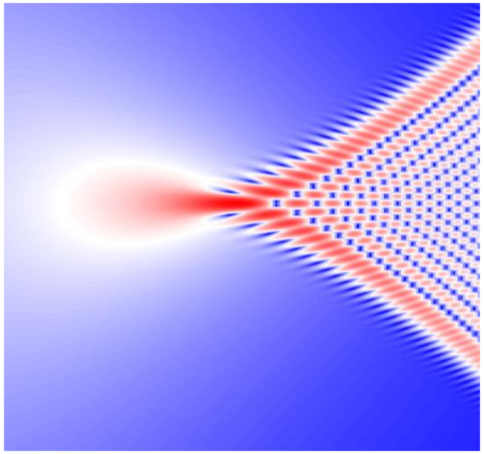

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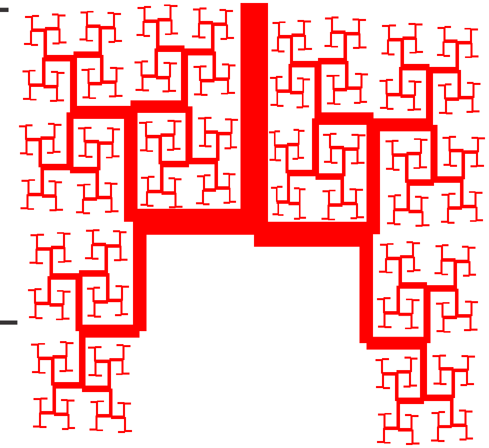

The Basics of the Radon transform for medical and quantum state tomography

Marko Ćosić

Laboratory of Physics 010, "Vinča" Institute of Nuclear Sciences
National Institute of the Republic of Serbia, University of Belgrade,
P. O. Box 522, 11001 Belgrade, Serbia

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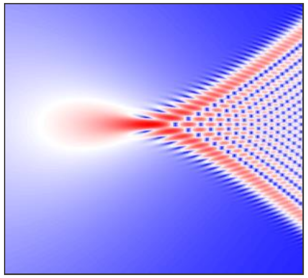

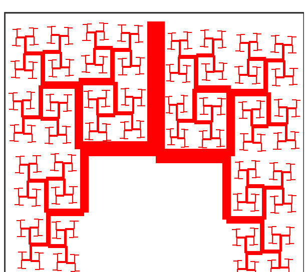

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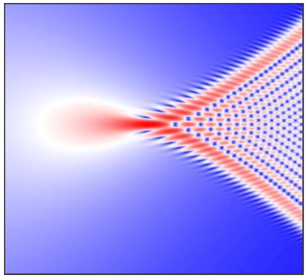
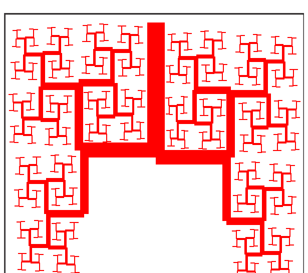


Introduction



X-rays are a type of electromagnetic radiation whose wavelengths range from 10 nm to 10 pm. They interact primarily with individual atoms, with absorption coefficient $\alpha \sim Z^3/E^3$.

When transmitting through an inhomogeneous medium, the radiation intensity changes according to the law

$$\frac{d}{dz}I(z) = -\alpha(z)I(z).$$

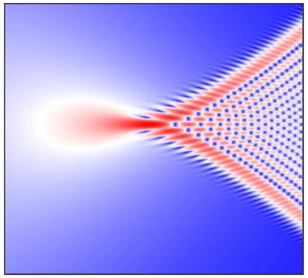
After passing through a slab of material of the thickness L, the radiation intensity of the high-energy beam is

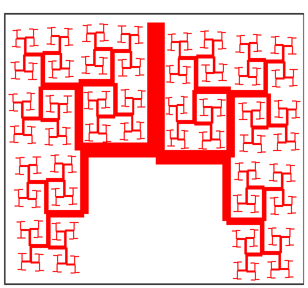
$$I = I_0 \exp\left[-\int_0^L \alpha(z)\mathrm{d}z\right] = \left(1 - \int_0^L \alpha(z)\mathrm{d}z\right) I_0.$$

Thus, a high-energy beam penetrates the human body easily, with the negative of the recorded image proportional to the cumulative mass density in the longitudinal direction.

X-rays are a very effective way of looking at fractured bones, examine organs. Surgeons often use them during therapeutic procedures, such as a coronary angioplasty to help guide equipment to the area being treated.

Figure 1. A negative of the recorded distribution of the X-ray intensity transmitted through a human hand. Adapted from www.ed.ac.uk/clinical-sciences





Reconstruction Problem

Mass density $\boldsymbol{M}(x, y)$ is discretized on the uniform grid of $N_x \cdot N_y$ points resolutions Δx and Δy, and represented by a matrix $\boldsymbol{M}_{mn} = \boldsymbol{M}\big((m - m_0)\Delta x, (n - n_0)\Delta y\big)$. The radiogram in the x direction is then

$$I_n = \sum_m \boldsymbol{M}_{m,n} \Delta x,$$

a system of N_y equations with $N_x \cdot N_y$ unknowns. This system is in general unsolvable!

