Genus Drop in Hyperelliptic Feynman Integrals

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The maximal cut of the nonplanar crossed box diagram with all massive internal propagators was long ago shown to encode a hyperelliptic curve of genus 3 in momentum space. Surprisingly, in Baikov representation, the maximal cut of this diagram only gives rise to a hyperelliptic curve of genus 2. To show that these two representations are in agreement, we identify a hidden involution symmetry that is satisfied by the genus 3 curve, which allows it to be algebraically mapped to the curve of genus 2. We then argue that this is just the first example of a general mechanism by means of which hyperelliptic curves in Feynman integrals can drop from genus g to $\lceil g/2 \rceil$ or $\lfloor g/2 \rfloor$, which can be checked for algorithmically. We use this algorithm to find further instances of genus drop in Feynman integrals.

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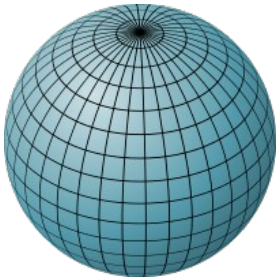
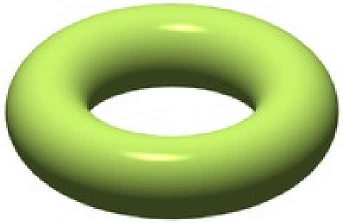
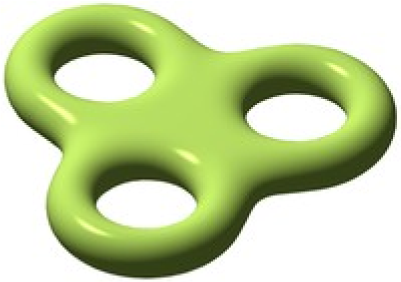
Speaker: Maxim Bezuglov

Genus

$g = 0$ $g = 1$ $g = 2$ $g = 3$







Polynomial equation

From Riemann–Roch theorem

$$f(x,y) = 0, \qquad \deg(f(x,y)) = d, \qquad g = \frac{(d-1)(d-2)}{2} - sing$$

$$y^2 - P_d(x) = 0$$

Hyperelliptic Integral

$$\int \frac{R(x)dx}{\sqrt{P_d(x)}}, \qquad d > 4$$

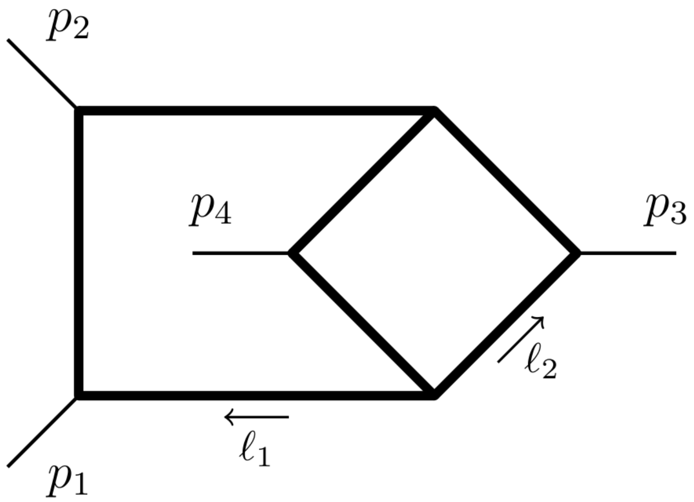


FIG. 1. The nonplanar crossed box diagram, with massive internal propagators.

Direct calculation in momentum space

$$\mathrm{MaxCut}_{\mathrm{MOM}}(I^{\mathrm{NPCB}}) \sim \int \frac{\mathrm{d}z\ z}{\sqrt{P_8(z)}}\,,$$

From loop-by-loop Baikov parametrization

$$\mathrm{MaxCut}_{\mathrm{LBL}}(I^{\mathrm{NPCB}}) \sim \int \frac{\mathrm{d}z}{\sqrt{P_6(z)}}\,,$$

$$P_6(z) = s\left(2z(s+2z) - 3m^2 s\right)\left(m^2 s + 2z(s+2z)\right)\left(s(s+t+2z)^2 - 4m^2 t(s+t)\right)\,,$$

$$\begin{aligned} P_8(z) = (s+t)^2\left(t^2 m^2 + s^2 z(sz+t)\right)\left(m^2(s+t)^2 + s^2 z(sz+s+t)\right)\times \\ \left(s^2 z m^2\left(-3s^3 z + s^2(2tz+t) + st^2(2z+3) + 2t^3\right) + t^2\left(m^2\right)^2(s+t)^2 + s^4 z^2(sz+t)(sz+s+t)\right) \end{aligned}$$

$$\mathcal{H} : y^2 = P(z),$$

A hyperelliptic curve is said to possess an extra involution $e_1 \in \overline{\mathrm{Aut}}\,(\mathcal{H})$ if there exists a $\gamma \in \mathrm{PGL}_2(\mathbb{C})$, such that we have

$$\hat{P}(\hat{z}) = Q(\hat{z}^2) \equiv c(\hat{z}^2 - \hat{\alpha}_1^2) \ldots (\hat{z}^2 - \hat{\alpha}_{g+1}^2)\,, \qquad (13)$$

involution

$$e_0 : y \to -y$$

extra involutions

$$e_1 : \hat{z} \to -\hat{z}\,.$$

$$e_2 = e_1 \circ e_0 : (\hat{y}, \hat{z}) \to (-\hat{y}, -\hat{z})$$

To our curve $\mathcal{H}$, we can associate the two curves

$$\mathcal{H}_1 : v_1^2 = Q(w) = c(w - \hat{\alpha}_1^2) \ldots (w - \hat{\alpha}_{g+1}^2)\,,$$
$$\mathcal{H}_2 : v_2^2 = wQ(w) = cw(w - \hat{\alpha}_1^2) \ldots (w - \hat{\alpha}_{g+1}^2)\,.$$

