter neutron's life...

Baryogenesis

Sakharov conditions of successful baryogenesis

- B-violation $(\Delta B \neq 0)\ XY \cdots \to X'Y' \ldots B$
- C- & CP-violation $(\Delta C \neq 0,\ \Delta CP \neq 0)\ \bar{X}\bar{Y} \cdots \to \bar{X}'\bar{Y}' \ldots \bar{B}$
- processes above are out of equilibrium $X'Y' \ldots B \to XY \ldots$

At 100 GeV$\lesssim T \lesssim 10^{12}$ GeV nonperturbative processes (EW-sphalerons) violate B, L_α, so that only three charges are conserved out of four, e.g.

$$B - L\,, \quad L_e - L_\mu\,, \quad L_e - L_\tau$$

and $\quad B = \alpha \times (B - L),\ L = (\alpha - 1) \times (B - L)$

Leptogenesis: Baryogenesis from lepton asymmetry of the Universe ... due to sterile neutrinos

Why $\Omega_B \sim \Omega_{DM}$? antropic principle?

Dark Matter Properties $p = 0$

(If) particles:

1. stable on cosmological time-scale
2. nonrelativistic long before RD/MD-transition (either Cold or Warm, $v_{RD/MD} \lesssim 10^{-3}$)
3. (almost) collisionless
4. (almost) electrically neutral

If were in thermal equilibrium: $M_X \gtrsim 1$ keV

If not: for bosons

$\lambda = 2\pi/(M_X v_X)$, in a galaxy $v_X \sim 0.5 \cdot 10^{-3} \longrightarrow M_X \gtrsim 3 \cdot 10^{-22}$ eV

for fermions

Pauli blocking: $M_X \gtrsim 750$ eV

$$f(\mathbf{p},\mathbf{x}) = \frac{\rho_X(\mathbf{x})}{M_X} \cdot \frac{1}{\left(\sqrt{2\pi} M_X v_X\right)^3} \cdot e^{-\frac{\mathbf{p}^2}{2M_X^2 v_X^2}} \bigg|_{\mathbf{p}=0} \leq \frac{g_X}{(2\pi)^3}$$

Dark Matter Candidates

- WIMPs (neutralino, . . .)
- sterile neutrinos
- gravitino
- axion
- Heavy relics
- (Topological) defects
- Massive Astrophysical Compact Halo Objects
- Primordial black hole remnants

Weakly Interacting Massive Particles

Assumptions:

1. no $X-\bar{X}$ asymmetry $n_X = n_{\bar{X}}$
2. @ $T < M_X$ in thermal equilibrium with plasma

$$n_X = n_{\bar{X}} = g_X \left(\frac{M_X T}{2\pi}\right)^{3/2} e^{-M_X/T}$$

$$X\bar{X} \longrightarrow \text{light particles}$$

freeze-out temperature T_f $M_{Pl}^* = M_{Pl}/1.66\sqrt{g_*}$

$$\frac{1}{n_X}\frac{1}{\langle\sigma_{ann} v\rangle} = H^{-1}(T_f) \longrightarrow T_f = \frac{M_X}{\ln\left(\frac{g_X M_X M_{Pl}^* \sigma_0}{(2\pi)^{3/2}}\right)} .$$

Bethe formulae: s-wave: $\sigma_{ann} = \frac{\sigma_0}{v}$

Weakly Interacting Massive Particles

density after freeze-out: $n_X(T_f) = \frac{T_f^2}{M_{Pl}^* \sigma_0}$

present density: $n_X(T_0) = \left(\frac{a(T_f)}{a(T_0)}\right)^3 n_X(T_f) = \left(\frac{s_0}{s(T_f)}\right) n_X(T_f) \propto \frac{1}{T_f} \propto \frac{1}{M_X}$

$X + \bar{X}$ contribution to critical density:

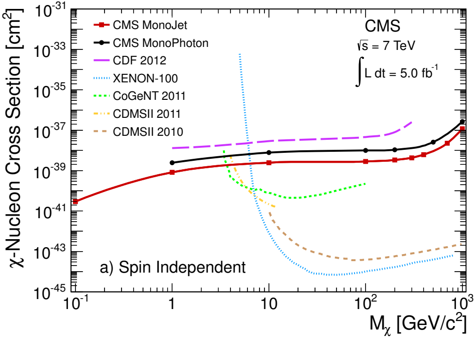
$$\Omega_X = 2\frac{M_X n_X(T_0)}{\rho_c} = 7.6\frac{s_0 \ln\left(\frac{g_X M_{Pl}^* M_X \sigma_0}{(2\pi)^{3/2}}\right)}{\rho_c \sigma_0 M_{Pl}\sqrt{g_*(T_f)}}$$

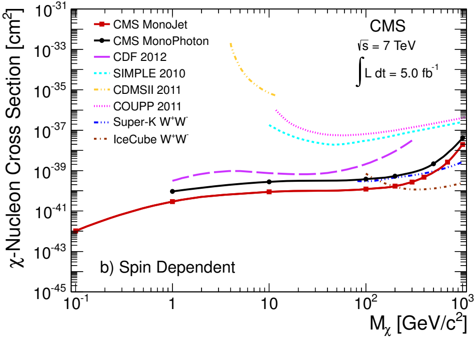
$$= 0.1 \cdot \left(\frac{(10\ \text{TeV})^{-2}}{\sigma_0}\right) \frac{0.3}{\sqrt{g_*(T_f)}} \ln\left(\frac{g_X M_{Pl}^* M_X \sigma_0}{(2\pi)^{3/2}}\right) \cdot \frac{1}{2h^2}$$

natural dark matter: $\sigma_0 \sim 0.01 \times \sigma_W$

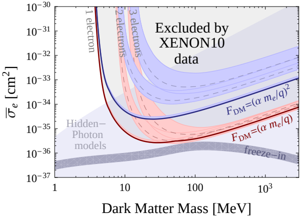
naturaly "light" $\sigma_0 \lesssim \frac{4\pi}{M_X^2} \longrightarrow M_X \lesssim 100$ TeV

Recent results of (in)direct searches





for WIMPs 1206.5663



there are analyses for lower mass ranges and other type of interactions:

e.g. 1206.2644

Decoupling of relativistic specia (DM?)

Thermal equilibrium is forbidden:

$T_d \gg M_X$, and then $n_X/s = \text{const}$

$$\Omega_{3/2} = \frac{m_X \cdot n_{X,0}}{\rho_c} = \frac{m_X \cdot s_0}{\rho_c}\frac{n_{X,0}}{s_0} = 0.2\frac{M_X}{100\text{ eV}}\left(\frac{g_X}{2}\right)\cdot\left(\frac{100}{g_*(T_d)}\right)\cdot\frac{1}{2h^2}$$

- If fermions: limit from Pauli-blocking
- Generally: too hot at Equality:
  from structure formation we need at $T_{Eq} \sim 1$ eV, $v_{DM} \lesssim 10^{-3}$

NB: for $M_X = 100$ eV at Equality ($T_{Eq} \sim 1$ eV) X-particle velocities are

Decoupling of relativistic specia (DM?)

Thermal equilibrium is forbidden:

$T_d \gg M_X$, and then $n_X/s = \text{const}$

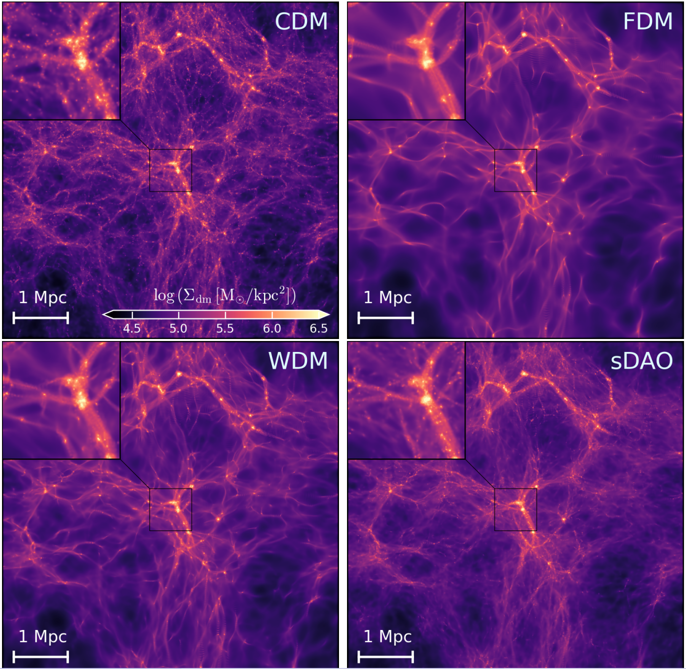
$$\Omega_{3/2} = \frac{m_X \cdot n_{X,0}}{\rho_c} = \frac{m_X \cdot s_0}{\rho_c}\frac{n_{X,0}}{s_0} = 0.2\frac{M_X}{100\text{ eV}}\left(\frac{g_X}{2}\right)\cdot\left(\frac{100}{g_*(T_d)}\right)\cdot\frac{1}{2h^2}$$