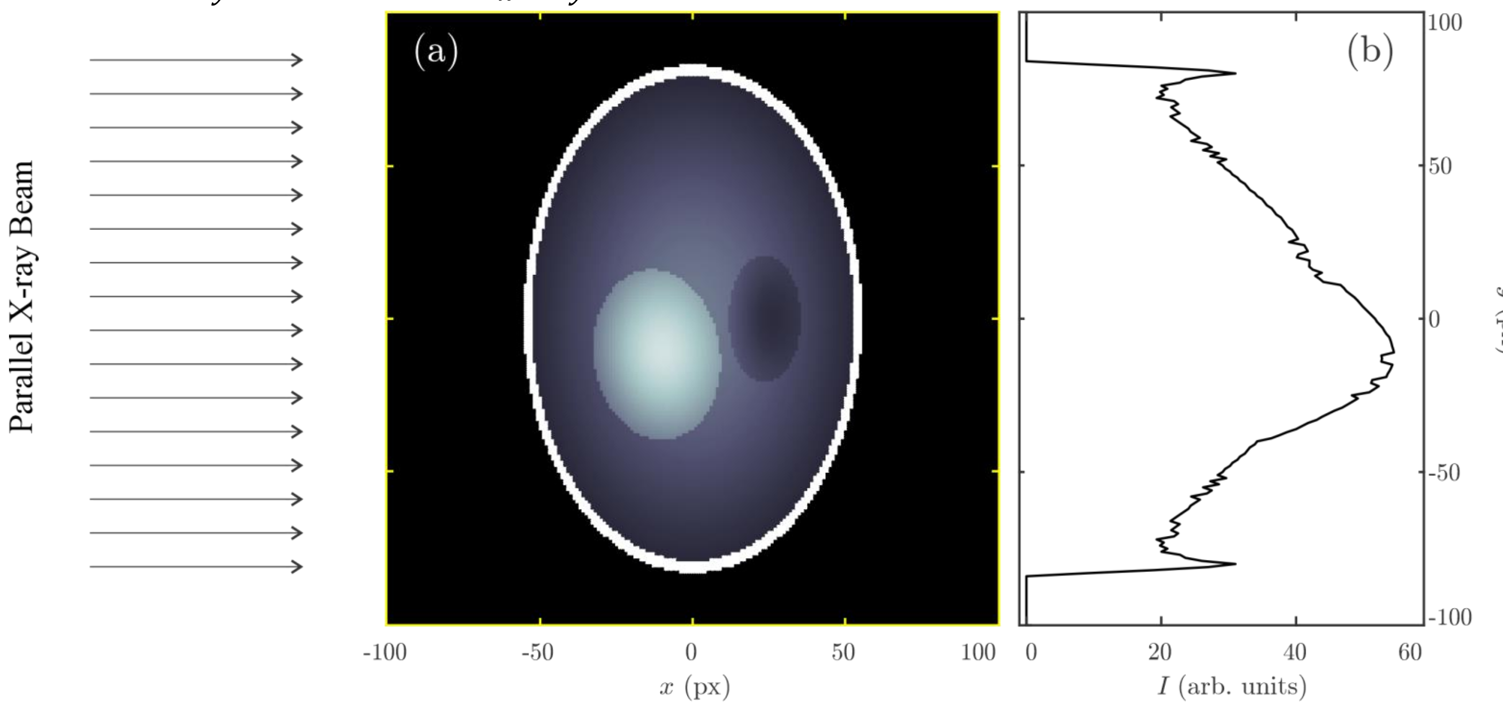
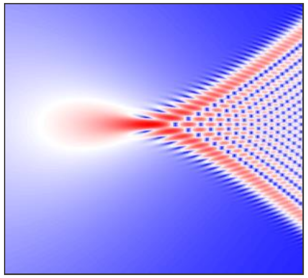


Figure 2. (a) Head phantom with two regions of different density. (b) A line integral through the matrix in x direction.



Radon Transform

For ideal reconstruction N_x additional linearly independent systems of N_y equations, are necessary. One possibility is to record a radiogram for N_x different orientations θ of the Phantom

$$\mathcal{R}\{M(x,y)\} = S(\eta,\theta) = \int M(\eta\cos\theta + \zeta\sin\theta\,, \eta\sin\theta - \zeta\cos\theta)\mathrm{d}\zeta\,,$$

But, how to know that the resulting system is solvable.

