High-precision numerical estimates of Mellin-Barnes integrals for the structure functions using a stationary phase contour

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Outline

- Motivation
- Overview of theoretical framework: contours
- Numerical estimation of accuracy
- From Q_0^2 to Q^2
- Conclusions

A list of physical task in high-energy physics solved using the Mellin-Barnes (MB) integrals – a family of integrals in the complex plane whose integrand is given by the ratio of products of Gamma functions (two-loop massive Bhabha scattering in QED, three-loop massless form factors, static potentials, massive two-loop QCD form factors, B-physics studies hadronic top-quark physics and muon magnetic moment anomaly from lepton vacuum polarization) can be supplemented by finding the structure functions, the fragmentation functions and parton distributions in QCD analysis of the deep inelastic scattering (DIS) data. The volume of experimental data which is included in the analysis quickly increases and their accuracy improves. Numerical processing of this experimental material requires the effective methods.

Recently, significant progress has been made in the numerical computation of the MB integrals, as example, the paper by Gluza, Jelinsky,Kosower[PRD 95 (2017)]. The choice of the integration contour is of great practical importance. The best efficiency in a numerical integration of the MB integrals can be achieved on the contour of the stationary phase where the oscillations of the integrand are minimal. However, the solution of the differential equation for the stationary phase contour and its subsequent application to calculate the MB integral requires big computing expenses. Instead, it is proposed to build such approximations of the stationary phase contour that would allow the effective application of the quadrature integration formulas.

Overview of theoretical framework: contours

The inverse Mellin transform method is widely used in calculations related to DIS [M. Gluck, E. Reya PRD 14 (1976)]. The general expression for the inverse Mellin transform is written as a contour integral in the complex z-plane

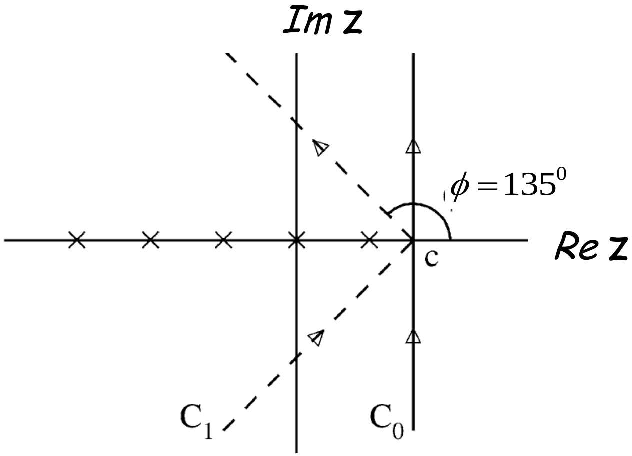
$$f(x,Q^2) = \frac{1}{2\pi i}\int_C dz\ x^{-z}\,\tilde{f}(z,Q^2) \qquad (1)$$

The moments of the structure function at some fixed momentum transfer Q^2 is usually expressed in terms of the ration of Gamma function . Then Eq. (1) is a typical MB integral.

In DIS-data processing the integral has to be calculated more than a few millions times. Therefore, optimal numerical integration depends on the number of terms in the quadrature formula.

Usually, the contour C in (1) is chosen parallel to the imaginary axis (C_0) to the right of the rightmost pole in the integrand, or a straight line at an angle (C_1) .

[M. Gluck, E. Reya and A. Vogt, Z. Phys. 48, 471 (1990); A. Vogt, Com. Phys. Commun. 170, 65 (2005)]



Standard contours

The efficient contour based on the saddle-point method of the integrand in (1) was suggested by Kosower [D.A. Kosower, Nucl. Phys. B 506, 439 (1997)].
This contour running through a saddle point of the integrand (1) in the complex z-plane has the parabolic shape.

We present a new approximation for the efficient contour, which close to the exact contour of the stationary phase [A. Sidorov, V. Lashkevich, OS, Phys. Rev. D97 (2018) 076009].

We will consider efficient numerical evaluation of the MB integrals for the structure functions, for two exactly solvably examples, and some MB integrals arising in the Feynman diagrams.

We will answer a question that is important from the practical point of view: how many polynoms in the quadratur formula for r.h.s. in (1) has to be used for evaluate the integral with a fixed relative error of $\sim 10^{-4}$ or 10^{-12} by using the efficient contours C_{as} and C_K .

Notation

C_{st} - the exact stationary phase contour

C_{as} - our contour [the asymptotic stationary phase contour]

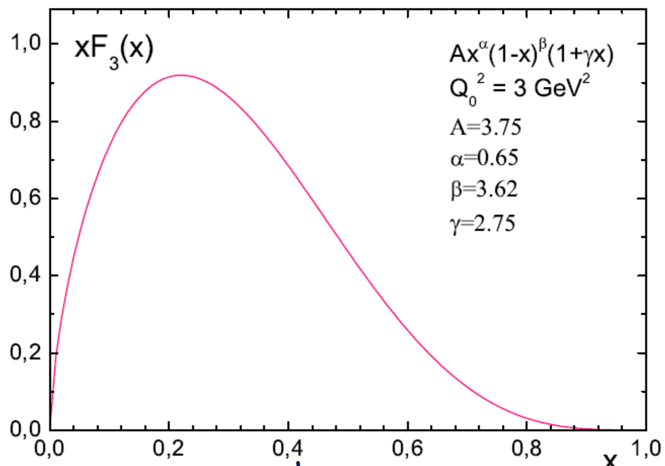
C_K - the Kosower contour

Note

The Mellin-Barnes integral widely used in calculations of Feynman diagrams, as massive propagator can be written as

$$\frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(a+z)\Gamma(-z)\frac{(-m^2)^z}{(p^2)^{a+z}}.$$

Input: DIS



x-shape

The parameterization of the F_3 structure function is typical and widely used in DIS for quarks and gluons densities

$$xF_3(x,Q_0^2) = Ax^{\alpha}(1-x)^{\beta}(1+\gamma x)$$

$$M_n(Q_0^2) = \int_0^1 x^{n-1} F(x,Q_0^2)dx$$

$$M(z,Q_0^2) = A\Gamma(\beta+1)\left[\frac{\Gamma(\alpha+z)}{\Gamma(\alpha+\beta+1+z)} + \gamma\frac{\Gamma(\alpha+1+z)}{\Gamma(\alpha+\beta+2+z)}\right]$$

Note

Earlier from QCD analysis of DIS data we extracted values of the form for NS structure functions at some fixed Q
[A. Sidorov, OS, Mod. Phys. Lett. A 29 (2014)].

The Q^2 evolution of the structure-function non-singlet (NS) moments has a simple form and, in the leading order, is fully determined by the strong-interaction constant and by the values of the nonsinglet anomalous dimensions $\gamma_{NS}^{(0)}(n)$.

$$M_n(Q^2) = \left[\frac{\alpha_S(Q_0^2)}{\alpha_S(Q^2)}\right]^{\frac{\gamma_{NS}^{(0)}(n)}{2\beta_0}} M_n(Q_0^2)$$

$$(\beta_0 = 11 - 2n_f/3)$$

Restoration of structure function

The inverse Mellin transformation (1) for the xF_3

$$xF_3(x,Q^2) = \frac{1}{2\pi i}\int_{C'} dz\, x^{-z} M_3(z,Q^2) \qquad (1a)$$

$$M_3(z,Q_0^2) = \int_0^1 dx\, x^{z-1} xF_3(x,Q_0^2) = A\,\Gamma(\beta+1)\underbrace{\left[\frac{\Gamma(\alpha+z)}{\Gamma(\alpha+\beta+1+z)} + \gamma\frac{\Gamma(\alpha+1+z)}{\Gamma(\alpha+\beta+2+z)}\right]}$$

$$xF_3(x,Q_0^{\,2}) = Ax^{\alpha}(1-x)^{\beta}(1+\gamma x)$$

We will calculate (1a) numerically and compare the result with exact value.