- If fermions: limit from Pauli-blocking
- Generally: too hot at Equality:
  from structure formation we need at $T_{Eq} \sim 1$ eV, $v_{DM} \lesssim 10^{-3}$

NB: for $M_X = 100$ eV at Equality ($T_{Eq} \sim 1$ eV) X-particle velocities are ■ $v \simeq 10^{-2}$

Other Dark Matter candidates are not in equilibrium!

- WIMPs (neutralino, . . .) $\Longleftarrow$ thermal !
- sterile neutrinos $\Longleftarrow$ Price: sensitive to mass and couplings!
- axion $\Longleftarrow$ Price: sensitive to mass and (=couplings)!
- gravitino $\Longleftarrow$ Price: sensitive to mass, couplings and reheating temperature !!!
- Heavy relics
- (Topological) defects
- Massive Astrophysical Compact Halo Objects
- Primordial black hole remnants


CDM
1 Mpc
log (Σdm [M⊙/kpc²])
4.5
5.0
5.5
6.0
6.5
FDM
1 Mpc
WDM
1 Mpc
sDAO
1 Mpc

2304.06742

DM keV sterile neutrino

Sterile neutrino of keV scale mass provides the Warm Dark Matter

Relevant parameters: mass $M_N \sim$1-10 keV and active-sterile neutrino mixing angle $\theta \ll 1$

Bounds on mass

- Phase space density (refined Pauli-blocking): $M_N \gtrsim 0.3$ keV
- Lyman-α forest: $M_N \gtrsim 10$ keV

Bound on mass M_N and mixing angle θ

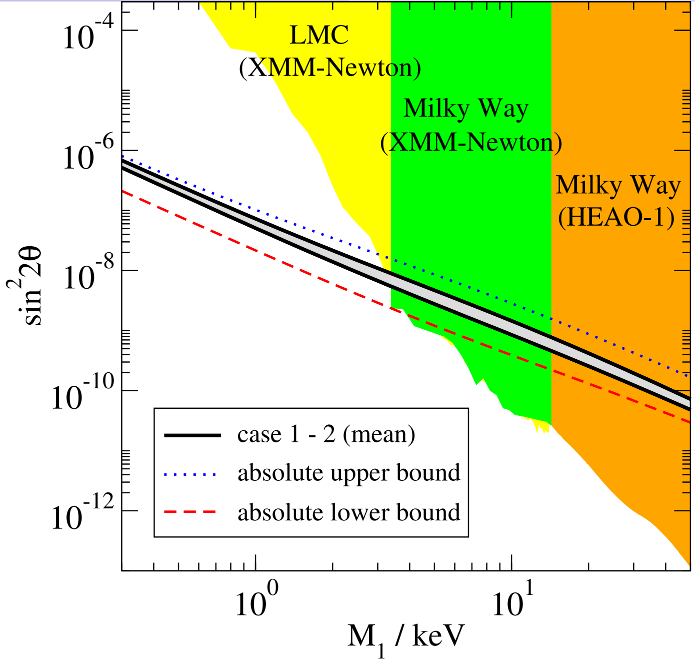
- X-ray observation: $N \to \nu + \gamma$, a peak at $\omega_\gamma = M_N/2$ of intensity $\propto \theta^2$

Production mechanism

- Dodelson-Widrow (thermal) scenario: $\nu_a \to N$ due to mixing,

$$\rho_N \propto \theta^2$$

- Primordial abundance: physics at higher energies
  - Lepton asymmetries
  - Production from inflaton decay
  - etc.