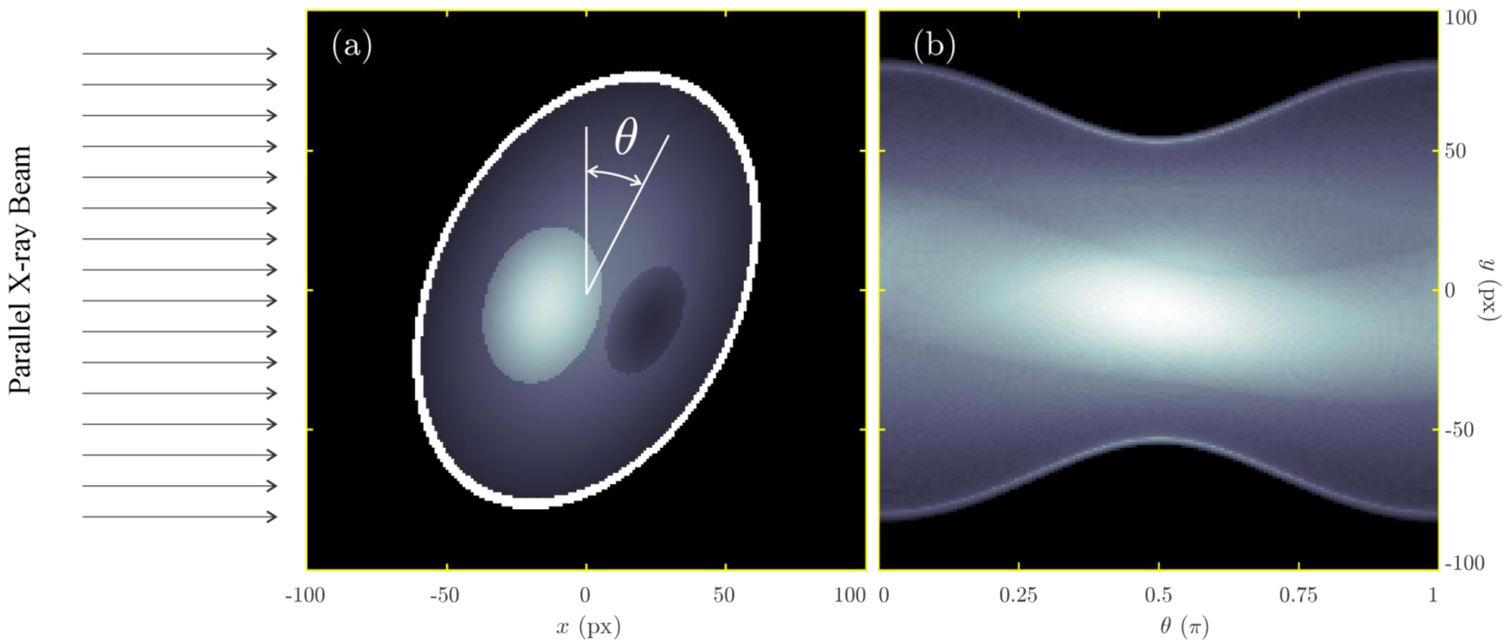
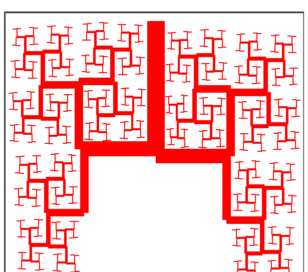
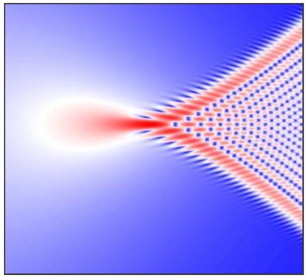
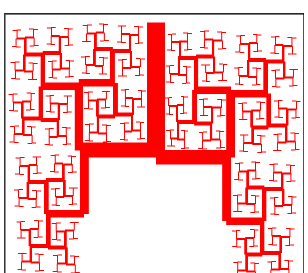


Figure 3. (a) Rotated head phantom from Fig. 2(a). (b) A collection of radiograms for different phantom orientations – *i.e.,* a collection of integral projections corresponding to different rotation angles.







Brute-Force Reconstruction

To show that the obtained system is consistent, let's introduce linear matrix index $J = (m-1)N_y + n$, and represent mass density matrix as large vector $\boldsymbol{M}_{nm} = M_J$. The rotation by an angle $\Delta\theta = \pi/N_x$ is then mapping $J \to K = (\bar{m}-1)N_y + \bar{n}$

$$\begin{bmatrix} \bar{n} \\ \bar{m} \end{bmatrix} = \begin{bmatrix} n_0 \\ m_0 \end{bmatrix} + \mathrm{round}\left\{ \begin{bmatrix} \cos\Delta\theta & \sin\Delta\theta \\ -\sin\Delta\theta & \cos\Delta\theta \end{bmatrix} \cdot \begin{bmatrix} n - n_0 \\ m - m_0 \end{bmatrix} \right\},$$

Which is in the characteristic subspaces represented by the block matrix $\boldsymbol{R}$

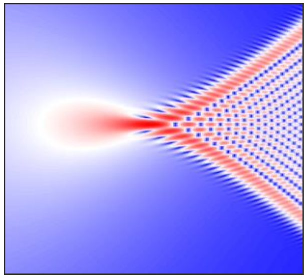
$$\boldsymbol{R}_{\Delta\theta} = \begin{bmatrix} \boldsymbol{I} & & \boldsymbol{O} & \\ \boldsymbol{O} & P_{\Delta\theta} & & \boldsymbol{O} \\ & \boldsymbol{O} & & P_{\mathrm{rand}} \end{bmatrix}_{N_xN_y \times N_xN_y}, \quad \begin{matrix} M_J \neq 0, & J \to J \\ M_J \neq 0, & \mathrm{J} \to \mathrm{K} \\ M_J = 0, & \end{matrix}$$

An additional system of equations is then

$$I_{\Delta\theta} = \boldsymbol{T} \cdot \boldsymbol{R}_{\Delta\theta} \cdot Q_{\Delta\theta} \cdot M = \Delta x \begin{bmatrix} 1, \dots, 1 & 0, \dots, 0 & \dots \\ 0, \dots, 0 & \ddots & 0, \dots, 0 \\ \vdots & 0, \dots, 0 & 1, \dots, 1 \end{bmatrix}_{N_y \times N_xN_y} \cdot \boldsymbol{R}_{\Delta\theta} \cdot T_{\Delta\theta} \cdot M.$$

The complete system of equations is

$$\begin{bmatrix} I_0 \\ \vdots \\ I_{(N_x-1)\Delta\theta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T} \cdot \boldsymbol{R}_0 \cdot Q_0 \\ \vdots \\ \boldsymbol{T} \cdot \boldsymbol{R}_{(N_x-1)\Delta\theta} \cdot Q_{(N_x-1)\Delta\theta} \end{bmatrix} \cdot M.$$



Projection-Slice Theorem

Fourier transform expresses arbitrary function f in the terms of trigonometric functions

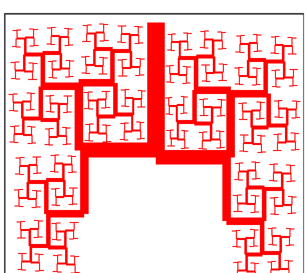
$$f(x,y)=\frac{1}{2\pi}\iint\tilde{f}(k_x,k_y)\exp\left[ik_xx+ik_yy\right]d^2\boldsymbol{k},$$