Note

$\Phi(z) \equiv e^{-z\ln x} M_3(z,Q_0^2)$

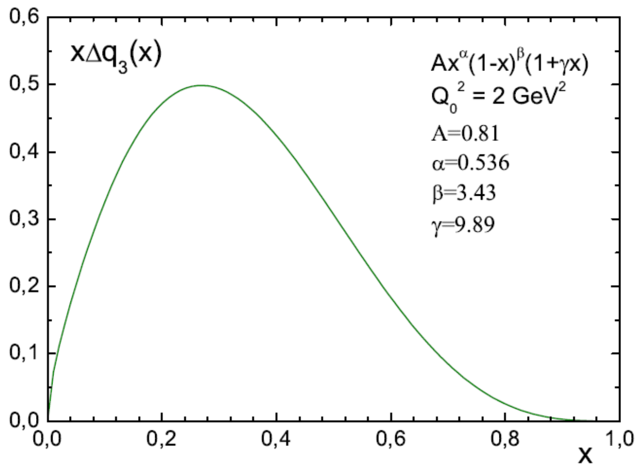
c_0 – the saddle point : $\Phi'(c_0) = 0$

$*$ – the complex point conjugation

The equation for the stationary phase contour C_{st}:

$$\frac{dx}{dy} = \frac{\mathrm{Re}\left\{\partial_z \ln\Phi\right\}^*}{\mathrm{Im}\left\{\partial_z \ln\Phi\right\}^*} \quad \text{with condition } x(0) = c_0,$$

$z = x + iy$



Non-singlet combination of polarized quark densities

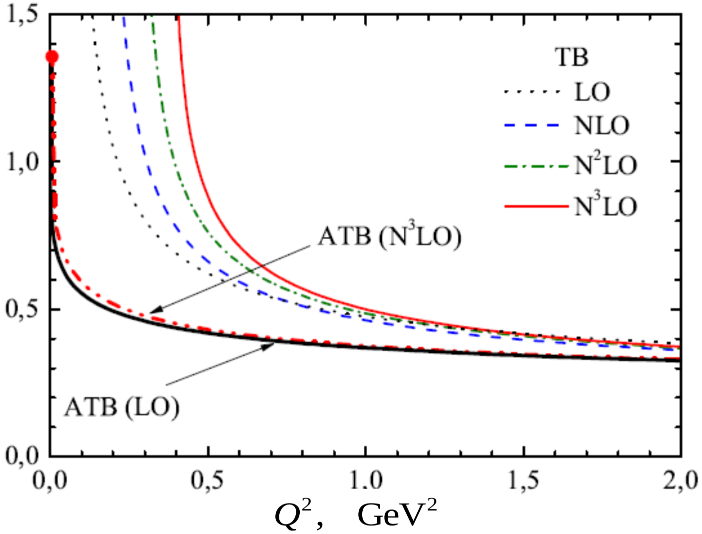
$$x\Delta q_3(x, Q_0^{\,2}) = A x^{\alpha}(1-x)^{\beta}(1+\gamma x)$$

$$x\Delta q_3(x, Q^2) =$$

$$= [x\Delta u(x, Q^2) - x\Delta \bar{u}(x, Q^2)] - [x\Delta d(x, Q^2) - x\Delta \bar{d}(x, Q^2)]$$

Note: Analytic QCD running coupling

APT = PT + RG + Q^2-analyticity



Semi-inclusive lepton-nucleon process (SIDIS)

$$\ell + p \rightarrow \ell' + h + X$$

$$zD_i^{\pi^+}(z) = \frac{N_0 z^{\alpha}(1-z)^{\beta}[1+\gamma(1-z)^{\delta}]}{B[\alpha+1, \beta+1] + \gamma B[1+\alpha, \beta+\delta+1]}$$

$B\ (a,b)$ - the Beta function

parameters
from the paper D.de Florianet *et.al.*, PRD 95 (2017)

$$\alpha_{\mathrm{PT}}^{LO}(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln\left(Q^2/\Lambda^2\right)}$$

$$\alpha_{APT}^{LO}(Q^2) = \frac{4\pi}{\beta_0}\left[\frac{1}{\ln\left(Q^2/\Lambda^2\right)} + \frac{1}{1-Q^2/\Lambda^2}\right]$$

D.V. Shirkov, I.L.Solovtsov,
Phys. Rev. Lett. 79 (1997) 1209

1st step

[D.A. Kosower, Nucl. Phys (1997)]

$$f(x,Q^2)=\frac{1}{\pi}\int_{C'}\mathbf{Re}\left[-i\,dz\;F(z,Q^2)\right],$$

$$F(z,Q^2)\equiv x^{-z}\tilde{f}(z,Q^2)$$

The saddle point: $F'(c_0,Q^2)=0$

$z(t)=\mathrm{x}(t)+i\mathrm{y}(t)$ with the conditions $\mathrm{x}(t_0=0)=c_0,\mathrm{y}(t_0)=0$ $[(\mathrm{x},\mathrm{y})-\text{real}]$.

(the dependence on values of the Bjorken $-x$ and Q^2 is omitted)

$$F(z(t))=F(c_0)-\frac{F''(c_0)}{2}t^2+\frac{1}{6}\left[-iF^{(3)}(c_0)+3iF''(c_0)x''(0)\right]t^3+\ldots$$

Im $F(z(t))=0$ to order $O(t^3)$:

$$z(t)=c_0+it+\frac{F^{(3)}(c_0)}{6F''(c_0)}t^2=c_0+it+\frac{c_3}{2}t^2 \qquad [C_K^{(3)}\text{-contour or } C_K]$$

$$f(x,Q^2)=\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\left(1-ic_3t\right)F(z(t),Q^2)\right]dt$$

$$z(t)=c_0+it+\frac{c_3}{2}t^2+c_6t^4 \qquad [C_K^{(6)}\text{-contour]}$$

2nd step New variable: $t = c_2\sqrt{u}$, where $c_2 = \sqrt{2F(c_0)/F''(c_0)}$

Using a new variable u one can write the inverse Mellin transform in the form

$$f(x,Q^2) = \frac{c_2}{2\pi}\int_0^\infty \frac{du}{\sqrt{u}}\, e^{-u} H_1(u),$$

where the function $H(u)$ up to order $O(u^2)$ reads as

$$H_1(u) = \mathrm{Re}\left[e^u\left(1 - ic_3c_2\sqrt{u} - 4ic_6c_2^3\, u^{3/2}\right)F\left(z(c_2\sqrt{u}), Q^2\right)\right]$$

$$z(u) = c_0 + ic_2\sqrt{u} + \frac{c_3}{2}c_2^2\, u + c_6c_2^4\, u^2$$

$$c_3 = \frac{F^{(3)}(c_0)}{3F''(c_0)},\ c_4 = \frac{c_3F^{(4)}(c_0)}{12F''(c_0)},\ c_5 = \frac{F^{(5)}(c_0)}{120F''(c_0)},\ \text{and } c_6 = \left(c_4 - c_5 - \frac{3}{8}c_3^3\right).$$

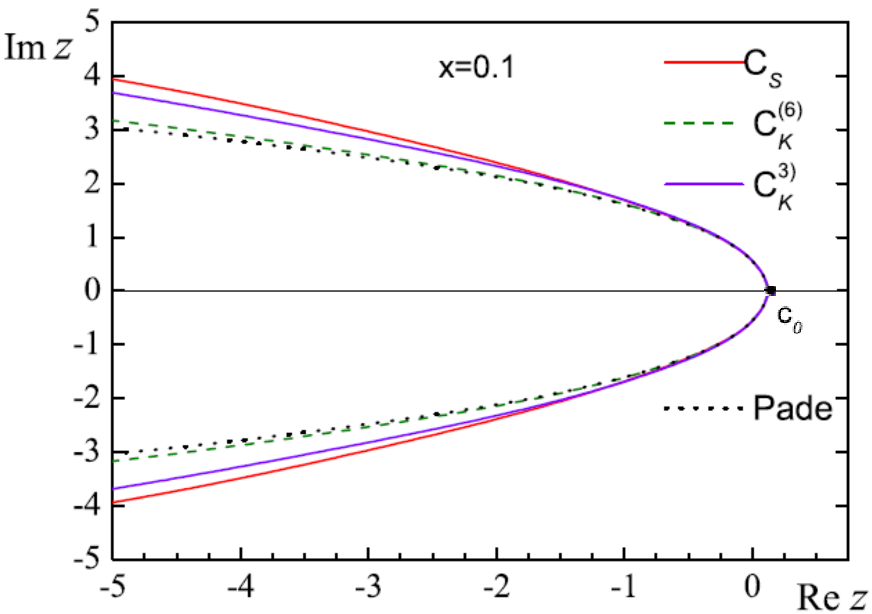
For the evaluating the integtal the generalized Gauss – Laguerre quadrature formula is used