LMC
(XMM-Newton)
Milky Way
(XMM-Newton)
Milky Way
(HEAO-1)
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
$\sin^2 2\theta$
10^0
10^1
M_1 / keV
case 1 - 2 (mean)
absolute upper bound
absolute lower bound

DM keV sterile neutrino

Sterile neutrino of keV scale mass provides the Warm Dark Matter

Relevant parameters: mass $M_N \sim$1-10 keV and active-sterile neutrino mixing angle $\theta \ll 1$

Bounds on mass

- Phase space density (refined Pauli-blocking): $M_N \gtrsim 0.3$ keV
- Lyman-α forest: $M_N \gtrsim 10$ keV

Bound on mass M_N and mixing angle θ

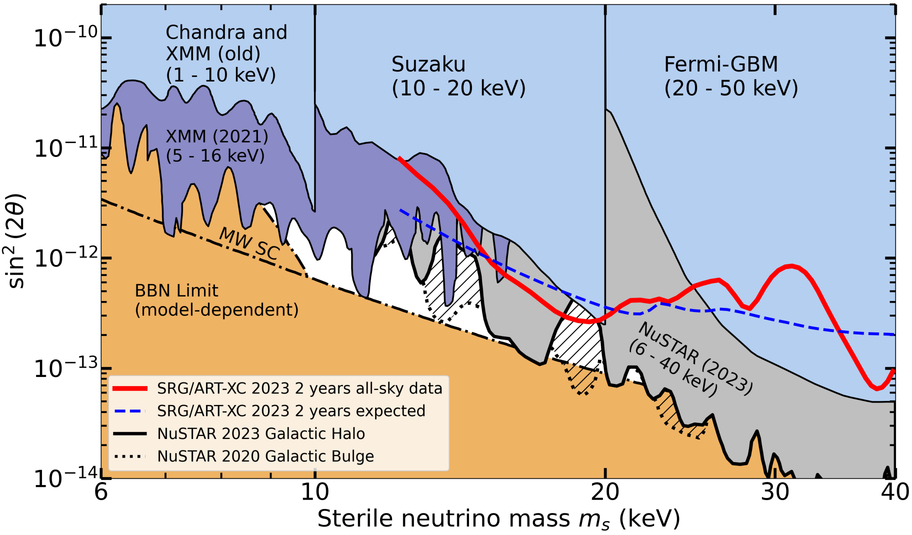
- X-ray observation: $N \to \nu + \gamma$, a peak at $\omega_\gamma = M_N/2$ of intensity $\propto \theta^2$

Production mechanism

- Dodelson-Widrow (thermal) scenario: $\nu_a \to N$ due to mixing,

$$\rho_N \propto \theta^2$$

  is ruled out
- Primordial abundance: physics at higher energies
  - Lepton asymmetries
  - Production from inflaton decay
  - etc.


Chandra and
XMM (old)
(1 - 10 keV)
Suzaku
(10 - 20 keV)
Fermi-GBM
(20 - 50 keV)
XMM (2021)
(5 - 16 keV)
MW SC
BBN Limit
(model-dependent)
NuSTAR (2023)
(6 - 40 keV)
SRG/ART-XC 2023 2 years all-sky data
SRG/ART-XC 2023 2 years expected
NuSTAR 2023 Galactic Halo
NuSTAR 2020 Galactic Bulge
sin^2 (2θ)
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
6
10
20
30
40
Sterile neutrino mass m_s (keV)

2303.12673

Free scalar field as Cold Dark Matter (axion)

Homogeneous scalar field

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$$

at $m \ll H$ no evolution: $\phi = \text{const}$, at $m \gg H$ it oscillates, so

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{m^2}{2}\phi^2 = \langle E_k\rangle + \langle E_p\rangle = 2\langle E_p\rangle\,, \quad p = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{m^2}{2}\phi^2 = \langle E_k\rangle - \langle E_p\rangle = 0\,,$$

behaves as nonrelativistic (dark) matter (dust-like component) !!