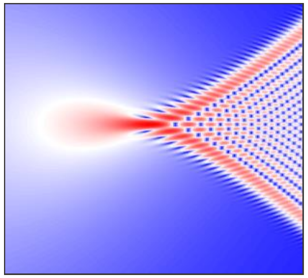
which allows the integral projection to be evaluated as

$$\begin{aligned}S(\eta,\theta)&=\int f(\eta\cos\theta+\zeta\sin\theta,\eta\sin\theta-\zeta\cos\theta)\mathrm{d}\zeta\\&=\iint\tilde{f}(k_x,k_y)e^{i(\cos\theta k_x+\sin\theta k_y)\eta}\delta(\sin\theta\,k_x-\cos\theta\,k_y)d^2\boldsymbol{k}\\&=\iint\tilde{f}(k_r\cos\phi,k_r\sin\phi)e^{i\cos(\theta-\phi)k_r\eta}\delta(\sin(\theta-\phi))dk_rd\phi\\&=\int\tilde{f}(k_r\cos\theta,k_r\sin\theta)e^{ik_r\eta}dk_r+\int\tilde{f}(-k_r\cos\theta,-k_r\sin\theta)e^{-ik_r\eta}dk_r,\end{aligned}$$

an inverse Fourier transform along the cross-section defined by the angle θ, or equivalently

$$\frac{1}{\sqrt{2\pi}}\int S(\eta,\theta)\exp\left[-ik_\eta\eta\right]\mathrm{d}\eta=\sqrt{2\pi}\tilde{f}\left(|k_\eta|\cos\theta,|k_\eta|\sin\theta\right).$$





Inverse Radon Transform

$$M(x,y)\xrightarrow{\mathcal{R}}S(\eta,\theta)\xrightarrow{\mathcal{F}}\tilde{S}\left(k_\eta,\theta\right)\rightarrow\widetilde{M}\left(k_x,k_y\right)\xrightarrow{\mathcal{F}^{-1}}M(x,y)$$

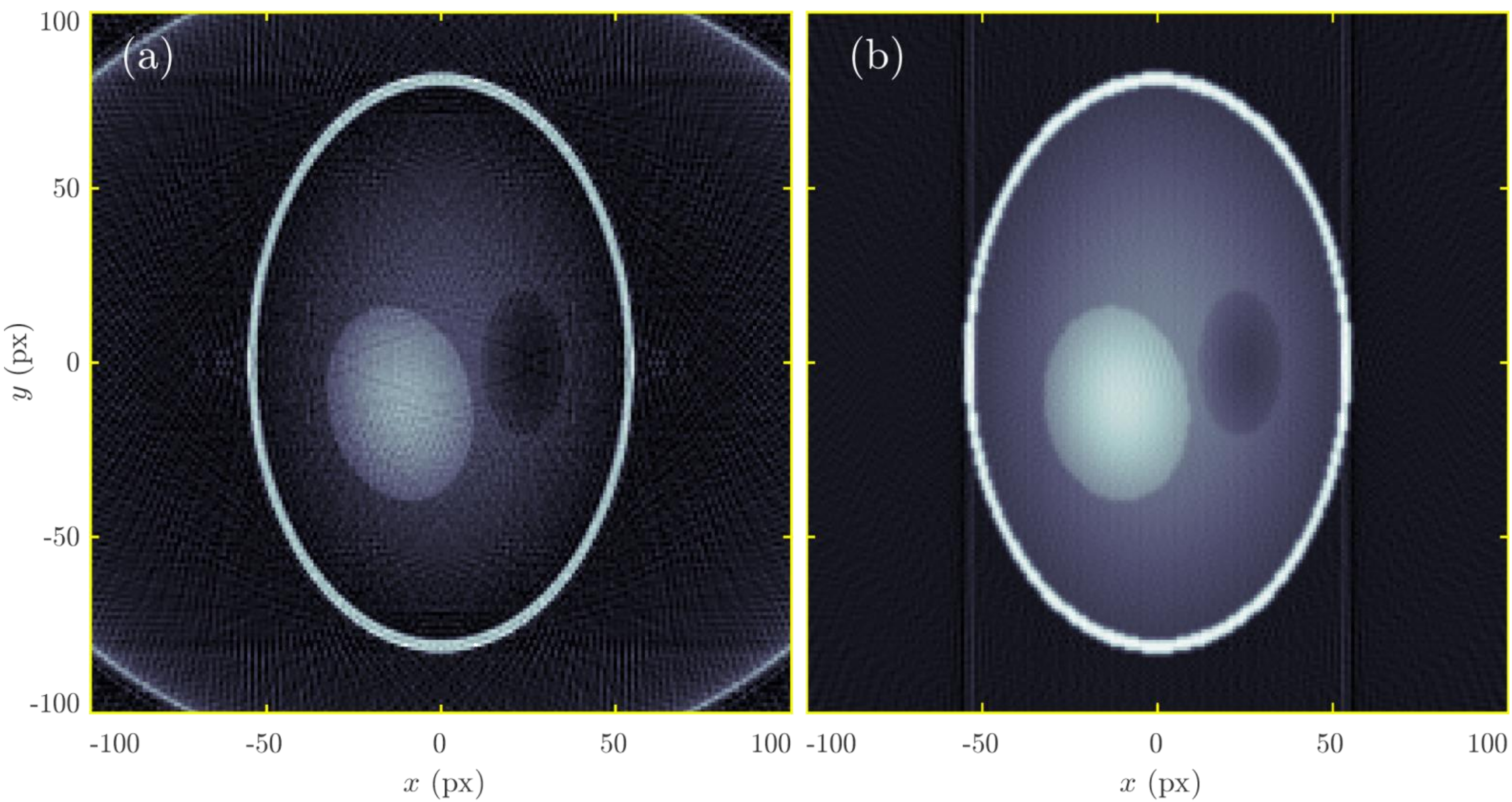
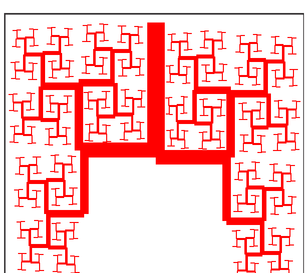
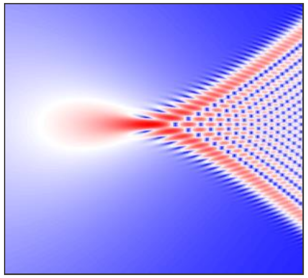