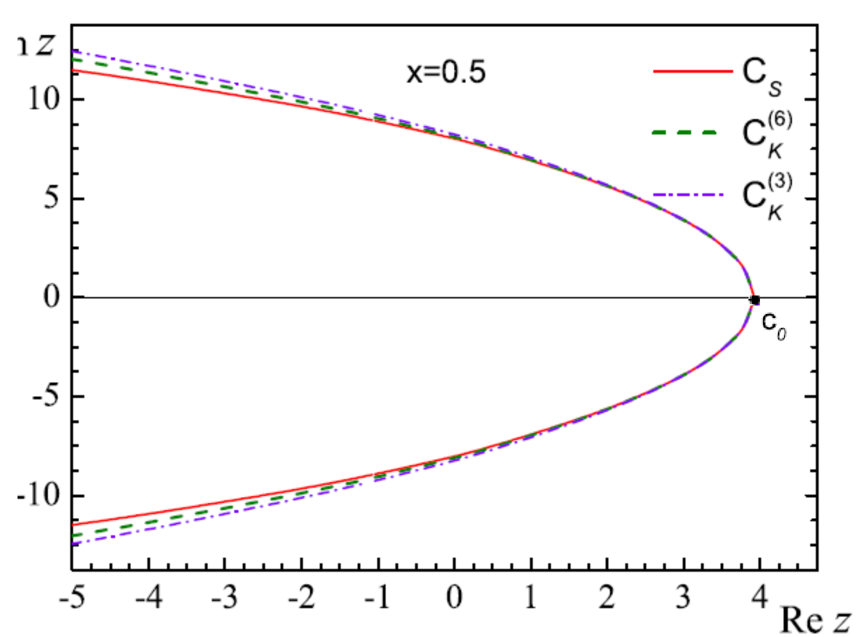
$$\int_0^\infty \frac{du}{\sqrt{u}}\, e^{-u} H_1(u) \cong \sum_{j=1}^{n} w_j H_1(u_j), \qquad (2)$$

where the u_j are the zeros of the generalized Gauss – Laguerre polynomials $L_n^{(-1/2)}(u)$.

This was the key achievement of the method proposed by Kosower. We'll see later that one point (n=1) quadrature formula (2) gives an accuracy of about a few percent. [in Table for $xF_3(x)$]

Comparison of contours





✓ the main contribution to the integral (1a) comes from the region near the saddle point, where the curves practically coincide

✓ the difference in the behavior of contours is more pronounced for small values of Bjorken x-variable

Relative accuracy

$$\varepsilon(n) = \frac{f_n(x,Q^2) - f^{exact}(x,Q^2)}{f^{exact}(x,Q^2)}, \text{ where } f_n = \sum_{j=1}^{n} w_j H_1(u_j), \; f^{exact} = xF_3(x,Q_0^2).$$

Table for $xF_3\,(x,Q_0^2)$: $C_K^{(3)}$-contour

x	$xF_3^{exact}(x)$	$n=1$	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
0.001	0.0430001482	0.030	$0.19\cdot10^{-2}$	$0.26\cdot10^{-4}$	$-0.16\cdot10^{-5}$	$-8.20\cdot10^{-8}$	$-5.3\cdot10^{-8}$
0.01	0.1890698425	0.025	$0.18\cdot10^{-2}$	$-0.66\cdot10^{-4}$	$-0.25\cdot10^{-4}$	$0.36\cdot10^{-5}$	$-3.7\cdot10^{-8}$
0.1	0.7365980291	-0.0034	$0.84\cdot10^{-3}$	$0.65\cdot10^{-5}$	$-0.31\cdot10^{-5}$	$-5.97\cdot10^{-7}$	$-7.1\cdot10^{-8}$
0.2	0.9151997245	-0.0043	$0.73\cdot10^{-3}$	$0.51\cdot10^{-5}$	$-8.12\cdot10^{-7}$	$-3.29\cdot10^{-8}$	$-7.32\cdot10^{-9}$
0.3	0.8636734952	0.0003	$0.11\cdot10^{-3}$	$-0.53\cdot10^{-5}$	$3.42\cdot10^{-7}$	$-2.12\cdot10^{-8}$	$2.14\cdot10^{-10}$
0.5	0.4625246243	0.0078	$-0.19\cdot10^{-3}$	$0.23\cdot10^{-5}$	$-6.77\cdot10^{-8}$	$1.95\cdot10^{-9}$	$2.16\cdot10^{-10}$
0.8	0.0306079020	0.012	$-0.46\cdot10^{-4}$	$-0.24\cdot10^{-5}$	$1.32\cdot10^{-7}$	$-1.31\cdot10^{-9}$	$-6.53\cdot10^{-10}$
0.95	0.0002552325	0.012	$-0.28\cdot10^{-4}$	$-0.35\cdot10^{-5}$	$0.35\cdot10^{-6}$	$-0.30\cdot10^{-9}$	$-0.39\cdot10^{-9}$

Table for the $xF_3\,(x,Q_0^2)$: $C_K^{(6)}$-contour

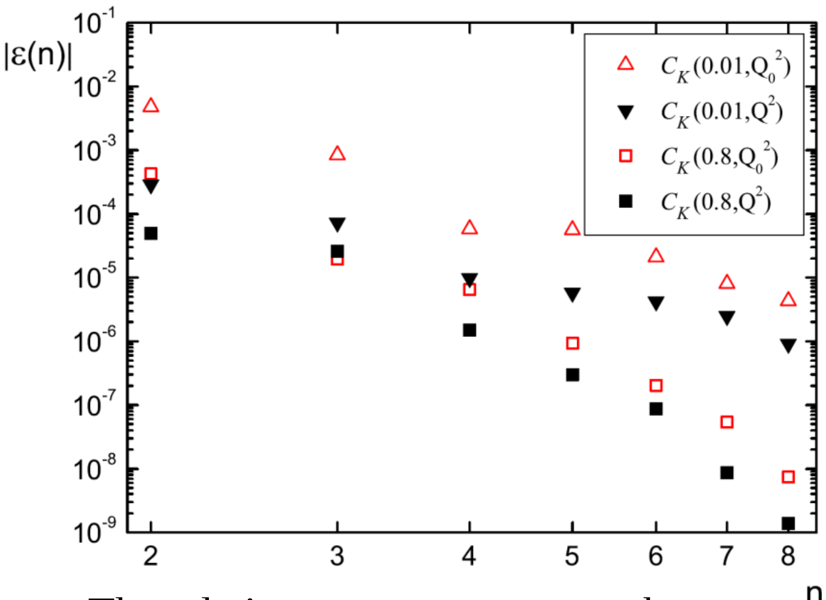
x	$n=1$	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
0.01	0.024	$0.59\cdot10^{-2}$	$0.16\cdot10^{-3}$	$-0.57\cdot10^{-4}$	$0.32\cdot10^{-4}$	$-0.16\cdot10^{-8}$
0.3	0.0003	$0.91\cdot10^{-4}$	$-0.17\cdot10^{-5}$	$1.97\cdot10^{-7}$	$-1.90\cdot10^{-8}$	$-2.55\cdot10^{-9}$
0.8	0.012	$-0.41\cdot10^{-3}$	$-1.06\cdot10^{-7}$	$0.50\cdot10^{-7}$	$-1.70\cdot10^{-9}$	$3.55\cdot10^{-11}$