We can recover $\mathcal{H}$ via the maps

$$\rho_1 : (v_1, w) \to (\hat{y}, \hat{z}^2)\,,$$
$$\rho_2 : (v_2, w) \to (\hat{y}\hat{z}, \hat{z}^2)\,,$$

It can be shown that there exists a $\gamma \in \mathrm{PGL}_2(\mathbb{C})$ such that $\gamma^{-1}[p_1^{(i)}] = -\gamma^{-1}[p_2^{(i)}]$ if and only if there exists a $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$ with $\nu_2^2 - 4\nu_1\nu_3 \neq 0$ such that

$$\begin{pmatrix} 2p_1^{(1)}p_2^{(1)} & p_1^{(1)} + p_2^{(1)} & 2 \\ \vdots & \vdots & \vdots \\ 2p_1^{(g+1)}p_2^{(g+1)} & p_1^{(g+1)} + p_2^{(g+1)} & 2 \end{pmatrix} \cdot \nu = \vec{0}\,, \qquad (18)$$

we need to conseder all roots pairings $(p_1^{(i)}, p_2^{(i)})$ for $g = 1, \ldots, g+1$

total $(2g+1)!!$ pairings

$$\gamma = \begin{cases} \begin{pmatrix} \nu_2 & -\lambda \frac{\nu_3}{\nu_2} \\ 0 & \lambda \end{pmatrix}, & \nu_1 = 0, \\ \begin{pmatrix} \nu_2 + \sqrt{\nu_2^2 - 4\nu_1\nu_3} & \lambda \frac{\sqrt{\nu_2^2 - 4\nu_1\nu_3} - \nu_2}{\nu_1} \\ -2\nu_1 & 2\lambda \end{pmatrix}, & \nu_1 \neq 0\,, \end{cases}$$

$$P_6(z) = s\left(2z(s+2z)-3m^2s\right)\left(m^2s+2z(s+2z)\right)\left(s(s+t+2z)^2-4m^2t(s+t)\right)\ ,$$

$$P_8(z) = (s+t)^2\left(t^2m^2+s^2z(sz+t)\right)\left(m^2(s+t)^2+s^2z(sz+s+t)\right)\times$$
$$\left(s^2zm^2\left(-3s^3z+s^2(2tz+t)+st^2(2z+3)+2t^3\right)+t^2\left(m^2\right)^2(s+t)^2+s^4z^2(sz+t)(sz+s+t)\right)$$

$$\gamma_1 = \begin{pmatrix} 1 & -1 \\ r & r \end{pmatrix}, \quad z = \gamma_1[\hat{z}] = \frac{1}{r}\frac{\hat{z}-1}{\hat{z}+1},$$

where $r = \sqrt{s^3/(m^2t(s+t))}$. The polynomial

$$Q_4(\hat{z}^2) = (r\hat{z}+r)^8\ P_8\left(\frac{1}{r}\frac{\hat{z}-1}{\hat{z}+1}\right)$$

the elliptic curve

$$v_1^2 = Q_4(w)$$

and the genus 2 hyperelliptic curve

$$v_2^2 = wQ_4(w) \equiv Q_5(w).$$

$$\gamma_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{m^4}{\sqrt[3]{2s^5(s+t)}}\begin{pmatrix} 1 & \frac{s+t}{2m^2}-\sqrt{\frac{(s+t)t}{sm^2}} \\ 1 & \frac{s+t}{2m^2}+\sqrt{\frac{(s+t)t}{sm^2}} \end{pmatrix}$$

which allows us to relate these curves via

$$(cw+d)^6Q_5(\gamma_2[w]) = P_6(w)\,.$$

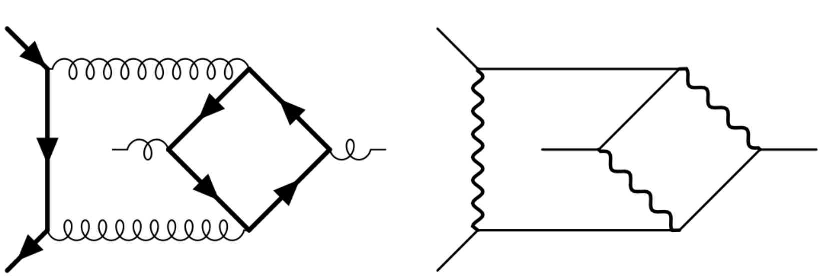


FIG. 3. Examples of hyperelliptic Feynman integrals in which genus drop via an extra involution can be observed. These integrals contribute to $gg \to t\bar{t}$ with a top loop, and Møller scattering $e^- e^- \to e^- e^-$ with the exchange of three Z bosons.

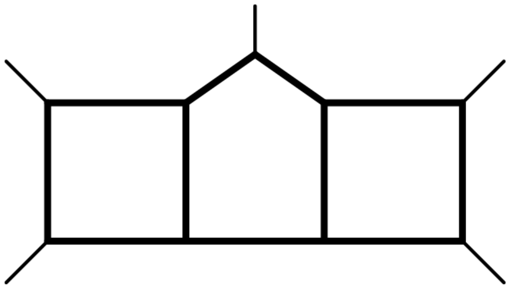


FIG. 4. The three-loop box-pentagon-box integral with equal internal masses and massless external momenta, which exhibits a genus drop from 5 to 3.

Thank you for your attention!