Figure 4. (a) Direct evaluation of the inverse Radon transform having computational complexity $O(N_\theta N_y^2)$. (b) Filtered Back-projection implementation of the inverse Radon transformation having computational complexity $O(N_\theta N_y^2)$



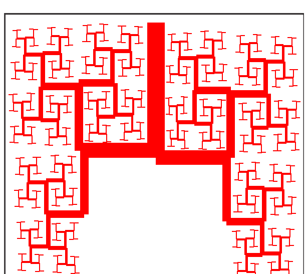


1. Introduction

2. Radon Transform

3. State Tomography

4. Conclusions



Evolution of Quantum State

In quantum mechanics, particles are initially represented by wave packets,

$$\psi_0(x, t=0) = \frac{1}{\sqrt[4]{2\pi\sigma_x^2}} \exp\left[-\frac{(x-b)^2}{4\sigma_x^2}\right],$$

whose subsequent evolution is governed by the Schrodinger equation

$$i\hbar\partial_t\psi(x,t) = \left[-\frac{\hbar^2}{2m_p}\partial_x^2 + V(x)\right]\psi(x,t).$$

However, only $|\psi(x,t)|^2$ is observable, and represents the probability density to find a particle at position x at time t. The corresponding state in momentum representation

$$\varphi(p_x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x,t) \exp\left[-\frac{ip_x x}{\hbar}\right] \mathrm{d}x,$$

Determines analogue probability density $|\varphi(p_x,t)|^2$ in the momentum space.

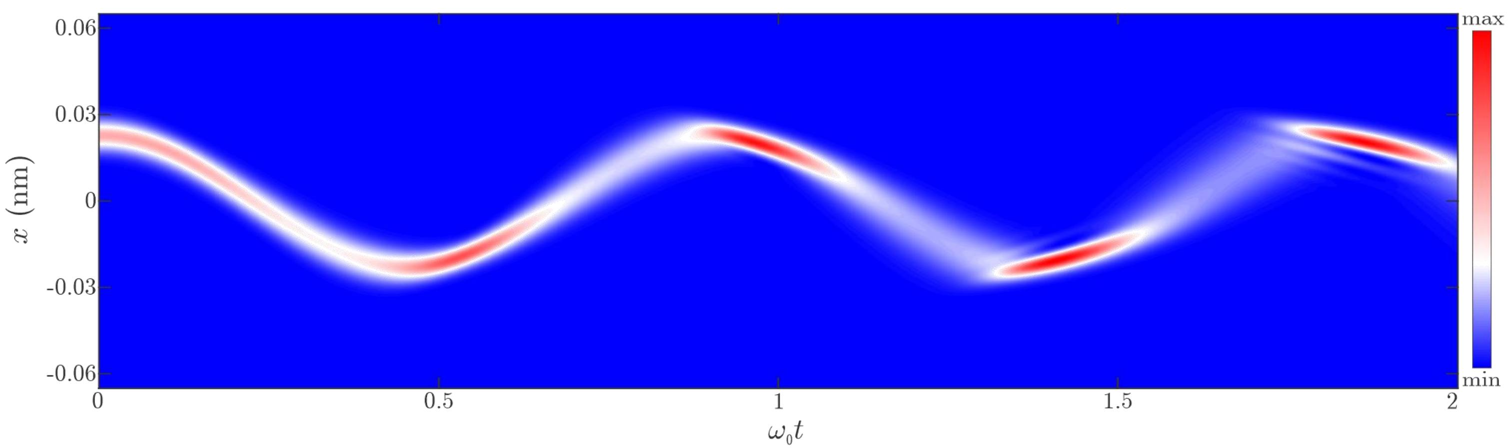
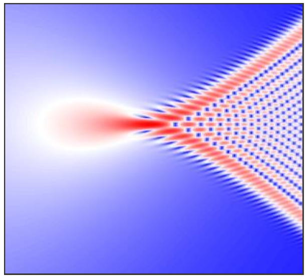


Figure 5. Evolution of $|\psi(x,t)|$^2 inside potential $V(x) \approx \frac{1}{2}m_p\omega_0 x^2 + \frac{\lambda}{4}x^4$ for $0 \leq \omega_0 t \leq 2$



Wigner Function

Wigner: Is there a function $W(x,p_x)$ with the following properties

$$\int \psi^* x \psi \mathrm{d}x = \iint x W(x,p_x) \mathrm{d}x \mathrm{d}p_x \,,$$

$$\int \psi^* \frac{\hbar}{i} \partial_x \psi \mathrm{d}x = \iint p_x W(x,p_x) \mathrm{d}x \mathrm{d}p_x \,.$$

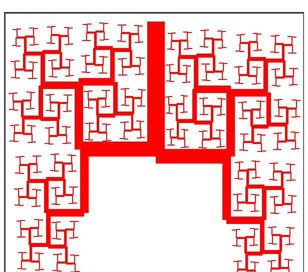
Szilard: The solution of the system above is Wigner's transform of the wave function :