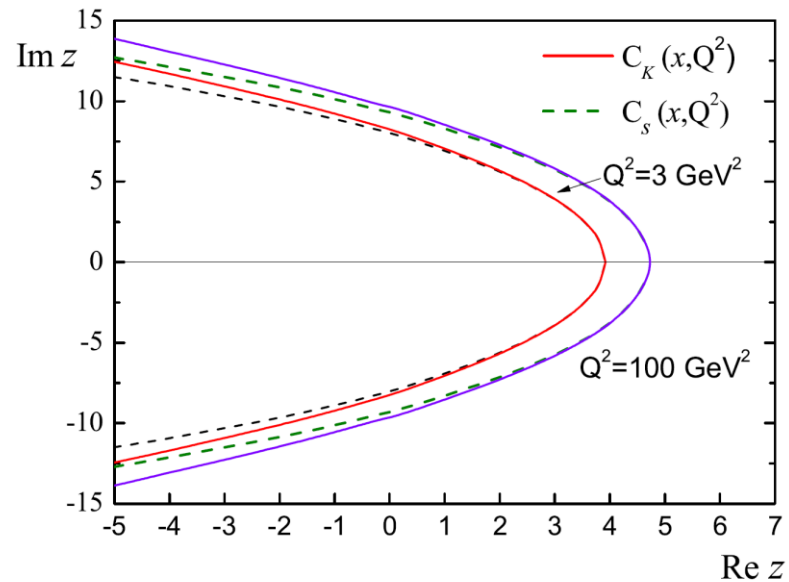
From Q_0^2 to Q^2

Q^2-evolution: $$M_3^{NS}(z,Q^2) = M_3(z,Q_0^2)\exp\left\{\ln\left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right]\frac{\gamma_{NS}^{(0)}(z)}{2\beta_0}\right\}$$

$$xF_3(x,Q^2) = \frac{1}{2\pi i}\int_{C'} dz\; x^{-z} M_3(z,Q^2)$$

Is it possible to use contour $C_K\left(Q_0^2\right)$ for calculation of structure functions at $Q^2 \neq Q_0^2$?





The relative accuracy vs. a number of polynoms N in the quadrature formula (2) at x=0.01 (triangles) and x=0.8 (squares) for the counter $C_K(x,Q_0^2=3\ \mathrm{GeV}^2)$ (open red triangles and squares) and for $C_K(x,Q^2=100\ \mathrm{GeV}^2)$ (full black triangles and squares).

Table. The number of polynomials which is needed for achieving $\varepsilon(n)$ better than 10^{-4} and 10^{-5} for $xF_3(x, Q^2 = 100\ \mathrm{GeV}^2)$ for two types of contour of integration.

	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-5}$
x	$C_K(x, Q^2)$	$C_K(x, Q_0^2)$	$C_K(x, Q^2)$	$C_K(x, Q_0^2)$
0.001	4	5	7	8
0.01	3	4	5	7
0.1	3	4	5	7
0.5	3	3	4	4
0.8	2	3	4	4

The contour $C_K(x, Q^2)$ works more accurately than the contour $C_K(x, Q_0^2)$, however this advantages are compensated if using the $C_K(x, Q_0^2)$ increase the number of terms in the quadrature formula (2) by 1-2 units.

At large x-values the number of terms in the sum (2) less, than at small x.

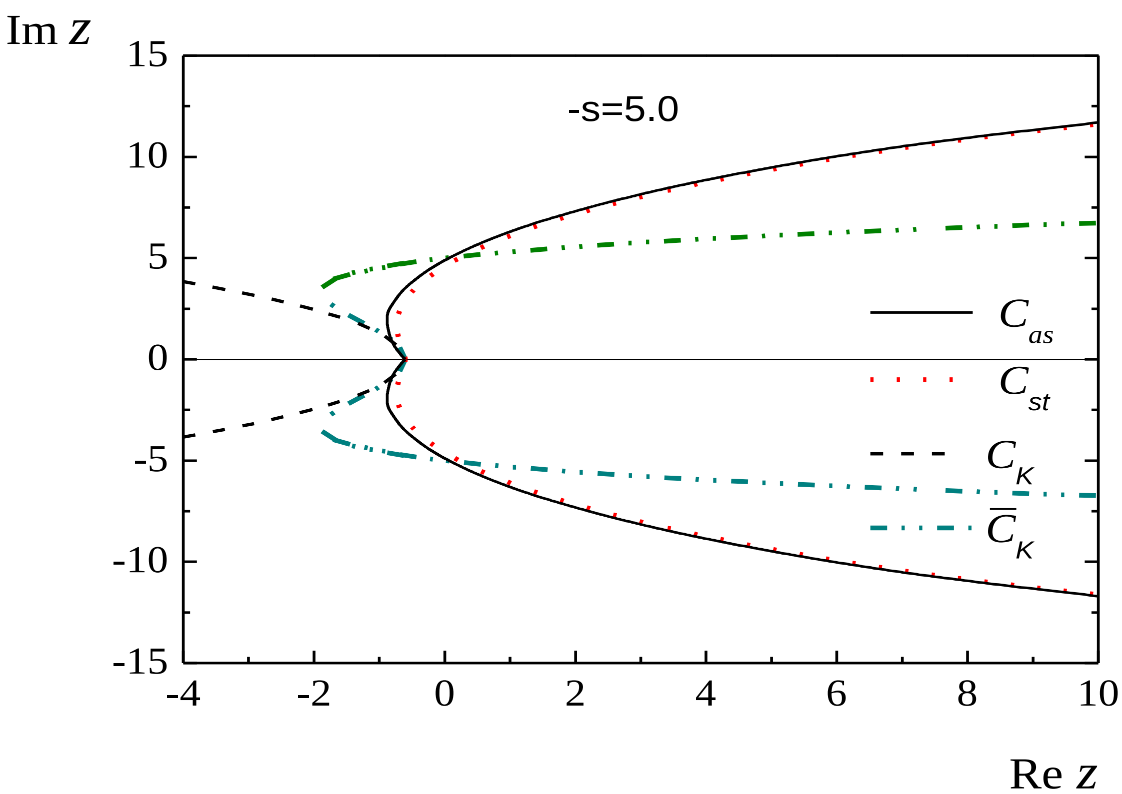
The contour $C_K(x, Q_0^2)$ can be considered as universal, i.e. applicable for any value of Q^2.

The similar result was obtained for other cases: $x\Delta q_3$, $x\Delta q_8$, $D_{u_V}^{\pi^+}$.

Some example for the Kosower contour

[Gluza, Jelinsky, Kosower, PRD 95 (2017)]

$$I_I(s) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dz\,(-s)^{-z}\,\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)\Gamma(1-z)\Gamma(2+z)} \qquad (-s \Rightarrow x_B\,)$$



The contour C_K does not take into account the asymptotic behavior of the stationary phase contour.

New construction a contour that is close to the exact contour of the stationary phase.

$$F(\mathrm{x}) = \mathrm{x}^{a} \qquad \mathrm{x} \in [0,1]$$

Mellin's moments $M(z) = \int_0^1 d\mathrm{x}\, \mathrm{x}^{n-1} F(\mathrm{x}) = \frac{1}{z+a} = \frac{\Gamma(\alpha+z)}{\Gamma(\alpha+1+z)}$

$$F(\mathrm{x}) = \frac{1}{2\pi i}\int_C dz\, \mathrm{x}^{-z} M(z) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dz\, \underbrace{\frac{e^{uz}}{z+a}}, \quad u \equiv -\ln(\mathrm{x})$$

C usually runs parallel to the imaginary axis and lies to the right of the rightmost pole and $\delta > -a$.

$$\Phi(z) \equiv \frac{e^{uz}}{z+a} \qquad z(y) = x(y) + iy$$

Selecting the imaginary part of Φ and imposing the condition on the contour $\mathrm{Im}[\Phi(z)] = 0$, we find the equation for the contour of the stationary phase C_{st}: $[2x(y)-1]\sin(uy) - 2y\cos(uy) = 0.$

