MOTIVATING MOTIVES

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Abstract. Underlying the Riemann Hypothesis there is a question whose full answer still eludes us: what do the zeros of the Riemann zeta function really mean? As a step toward answering this, André Weil proposed a series of conjectures that include a simplified version of the Riemann Hypothesis in which the meaning of the zeros becomes somewhat easier to understand. Grothendieck and others worked for decades to prove Weil's conjectures, inventing a large chunk of modern algebraic geometry in the process. This quest, still in part unfulfilled, led Grothendieck to dream of "motives": mysterious building blocks that could explain the zeros (and poles) of Weil's analogue of the Riemann zeta function. This exposition by a complete amateur tries to sketch some of these ideas in ways that other amateurs can enjoy.

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Speaker: Maxim Bezuglov

Among all the mathematical discoveries which I’ve been privileged to make, the concept of the motive still impresses me as the most fascinating, the most charged with mystery—indeed at the very heart of the profound identity of geometry and arithmetic.

Alexander Grothendieck

The Riemann Hypothesis

Riemann zeta function

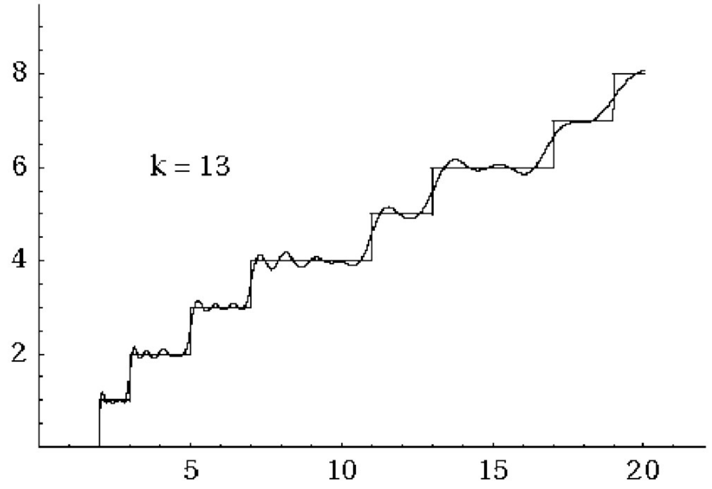
$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \frac{1}{(k-1)!} \int_0^{\infty} \frac{x^{k-1}}{e^x - 1} dx$$

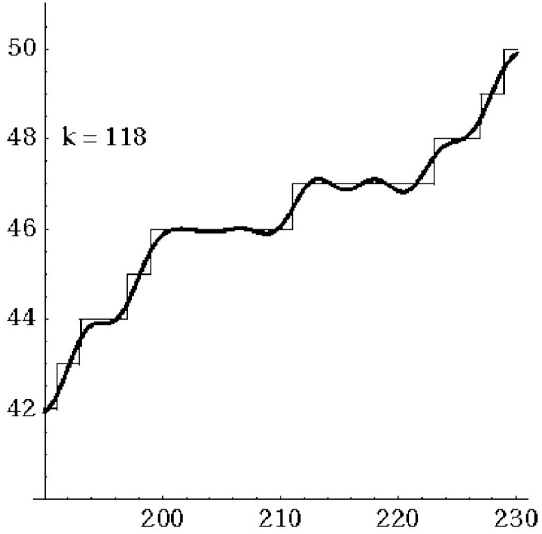
the Riemann hypothesis is the conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part 1/2

Why the Riemann hypothesis is so important ?

It is often claimed that there is no useful formula for the nth prime number. But Riemann came up with something just as good: a formula for the "prime counting function" $\pi(n)$, which is the number of primes $\leq n$:

Riemann showed that the prime counting function can be written as a main term plus a sum of oscillating "corrections", one for each nontrivial zero of the Riemann zeta function.





Since the prime counting function $\pi(n)$ equals this "main term":

$$\mathrm{li}(n) = \int_0^n \frac{dt}{\ln t}$$

plus corrections coming from the nontrivial Riemann zeta zeros, knowing the location of these zeros would give more information about the prime counting function. Indeed, the Riemann Hypothesis:

All nontrivial Riemann zeta zeros lie on the line $\mathrm{Re}(z) = \frac{1}{2}$.

is equivalent to this claim:

For some $C > 0$ and all $n \geq 1$, $|\pi(n) - \mathrm{li}(n)| \leq C\sqrt{n}\ln n$.