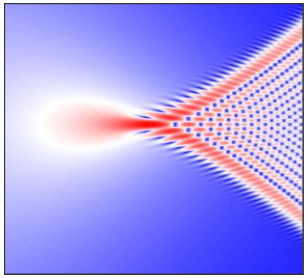
$$W(x,p_x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi^* \left(x - \frac{\xi}{2}\right) \psi \left(x + \frac{\xi}{2}\right) \exp[-i p_x \xi/\hbar] \mathrm{d}\tau;$$

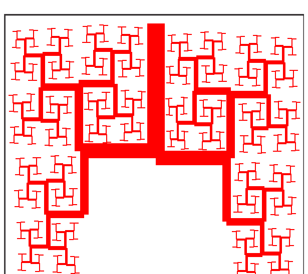
The most important properties of the Wigner function are:

$$W(x,p_x) = W^*(x,p_x), \qquad -\frac{1}{\pi\hbar} \leq W \leq \frac{1}{\pi\hbar},$$

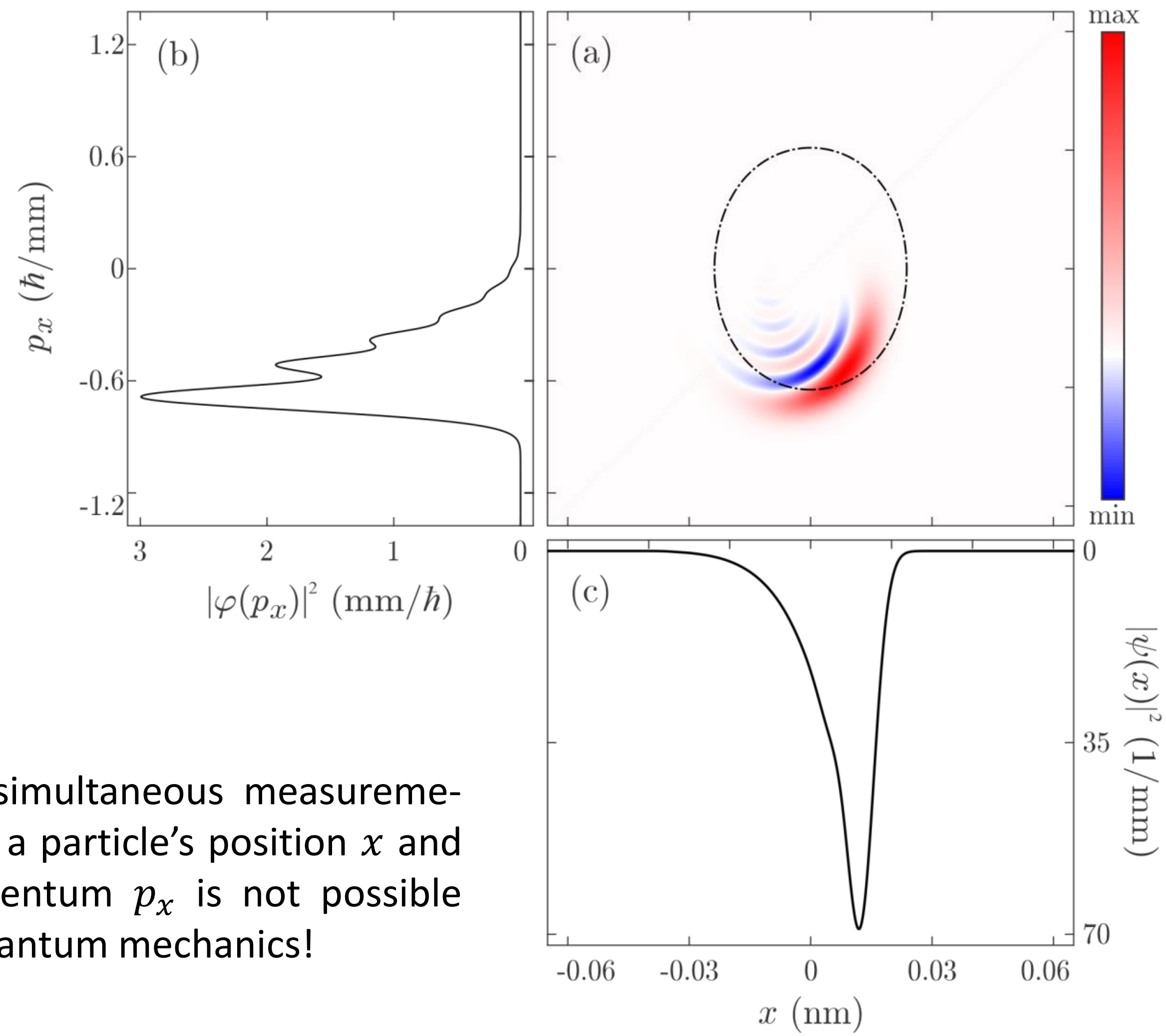
$$\int W(x,p_x) \mathrm{d}p_x = |\psi(x)|^2 \,, \qquad \int W(x,p_x) \mathrm{d}x = |\varphi(p_x)|^2 \,.$$





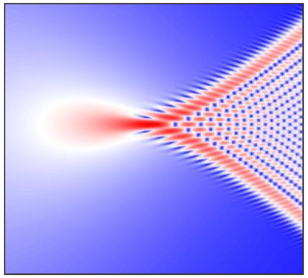


Wigner Function



The simultaneous measurement of a particle's position x and momentum p_x is not possible in quantum mechanics!

Figure 6. (a) Wigner function of the state from Fig. 5 for $\omega_0 t$=2. The dot-dash line represents corresponding classical trajectory. (b, c) Projections of the Wigner function in x and p_x direction.