The solution of this equation which provides continuity of the contour at $y = 0$ has the form

$$\begin{cases} x_{st}(y) = -a + y\,\mathrm{ctg}(uy), \; y \neq 0 \\ \lim\limits_{y\to 0} x_{st}(y) = -a + \dfrac{1}{u} \equiv c_0, y = 0 \end{cases}$$

c_0 – the saddle point : $\Phi'(c_0) = 0$

For $\mathrm{Re}\,|z| \to \infty$ $\quad x_{as}(y) = y\,\mathrm{ctg}(uy) \quad y_{as} = \dfrac{\pi}{u}\,\mathrm{sign}(y)$

Asymptotics of the contour C_{st} is bounded and is parallel to the real x-axis

To calculate the integral numerically, we apply the Gauss-Legendre quadrature formula

$$\int_0^{|y_{as}|} dy H_2(y) = \frac{|y_{as}|}{2} \sum_{j=1}^{N} w_j H_2(y_j) \qquad (3)$$

$$H_2(y) = \mathrm{Re}\left[\Phi(n_{as}(y)) \left(1 - i \frac{d\, x_{as}(y)}{dy} \right) \right] / \pi$$

$$y_j = \frac{|y_{as}|}{2}(x_j + 1), \quad x_j - \text{roots}$$

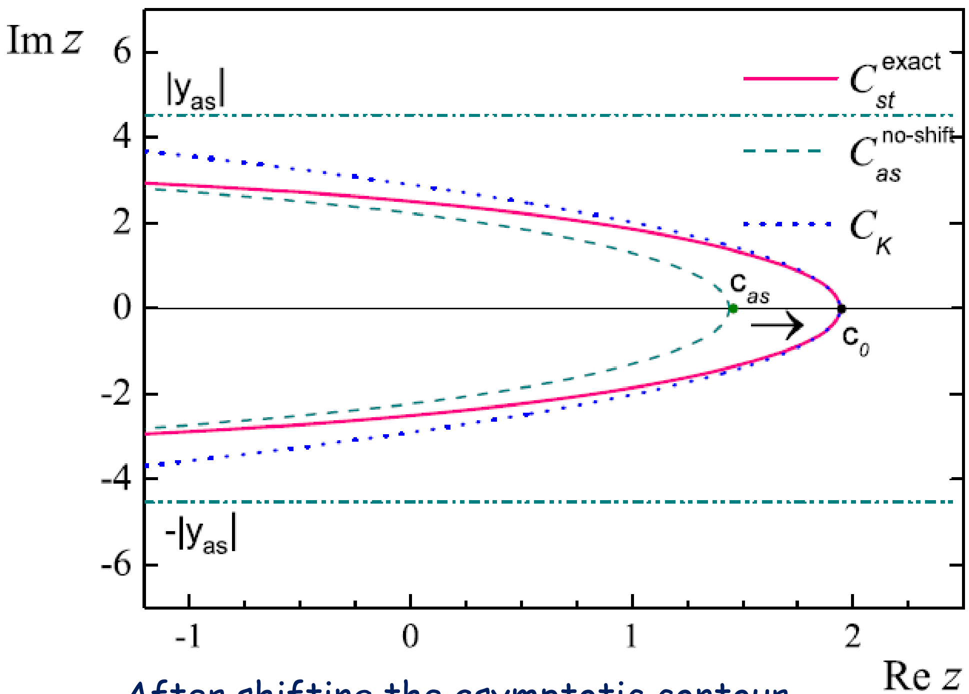
$$w_j = \frac{2}{(1 - x_j^2)[P_n'(x_j)]^2} \text{ - weight coefficients}$$

The asymptotics of the contour of the stationary phase can be found without solving above equation, but only considering the asymptotics of the integrand

$$\Phi(z) \sim \frac{e^{uz}}{z}.$$

Calculating the argument of Φ and equating its imaginary part to zero

for $\mathrm{Re}\,|z| \to \infty$ $\quad uy - \mathrm{arctg}\,\dfrac{y}{x(y)} = \pi$ $\quad \Rightarrow \quad x_{as}(y) = y\,\mathrm{ctg}(uy) \qquad y_{as} = \dfrac{\pi}{u}\,\mathrm{sign}(y)$



After shifting the asymptotic contour coincides with an exact contour: $C_{as} = C_{st}$

$$x_{st}(y) = -a + x_{as}(y)$$

The efficient asymptotic contour C_{as} is written as

$$z_{as}(y) = x_{as}(y) + \Delta c_0 + iy$$

$$\Delta c_0 = c_0 - c_{as}\,, \quad c_{as} \equiv x_{as}(y=0)$$

$$x_{as}(y=0) = 1/u$$

$$z_K(y) = c_0 - \frac{u}{3} y^2 + i\,y$$

$$y\mathrm{ctg}(uy) \simeq \frac{1}{u} - \frac{uy^2}{3} - \frac{u^4 y^4}{45} + \ldots$$

Example of two saddle points

$$F(\mathrm{x}) = \mathrm{x}^{a}(1-\mathrm{x}) \qquad 0 < \mathrm{x} < 1$$

Mellin's moments $M(z) = \int_0^1 d\mathrm{x}\; \mathrm{x}^{n-1} F(\mathrm{x}) = \dfrac{1}{(a+z)(a+1+z)} = \dfrac{\Gamma(\alpha+z)}{\Gamma(\alpha+2+z)}$

$$F(\mathrm{x}) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dz\, \frac{e^{uz}}{(a+z)(a+1+z)}, \qquad u \equiv -\ln(\mathrm{x})$$

$$\Phi(z) \equiv \frac{e^{uz}}{(a+z)(a+1+z)} \qquad\qquad z(y) = x(y) + iy$$

Selecting the imaginary part of Φ and imposing the condition on the contour

$$\mathrm{Im}\left[\Phi(z)\right] = 0,$$

we find the equation for the contour of the stationary phase C_{st}

$$4\sin(uy)x^{2}(y) - 8\, y\cos(uy)x(y) - (1+4y^{2})\sin(uy) = 0$$

$$\begin{cases} x^{(1)}(y) = -a - \dfrac{1}{2} + y\,ctg(uy) + \dfrac{\sqrt{\sin(uy)^2 + 4y^2}}{2\sin(uy)}\,\mathrm{sign}(y) \\ c_0^{(1)} = -a - \dfrac{1}{2} + \dfrac{1}{u} + \dfrac{\sqrt{4+u^2}}{2u} \\ x_{as}^{(1)}(y) = y\;ctg\left(\dfrac{uy}{2}\right) \qquad y_{as} = \dfrac{2\pi}{u}\,\mathrm{sign}(y) \end{cases}$$

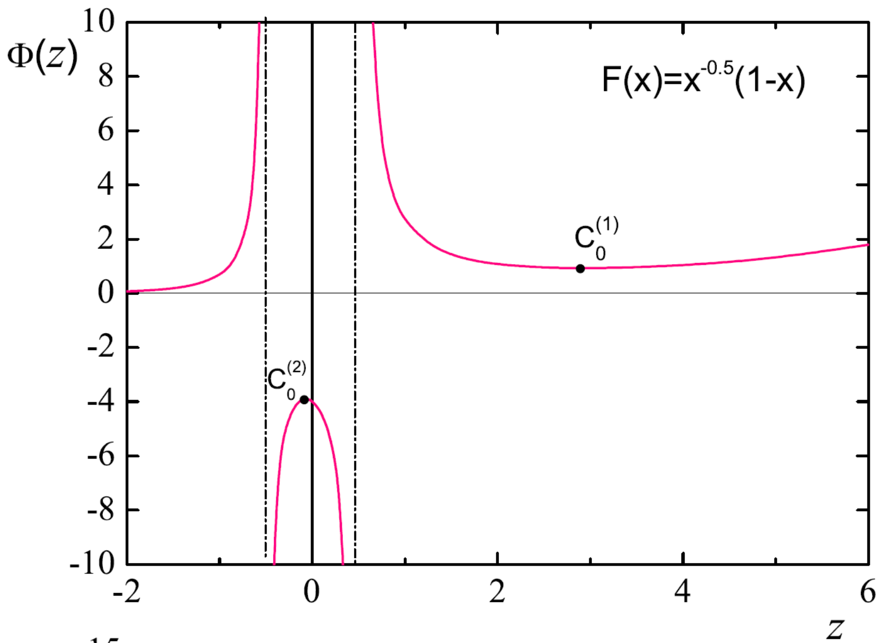
$$\begin{cases} x^{(2)}(y) = -a - \dfrac{1}{2} + y\,ctg(uy) - \dfrac{\sqrt{\sin(uy)^2 + 4y^2}}{2\sin(uy)}\,\mathrm{sign}(y) \\ c_0^{(2)} = -a - \dfrac{1}{2} + \dfrac{1}{u} - \dfrac{\sqrt{4+u^2}}{2u} \\ x_{as}^{(2)}(y) = -y\;tg\left(\dfrac{uy}{2}\right) \qquad y_{as} = \dfrac{\pi}{u}\,\mathrm{sign}(y) \end{cases}$$