nonperturbative CP-violation in QCD

$$L_\theta = \frac{\alpha_s}{8\pi}\left(\theta_0 + \text{Arg}\left(\text{Det}\hat{M}_q\right)\right) G^a_{\mu\nu}\tilde{G}^{\mu\nu\,a} \equiv \frac{\alpha_s}{8\pi}\cdot\theta\cdot G^a_{\mu\nu}\tilde{G}^{\mu\nu\,a}\,.$$

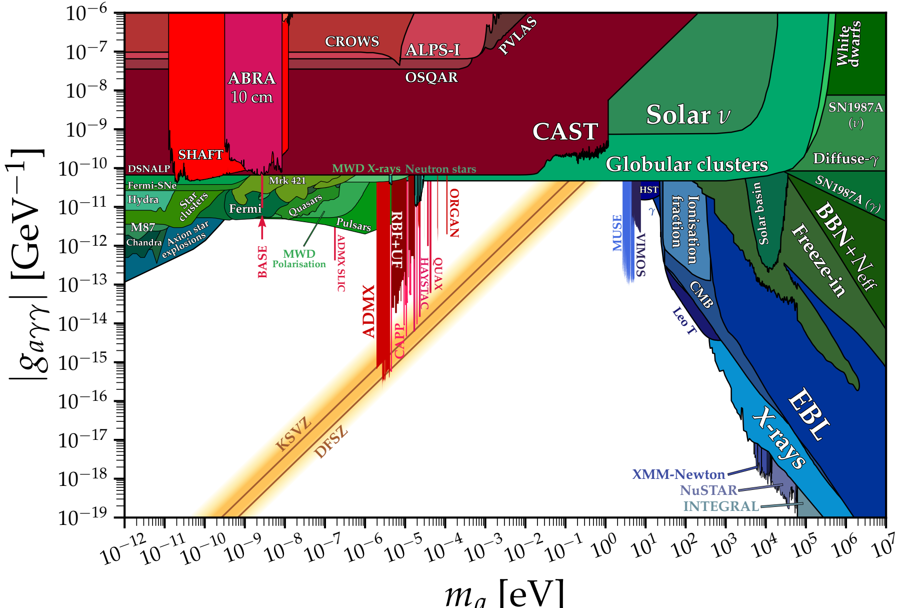
$$\theta \to \bar{\theta}(x) = \theta + C_g\frac{a(x)}{f_{PQ}}\,.$$

$$\mathscr{L} = \frac{f_{PQ}^2}{2}\cdot\left(\frac{d\bar{\theta}}{dt}\right)^2 - \frac{m_a^2(T)}{2}f_{PQ}^2\bar{\theta}^2\,,$$

$$m_a(T) \simeq 0\,, T > \Lambda_{QCD} \quad \text{and} \quad m_a(T) \simeq m_a \simeq m_\pi f_\pi / f_{PQ}$$

Check this $\Longrightarrow$

$$\Omega_a \simeq 0.2\cdot\bar{\theta}_i^2\cdot\left(\frac{4\cdot 10^{-6}\text{ eV}}{m_a}\right)\cdot\frac{1}{2h^2}$$



https://cajohare.github.io/AxionLimits/

Phenomenological problems of the Standard Model

Gauge fields (interactions) – γ, $W^{\pm}$, Z, g

Three generations of matter: $L = \binom{\nu_L}{e_L}$, e_R; $Q = \binom{u_L}{d_L}$, d_R, u_R

- Describes
  - all experiments dealing with electroweak and strong interactions
- Does not describe
  - Neutrino oscillations
  - Dark matter (Ω_{DM})
  - Baryon asymmetry

- BBN, CMB, structure formation:
  $\sum m_\alpha$, mass hierarchy
  (Planck, CMBPole, BOSS)
  ...light sterile neutrinos?
- Many models, but new ideas are welcome !

Phenomenological problems of the Standard Model

Gauge fields (interactions) – γ, $W^{\pm}$, Z, g

Three generations of matter: $L = \binom{\nu_L}{e_L}$, e_R; $Q = \binom{u_L}{d_L}$, d_R, u_R

- Describes
  - all experiments dealing with electroweak and strong interactions
- Does not describe
  - Neutrino oscillations
  - Dark matter (Ω_{DM})
  - Baryon asymmetry
  - BBN, CMB, structure formation:

    $\sum m_\alpha$, mass hierarchy

    (Planck, CMBPole, BOSS)

    . . . light sterile neutrinos?
  - Many models, but new ideas are welcome !