Quantum Tomography

The trick is not to measure distributions of variables x, and p_x, but of

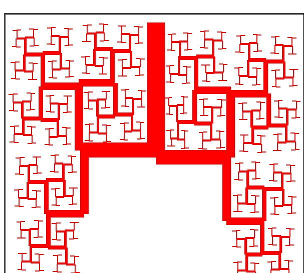
$$x_\theta = x\cos\theta + p_x \sin\theta\,, \qquad p_\theta = -x\sin\theta + p_x\cos\theta.$$

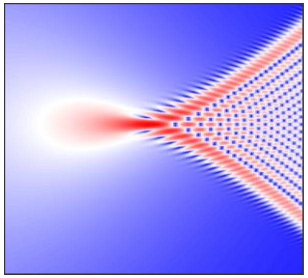
The corresponding probability distributions are

$$|\psi_\theta(x_\theta)|^2 = \int W(x_\theta\cos\theta - p_\theta\sin\theta, x_\theta\sin\theta + p_\theta\cos\theta)\,\mathrm{d}p_\theta,$$

$$|\varphi_\theta(p_\theta)|^2 = \int W(x_\theta\cos\theta - p_\theta\sin\theta, x_\theta\sin\theta + p_\theta\cos\theta)\,\mathrm{d}x_\theta,$$

Thus, if many measurements of $|\psi_\theta|^2$ or $|\varphi_\theta|^2$ are available, corresponding to different values of the angle θ, Wigner function W is the inverse Radon transformation of distribution $|\psi_\theta|^2$ or $|\varphi_\theta|^2$.





Photon Tomography

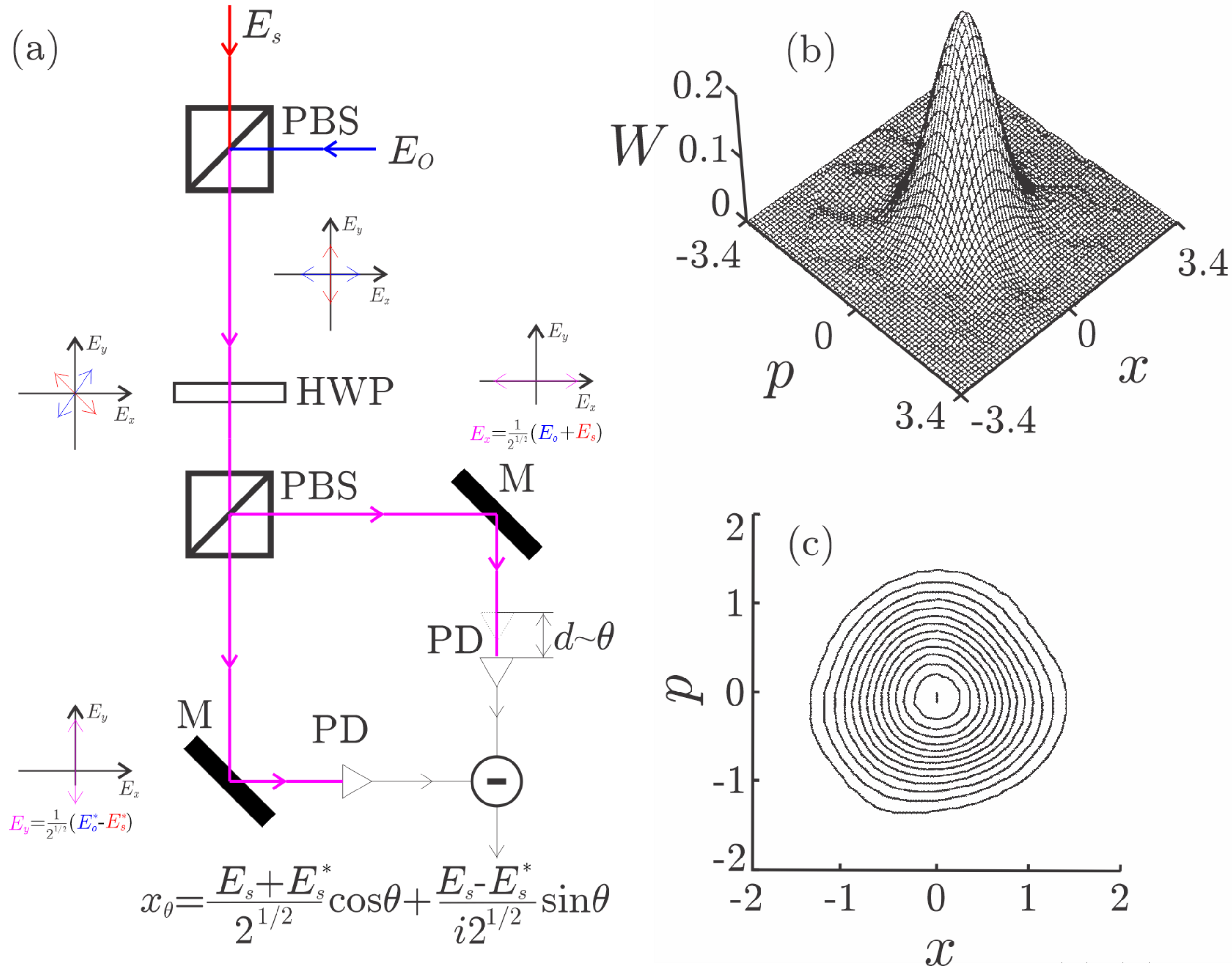
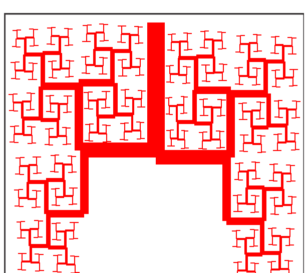
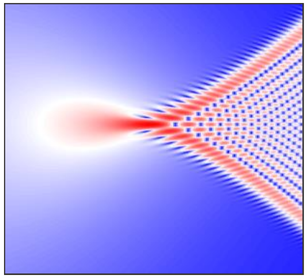
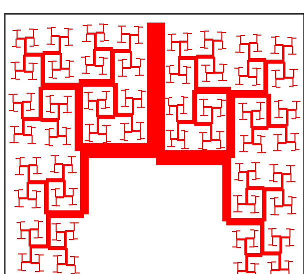