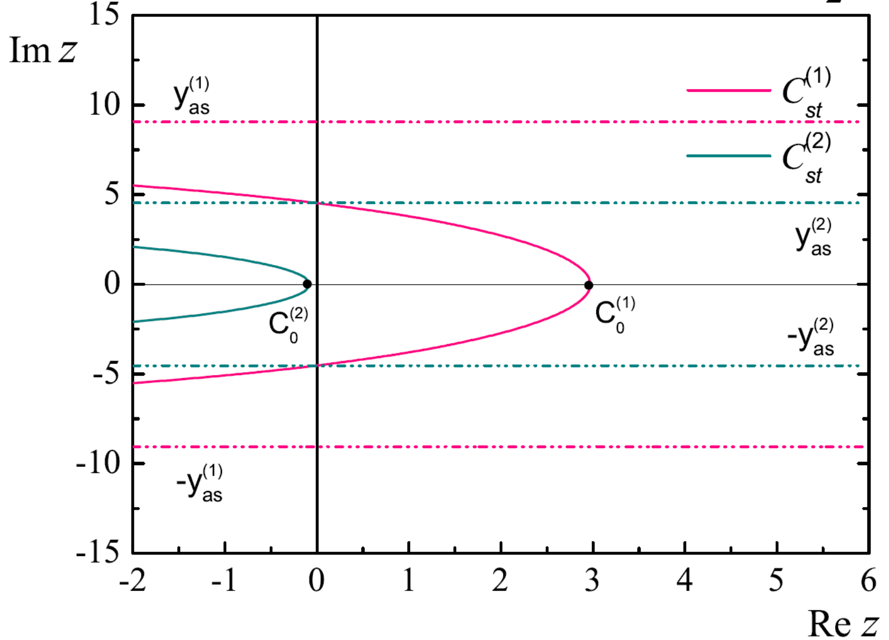
$$\Downarrow$$

Note

$$\mathrm{y} = \frac{2}{u}\,\mathrm{W}_t\left(\frac{u}{2}x\right)$$

A generalization the classical Lambert function: tg-Lambert function [Markushin, Rosenfelder, Schreiber, Nov. Cim. B 117 (2002)]





$$\Phi(z) \sim \frac{e^{uz}}{z^2}$$

$$\mathrm{Re}\,|z| \to \infty$$

$$z_{as}^{(1)} = y\ ctg\left(\frac{uy}{2}\right) + c_0^{(1)} - \frac{2}{u} + iy$$

$$z_{as}^{(2)} = -y\ tg\left(\frac{uy}{2}\right) + c_0^{(2)} + iy$$

The asymptotic contours after shifting practically merge with the corresponding exact contours.

New contour for structure function

$\Phi^{\rm DIS}(z) \equiv e^{\omega_B z} M_3(z)$, one can get the asymptotic behavior $\mathrm{Re}\,|z| \to \infty$

$$\Phi^{\rm DIS}(z) \sim e^{\omega_B z} A\,\Gamma(\beta+1)\,\frac{1+\gamma}{z^{\beta+1}}, \qquad \omega_B \equiv -\ln(x_B)$$

Calculating the argument of the Φ – function and equating its imaginary part to zero, we arrive at the equation $\omega_B y - (\beta+1)\arg z = 0.$
From this equation it follows that as z tends to -infinity, the argument z tends to $\pm\pi$, and we get the asymptotic behavior

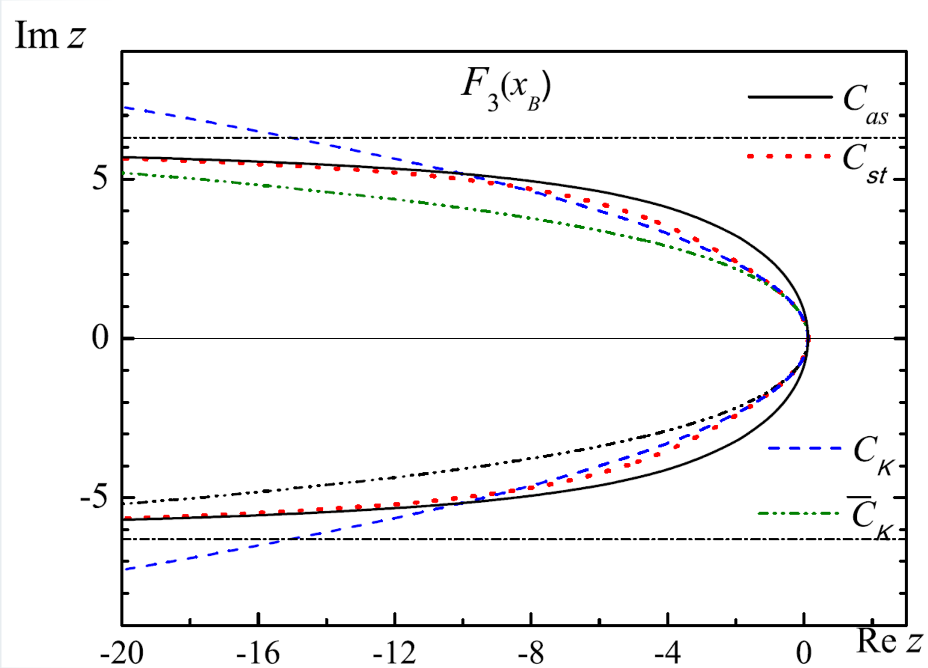
$$\boxed{x_{as}^{\rm DIS}(y) = y\,\mathrm{ctg}\left(\frac{\omega_B\, y}{\beta+1}\right), \qquad y_{as}^{\rm DIS} = \frac{(\beta+1)\pi}{\omega_B}\,\mathrm{sign}(y)}$$

The asymptotics of the stationary phase contour are parallel to the real axis.

$$\boxed{z_{as}^{\rm DIS} = y\,\mathrm{ctg}\left(\frac{\omega_B\, y}{\beta+1}\right) + c_0 - \frac{\beta+1}{\omega_B} + iy}$$

$$x_B F_3(x_B) = \int_0^{|y_{as}^{DIS}|} dy H_2^{DIS}(y),$$

$$H_2(y) \equiv \mathrm{Re}\left[\left(1 - i\,\frac{d\,x_{as}(y)}{dy}\right)\Phi(z_{as}(y))\right]/\,\pi$$



C_{st} - the exact stationary phase contour,
C_{as} - our contour,
C_K - the Kosower contour [or $C_K^{(3)}$]
$\bar{C}_K$ - the Kosower contour [or $C_K^{(6)}$].

	$x_B = 0.01$		$x_B = 0.1$		$x_B = 0.5$	
N	C_K	C_{as}	C_K	C_{as}	C_K	C_{as}
16	2.5×10^{-8}	3.2×10^{-7}	1.3×10^{-9}	2.9×10^{-8}	1.0×10^{-13}	1.4×10^{-11}
20	3.7×10^{-10}	4.1×10^{-9}	1.0×10^{-10}	1.6×10^{-10}	1.8×10^{-17}	2.2×10^{-14}
30	2.7×10^{-11}	7.6×10^{-13}	5.7×10^{-13}	3.6×10^{-16}	3.5×10^{-19}	8.7×10^{-21}
35	9.7×10^{-13}	1.5×10^{-16}	5.1×10^{-14}	1.5×10^{-16}	4.3×10^{-20}	3.6×10^{-23}

The contour C_{as} works more accurately than the contour C_K for N>30.