More simply put, the Riemann Hypothesis says that the wavelike corrections to a simple approximation to the prime counting function are not very large.

The Weil Conjectures

The Weil Conjectures include a variant of the Riemann Hypothesis that, while still difficult, has actually been proved. In this variant, the count of solutions of some polynomial equations in several variables has a “main term” and some oscillatory “correction terms”. One difference is that there are only finitely many correction terms. Another is that we know more about the meaning of the terms: they come from things called **“motives”**.

$$y^2 + y = x^3 + x \qquad x, y \in \mathbb{F}_q. \qquad q = p^n$$

n	$q = 2^n$	number of solutions
1	2	4
2	4	4
3	8	4
4	16	24
5	32	24
6	64	64
7	128	144
8	256	224
9	512	544
10	1024	1024
11	2048	1984
12	4096	4224

n	correction term
1	2
2	0
3	−4
4	8
5	−8
6	0
7	16
8	−32
9	32
10	0

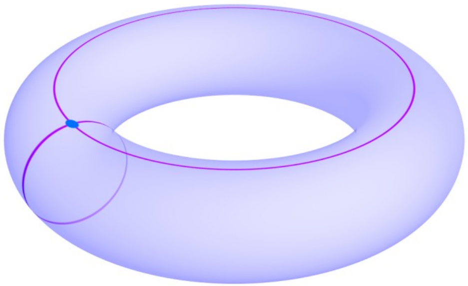
Theorem 1 (Hasse's Theorem). *Given a polynomial equation with integer coefficients in two variables that gives an elliptic curve, for any prime power $q = p^n$, the number of solutions in $\mathbb{F}_q$ of this equation is*

$$p^n - \alpha^n - \overline{\alpha}^n$$

where $\alpha \in \mathbb{C}$ has $|\alpha| = \sqrt{p}$.

$$p^n - \alpha^n - \overline{\alpha}^n + 1.$$

Now comes something amazing. The four terms in the above formula correspond, in some subtle and mysterious way, to four pieces of the elliptic curve over $\mathbb{C}$, which is a torus:



Theorem 2 (Weil's Theorem). *Given a polynomial equation with integer coefficients in two variables that gives an algebraic curve of genus g, for any prime power $q = p^n$ the number of points of this curve over $\mathbb{F}_q$ is*

$$p^n - \alpha_1^n - \cdots - \alpha_{2g}^n + 1$$

where all the numbers $\alpha_i \in \mathbb{C}$ have $|\alpha_i| = \sqrt{p}$.

Theorem 3 (Riemann Hypothesis for Varieties over Finite Fields). *Given a collection of polynomial equations with integer coefficients defining a smooth projective variety, for any prime power $q = p^n$, the number of points of this variety over $\mathbb{F}_q$ is*

$$\sum_{k=0}^{2d} \sum_{i=1}^{\beta_k} (-1)^k \alpha_{ik}^n$$

where $|\alpha_{ik}| = p^{k/2}$ and β_k is the dimension of the kth cohomology group of the corresponding projective variety over $\mathbb{C}$.

Motives

$$h\colon \mathsf{Var} \to \mathsf{Mot}^{\mathrm{op}},$$

- They are "linear categories": the hom-sets are vector spaces, and composition is bilinear.
- They are "Karoubian" or "Cauchy complete": they have direct sums, and any $\pi\colon X \to X$ with $\pi^2 = \pi$ is projection onto Y for some direct sum decomposition $X \cong Y \oplus Z$.
- They are "symmetric monoidal": they have a well-behaved tensor product $\otimes$, coming from the cartesian product of smooth projective varieties.

Grothendieck showed the Riemann Hypothesis for finite fields would follow from the so-called “Standard Conjectures”. Among other things, these conjectures would imply:

- Every variety X has $h(X) \cong X_0 \oplus \cdots \oplus X_n$ where the motive X_k has dimension k, or technically speaking “weight” k, meaning that it contributes terms proportional to α^n with $|\alpha| = p^{k/2}$ to the count of points of X over $\mathbb{F}_{p^n}$.

- The category Mot is “abelian”: it has well-behaved kernels, cokernels, subobjects and quotient objects.

- The category Mot is “semisimple”: every motive is a finite direct sum of so-called “simple” motives that have only two subobjects, 0 and that motive itself. Thus, each motive X_k above can be further decomposed into a direct sum of simple motives.

Thank you for your attention!