Phenomenological problems of the Standard Model

Gauge fields (interactions) – γ, $W^{\pm}$, Z, g

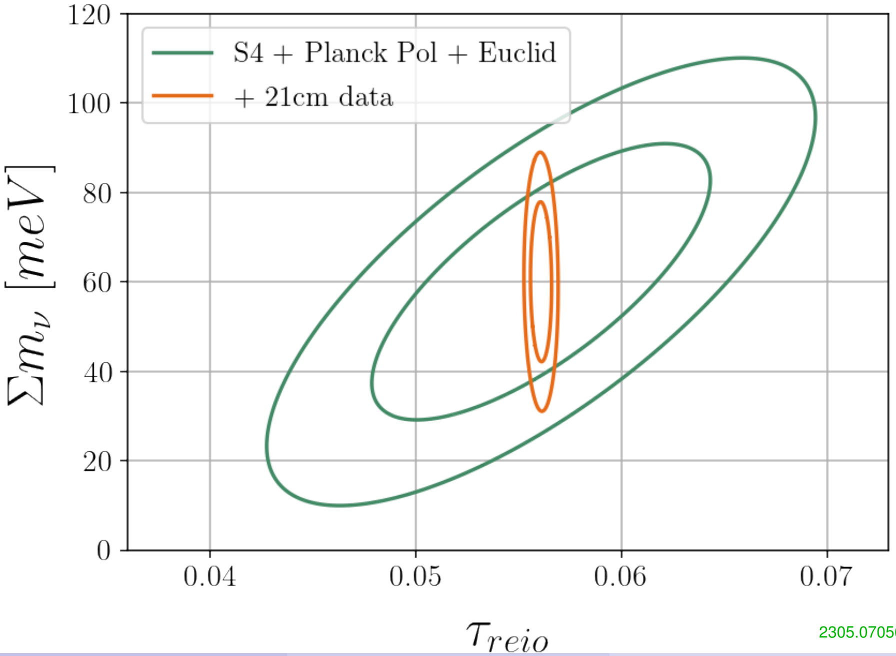
Three generations of matter: $L = \binom{\nu_L}{e_L}$, e_R; $Q = \binom{u_L}{d_L}$, d_R, u_R

- Describes
  - all experiments dealing with electroweak and strong interactions
- Does not describe
  - Neutrino oscillations
  - Dark matter (Ω_{DM})
  - Baryon asymmetry
  - BBN, CMB, structure formation:

    $\sum m_\alpha$, mass hierarchy

    (Planck, CMBPole, BOSS)

    . . . light sterile neutrinos?
  - Many models, but new ideas are welcome !


S4 + Planck Pol + Euclid
+ 21cm data
120
100
80
60
40
20
0
0.04
0.05
0.06
0.07
Σm_ν [meV]
τ_{reio}

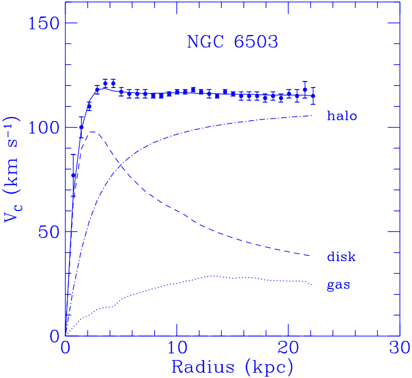
2305.07056

BACKUP SLIDES

Galactic dark halos: flat rotation curves

$$v(R) = \sqrt{G\frac{M(R)}{R}}$$

$$M(R) = 4\pi \int_0^R \rho(r) r^2 dr$$



observations: $v(R) \simeq \text{const}$

visible matter: internal regions $v(R) \propto \sqrt{R}$

external ("empty") regions $v(R) \propto 1/\sqrt{R}$

Dark Matter in clusters

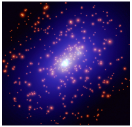
X-rays from hot gas in clusters

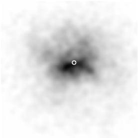
$$\frac{dP}{dR} = -\mu n_e(R) m_p \frac{GM(R)}{R^2}\,, \quad M(R) = 4\pi \int_0^R \rho(r) r^2 dr\,, \quad P(R) = n_e(R) T_e(R)$$