Q^2-evolution

$$x_{as}(y, Q^2) = y\,\mathrm{ctg}\left[\frac{\omega_B y}{(\beta+1)+\Delta(Q^2)}\right]$$

$$\Delta(Q^2) \equiv \frac{16}{33-2n_f}\ln\left[\frac{\alpha_s^{LO}(Q_0^2)}{\alpha_s^{LO}(Q^2)}\right]$$

$$y(x, Q^2) = \pm\frac{\left[(\beta+1)+\Delta(Q^2)\right]}{\omega_B} W_{ct}\left(\frac{\omega_B\, x}{(\beta+1)+\Delta(Q^2)}\right) \qquad \omega_B \equiv -\ln(x_B)$$

$$y_{as}^{DIS}(Q^2) = \pm\frac{\left[(\beta+1)+\Delta(Q^2)\right]\pi}{\omega_B}$$

The contour $C_{as}(Q^2)$ in higher orders of the perturbation theory QCD will coincides with the LO expression with the replacement only of $\alpha_s^{LO}(Q^2)$ by an expression for running coupling in the corresponding order of the perturbation theory.

Example I: MB integral arising in Feynman diagrams

$$I_I(s) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dz\,(-s)^{-z}\,\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)\Gamma(1-z)\Gamma(2+z)} \qquad (-s \Rightarrow x_B\ )$$

In the region $0 < -s < 4$ $\quad I_I(s) = -s$

$$\textit{for}\quad -s > 4 \quad I_I(s) = 2\ln(-s) - s\left(1-\sqrt{1+\frac{4}{s}}\right) + 4\ln\left[\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4}{s}}\right]$$

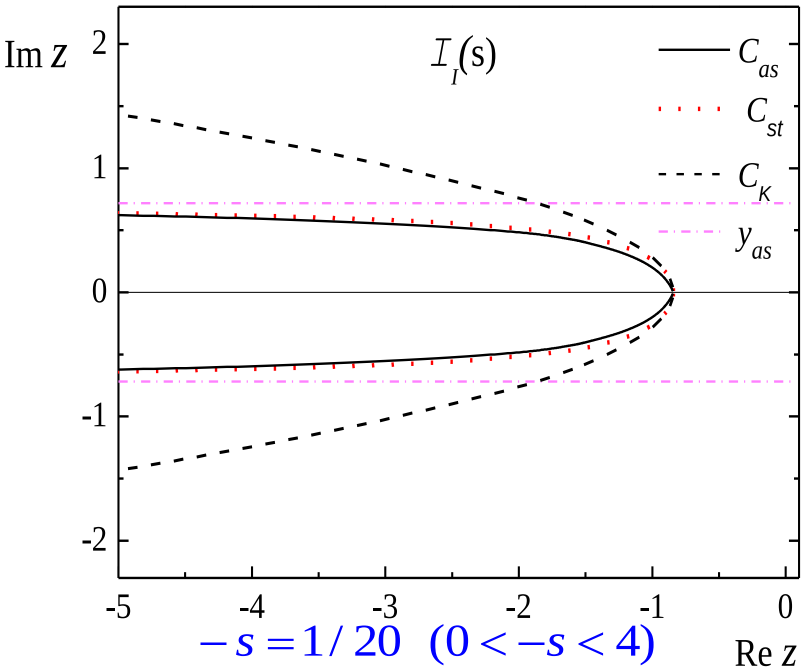
Asymptotic stationary phase integration contour

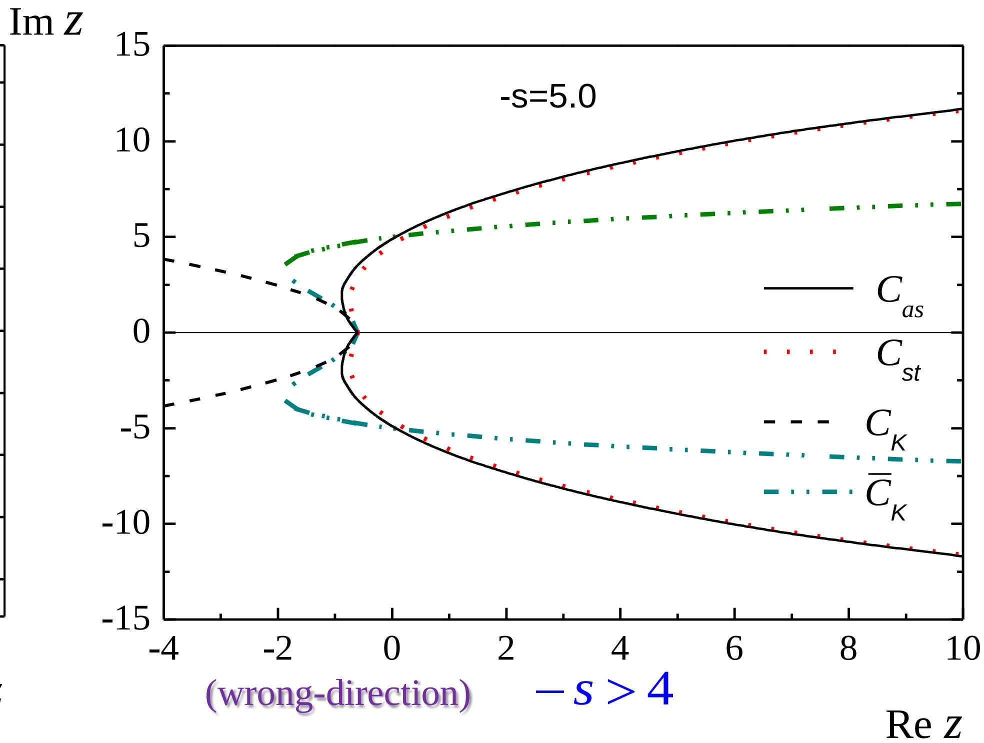
$$z_{as}(y) = x_{as}(y) + iy$$

$$x_{as}(y) = y\mathrm{ctg}\left\{\frac{2}{5}\left[-\ln(-s)+\ln 4\right]\cdot y + \frac{3}{2}\pi\mathrm{sign}(y)\right\}, \qquad \mathrm{Re}\,|z| \to \infty$$

$$y_{as} = \pm\frac{\pi}{-\ln(-s)+\ln 4} \qquad 0 < -s < 4$$

$$y_{as} = \pm\frac{3}{2}\frac{\pi}{|-\ln(-s)+\ln 4|} \qquad -s > 4$$





The proposed contour C_{as}, whose a construction is quite simple, reproduces the behavior of the exact contour C_{st} well, and the use of this contour can provide the required high relative accuracy.