Figure 7. (a) Apparatus for balanced homodyne detection. (b) Surface plot of the reconstructed photon's Wigner Function (c) constant-height contours of the surface from Fig. (b). Adapted from D. T. Smithey, M. Beck, and M. G. Raymer, **70**, 9 Phys. Rev. Lett. (1993)







Particle Beam Tomography

In potential $V(x) = \frac{1}{2} m_p \omega_0^2 x^2$, position and momentum operators evolve according to the law

$$\hat{x}(t) = \hat{x}\cos\omega_0 t + \frac{\hat{p}_x}{m_p\omega_0}\sin\omega_0 t\,,\;\; \hat{p}_x(t) = -\hat{x}\sin\omega_0 t + \frac{\hat{p}_x}{m_p\omega_0}\cos\omega_0 t,$$

thus, interaction τ sets the phase angle $\theta = \omega_0\tau$ of measured the Fairfield distributions $p_\theta = p_x(\theta/\omega_0)$.

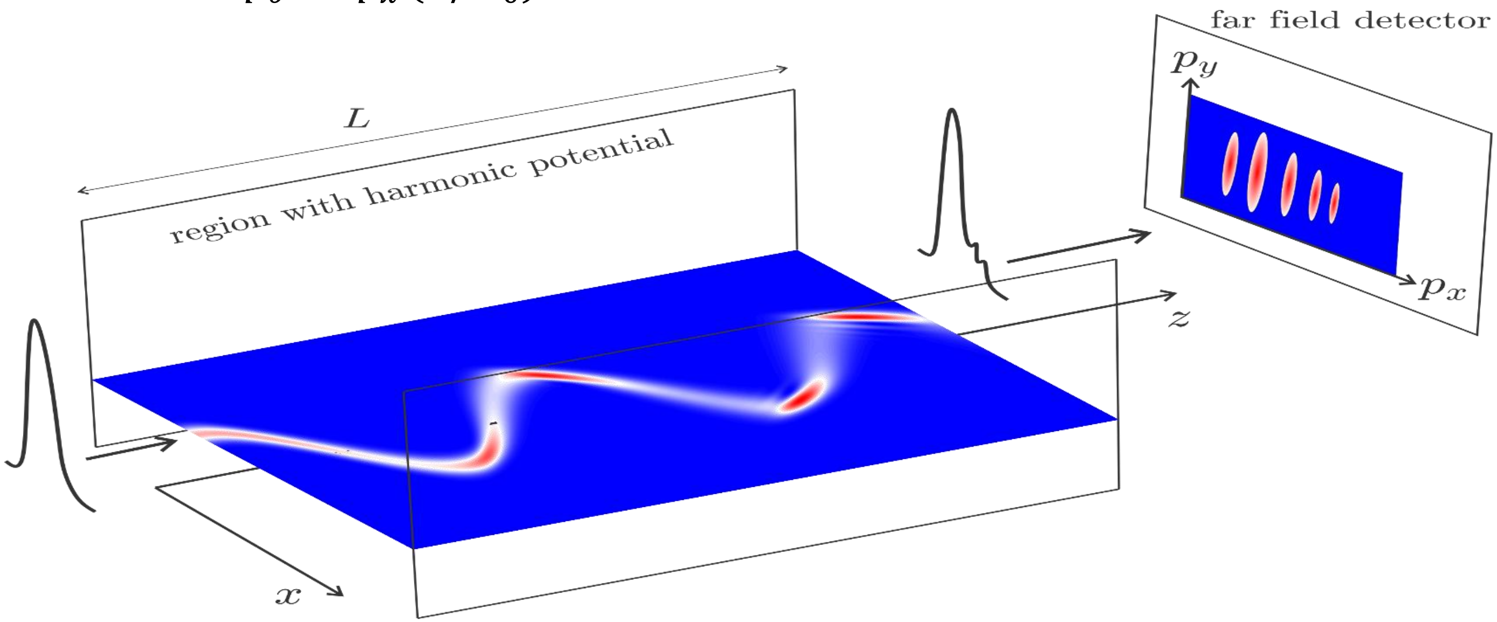
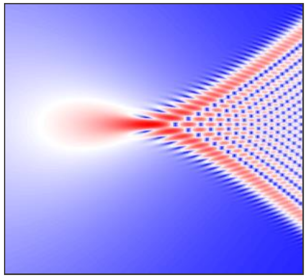


Figure 8. Quantum beam tomography setup. Adapted from S. H. Kienle, M. Freyberger, W .P. Schleich and M. G. Raymer, Quantum Beam Tomography, (Kluwer Academic Publishers, Dordrecht, 1997)



Conclusions

- A set of integral projections for different orientations of the specimen – *i.e.*, the Radon transform - is easy to record experientially or calculate theoretically. The inverse problem, although well posed, is much more difficult.

- Fortunately, the inverse Radon transform can be accomplished by two Fourier transforms, which can be efficiently implemented using a fast discrete Fourier transform algorithm.

- This approach actually reconstructs unknown mass density by using interpolation of the spectrum on the radial grid and provides the solution to the inverse problem with the minimal number of input radiograms.

- Similar problems are encountered in quantum mechanics where Heisenberg's uncertainty relations forbid simultaneous measurement of the amplitude of the wave function and its phase.

- If a high-quality particle source is available, then the interaction with homodyne detection apparatus or interaction with harmonic potential implements quantum measurement of linear combinations of the position and momentum operator. The quantum state is reconstructed then using the inverse Radon transformation.