galaxies in clusters — virial theorem

$$U + 2E_k = 0$$

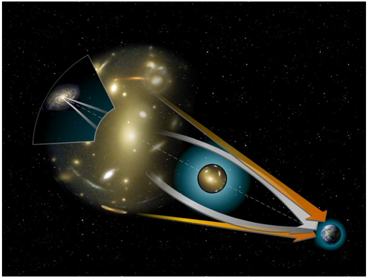
$$3M\langle v_r^2 \rangle = G\frac{M^2}{R}$$



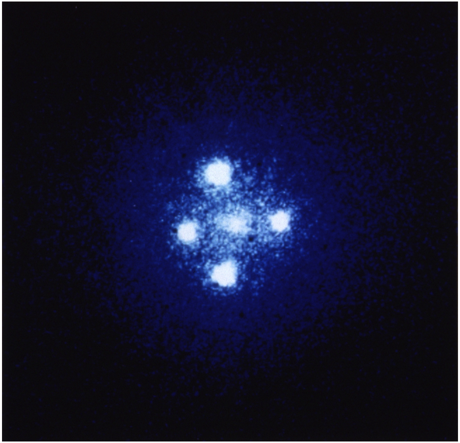


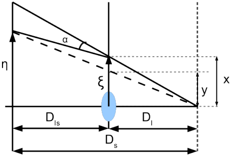
Milky Way: Virgo infall

Gravitational lensing in GR: $\alpha = 4GM/(c^2 b)$



Einstein Cross





$$\vec{\eta} = \frac{D_s}{D_l}\vec{\xi} - D_{ls}\vec{\alpha}\left(\vec{\xi}\right)$$

common lens with specific refraction coefficient

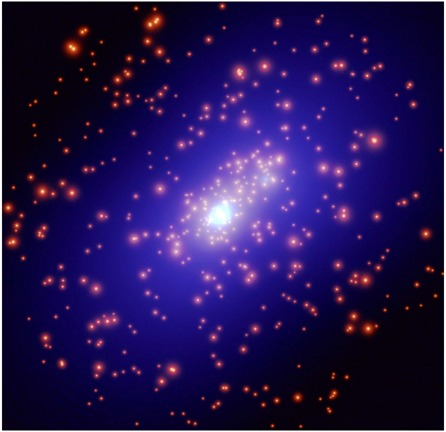
$$\vec{\alpha}\left(\vec{\xi}\right) = \frac{4G}{c}\int \frac{\vec{\xi}-\vec{\xi}'}{\left|\xi-\vec{\xi}'\right|^2} d^2\xi' \int \rho\left(\vec{\xi}',z\right) dz$$

source: quasar $D_s = 2.4$ Gpc

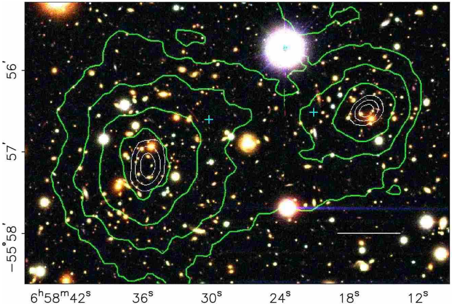
lens: galaxy $D_l = 120$ Mpc

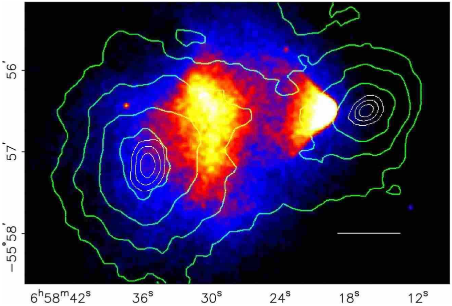
Dark Matter in clusters

gravitational lensing $\rho_B \approx 0.25\rho_{DM}$



Colliding clusters (Bullet clusters 1E0657-558)





gravitational lensing

Observations in X-rays

$M \simeq 10 \times m$

scale is 200 kpc

clusters are at 1.5 Gpc