	$-s = 1/20$		$-s = 2.0$	
N	C_K	C_{as}	C_K	C_{as}
16	1.2×10^{-8}	1.3×10^{-7}	1.2×10^{-6}	6.5×10^{-5}
20	6.7×10^{-10}	8.5×10^{-12}	4.0×10^{-6}	3.9×10^{-6}
30	1.2×10^{-11}	5.7×10^{-14}	5.1×10^{-7}	1.2×10^{-8}
35	5.8×10^{-13}	5.7×10^{-16}	1.4×10^{-7}	5.8×10^{-11}

Example II

$$I_{II}(s)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dz\,(-s)^z\,\frac{\Gamma^4(-z)\Gamma(1+2z)}{\Gamma^2(-2z)\Gamma^2(1+z)}$$

In the region $0<-s<4^3$ $\quad I_{II}(s)=\frac{2}{\pi}\sqrt{-s}\cdot\left[K\left(\frac{1}{2}\cdot\left(1-\sqrt{1+\frac{s}{64}}\right)\right)\right]^2$

K- the complete elliptic integral of the first kind

$-s>4^3$ $\quad I_{II}(s)=4\ln(-s)-4\cdot\sum_{n=1}^{\infty}\frac{(2n)!^3\cdot(-s)^{-n}\left[6\Psi(2n+1)-6\Psi(n+1)-7\ln(-s)\right]}{n!^6}$

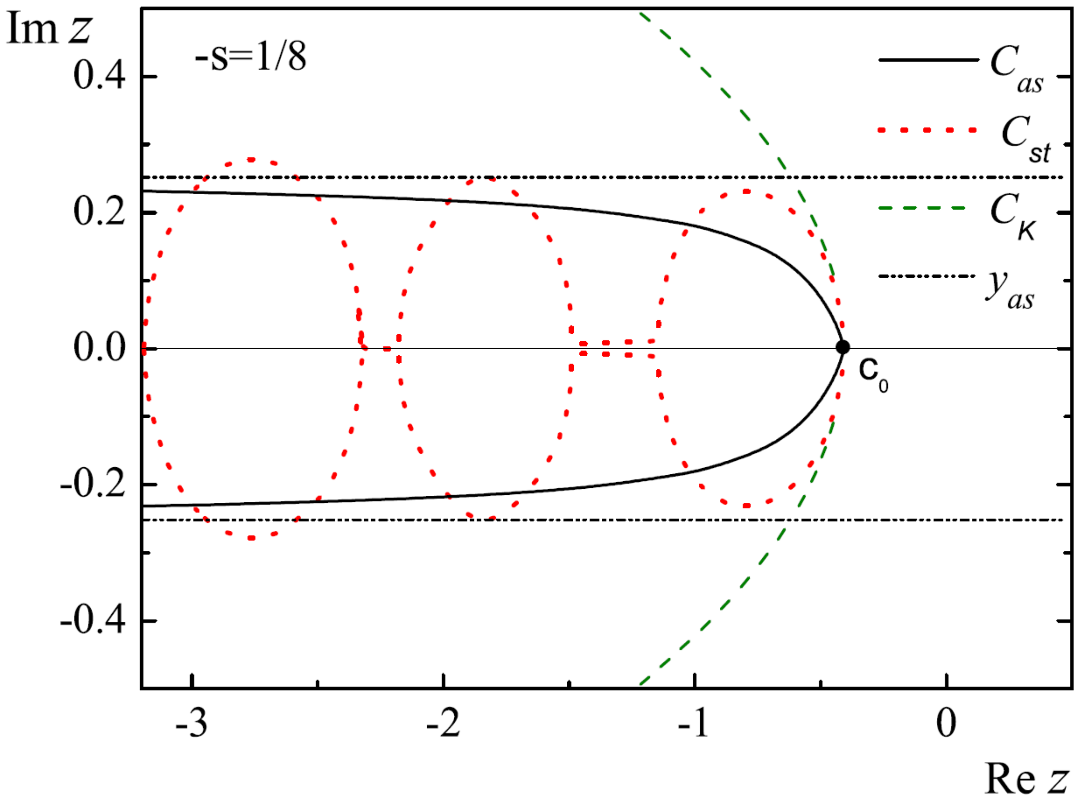
The asymptotic behavior of the integrand: $\qquad \Psi(z)=\frac{d}{dz}\ln\Gamma(z)$

$$\Phi(z)\sim\exp\left[uz-\ln(-z)+\left(6z-\frac{3}{2}\right)\ln(2)-\frac{1}{2}\ln z\right],\qquad Re\,|z|\to\infty$$

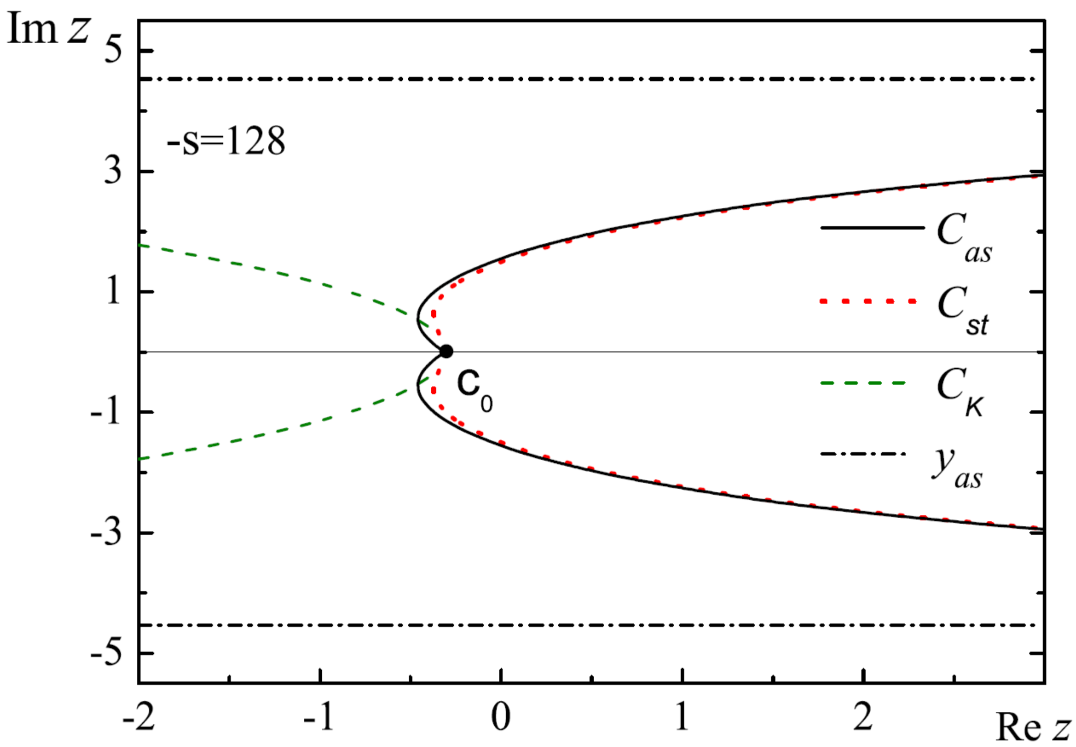
$$uy-\frac{3}{2}\arg(z)\pm\pi\,\mathrm{sign}(y)+3y\ln 4=0$$

$$x_{as}(y)=y\,\mathrm{ctg}\left[\frac{2}{3}(\omega+3\ln 4)y+\mathrm{sign}(y)\pi\right]$$

The case of a complicated shape of the stationary phase contour



The use of the asymptotic contour of the stationary phase proves to be effective even in this complicated case.



N	$-s = 20$		$-s = 128$	
	C_{as}	C_K	C_{as}	C_K
10	1.5×10^{-3}	2.1×10^{-3}	-8.3×10^{-3}	1.1×10^{-2}
12	-6.8×10^{-4}	1.4×10^{-3}	-1.1×10^{-2}	3.9×10^{-2}
16	-8.5×10^{-7}	5.3×10^{-4}	2×10^{-3}	8.8×10^{-2}
20	-3.4×10^{-6}	1.9×10^{-4}	-2×10^{-4}	0.1
30	1.3×10^{-6}	1.5×10^{-5}	-4.4×10^{-6}	0.22
35	-3.6×10^{-7}	4.2×10^{-6}	6.3×10^{-7}	0.26

Conclusions

We proposed the method for constructing an effective contour to calculate with high accuracy one-dimension Mellin-Barnes integrals. This contour is the approximation of the stationary phase contour in the case of their finite asymptotic behavior.

The construction of contour C_{as} is much easier because it is not requires computation of higher derivatives. This gives the advantage for the contour C_{as} at intensive computations related to fitting of experimental data.

It was compared the efficiency of application of the asymptotic stationary phase contour C_{as} and the contour C_K. The contour C_K turned out to be more effective for a small number of N terms in the Gauss-Laguerre quadrature formula, when the nodes of the quadrature formulas are located near the saddle point. The advantage of the contour C_{as} is manifested at large values of N. The 'regime change' occurs at N~20 for structure function.

It was shown that the contour C_{as} makes it possible to calculate the MB integrals effectively even in the case of a complicated shape of the stationary phase contour.