Algebraic solutions of linear differential equations: an arithmetic approach

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Abstract

Given a linear differential equation with coefficients in $\mathbb{Q}(x)$, an important question is to know whether its full space of solutions consists of algebraic functions, or at least if one of its specific solutions is algebraic. After presenting motivating examples coming from various branches of mathematics, we advertise in an elementary way a beautiful local-global arithmetic approach to these questions, initiated by Grothendieck in the late sixties. This approach has deep ramifications and leads to the still unsolved Grothendieck-Katz p-curvature conjecture.

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We consider in this text linear differential equations of order r

$$a_r(x)y^{(r)}(x) + a_{r-1}(x)y^{(r-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0,$$

where the a_i's are known rational functions in $\mathbb{Q}(x)$ and $y(x)$ is an unknown "function".

function $y \in \mathbb{Q}[[x]]$ is differentially finite (in short, D-finite) if it satisfies

a linear differential equation like (1).

A function $y \in \mathbb{Q}[[x]]$ is called *algebraic* if it is algebraic over $\mathbb{Q}(x)$, that is, if $y(x)$ satisfies a polynomial equation of the form $P(x, y(x)) = 0$, for some $P \in \mathbb{Q}[x, y] \setminus \{0\}$.

$$P_x(x, y(x)) + y'(x)P_y(x, y(x)) = 0.$$

any algebraic function is D-finite

Proposition 1.4 ("Eisenstein's criterion" (1852)). *If the function* $y(x) = \sum_{k\geq 0} a_k x^k \in \mathbb{Q}[[x]]$ *is algebraic, then there exists* $N \in \mathbb{N} \setminus \{0\}$ *such that* $y(Nx) - y(0) \in \mathbb{Z}[[x]]$. *In particular, only a finite number of prime numbers can divide the denominators of the coefficients* a_k.

$$\mathscr{L} = a_r(x)\cdot\partial_x^r + a_{r-1}(x)\cdot\partial_x^{r-1} + \cdots + a_1(x)\cdot\partial_x + a_0(x) \;\in \mathbb{Q}(x)\left\langle\partial_x\right\rangle$$

Proposition 1.5. *The ring* $\mathbb{Q}(x)\langle\partial_x\rangle$ *is left Euclidean,* i.e., *for all* $A, B \in \mathbb{Q}(x)\langle\partial_x\rangle$ *with* $B \neq 0$, *there exist* Q *and* R *in* $\mathbb{Q}(x)\langle\partial_x\rangle$ *such that* $A = BQ + R$ *and* $\deg R < \deg B$. *Moreover, the pair* (Q, R) *is unique with these properties.*

$$a_r(x)y^{(r)}(x) + a_{r-1}(x)y^{(r-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0, \tag{1}$$

Proposition 1.6 ("Cartier's lemma"). *If all solutions of* (1) *are algebraic functions, then for all but a finite number of prime numbers* p, *the remainder of the left Euclidean division of* ∂_x^p *by* $\mathscr{L}$ *has all its coefficients divisible by* p.

Example 1.7. The generating function of the Catalan numbers, $y(x) = \sum_{k\geq 0} C_k x^k$, satisfies the differential equation $(4x^2 - x)y''(x) + (10x - 2)y'(x) + 2y(x) = 0$, which is easily deduced, either from the inhomogeneous differential equation of order 1 in Example 1.1, or directly from the recurrence $(k+2)C_{k+1} - (4k+2)C_k = 0$. The associated differential operator is $\mathscr{L} = \left(4x^2 - x\right)\partial_x^2 + (10x-2)\,\partial_x + 2$, and the remainders of the left Euclidean divisions of ∂_x^p by $\mathscr{L}$ for $p \in \{2, 3, 5\}$ are

$$\partial_x^2 \bmod \mathscr{L} = -\frac{2\,(5x-1)}{x\,(4x-1)}\partial_x - \frac{2}{x\,(4x-1)},$$

$$\partial_x^3 \bmod \mathscr{L} = \frac{6\left(22x^2 - 9x + 1\right)}{x^2\,(4x-1)^2}\partial_x + \frac{6(6x-1)}{x^2\,(4x-1)^2},$$

$$\partial_x^5 \bmod \mathscr{L} = \frac{120\left(386x^4 - 325x^3 + 110x^2 - 17x + 1\right)}{x^4\,(4x-1)^4}\partial_x + \frac{120(130x^3 - 69x^2 + 14x - 1)}{x^4\,(4x-1)^4}.$$

Note that indeed, we have $\partial_x^p \bmod \mathscr{L} = 0$ modulo p, in the three cases.

Conjecture 1.10 (Christol-André conjecture). *Assume that* $y(x) = \sum_{k\geq 0} a_k x^k \in \mathbb{Q}[[x]]$ *is D-finite, such that:*

(1) *the sequence* $(a_k)_{k\geq 0}$ *has at most geometric growth;*

(2) *there exists* $N \in \mathbb{N}$ *such that* $y(Nx) - y(0) \in \mathbb{Z}[[x]]$*;*

(3) *in the minimal-order monic linear differential equation satisfied by* $y(x)$*, the point* $x = 0$ *is not a pole of any of the coefficients* $a_i(x)$*.*

Then, $y(x)$ *is algebraic.*

Conjecture 1.11 (Grothendieck's conjecture, version 1). *Let* $\mathscr{L} \in \mathbb{Q}(x)\langle\partial_x\rangle$ *be the differential operator attached to* (1)*. If for all but a finite number of prime numbers* p*, the remainder of the left Euclidean division of* ∂_x^p *by* $\mathscr{L}$ *has all its coefficients divisible by* p*, then all solutions of* (1) *are algebraic functions.*

Gauss hypergeometric function with parameters $a, b, c \in \mathbb{Q}$, $c \notin \mathbb{Z}_{\leq 0}$, defined by

$$ {}_2F_1([a,b],[c];x) = \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} x^k, $$

In general, $y(x) = {}_2F_1([a,b],[c];x)$ satisfies the second-order differential equation

$$ x(x-1)y''(x) + ((a+b+1)x - c)y'(x) + aby(x) = 0 $$

More precisely, let us assume that none of a, b, $c-a$ and $c-b$ is an integer (equivalently, the operator $H(a,b;c) := x(1-x)\partial_x^2 + (c-(a+b+1)x)\partial_x - ab$ is irreducible) and let D be the common denominator of a, b and c. Then, the Landau-Errera criterion says that the following assertions are equivalent:

1. the hypergeometric function ${}_2F_1([a,b],[c];x)$ is algebraic;
2. the operator $H(a,b;c)$ admits only algebraic solutions;
3. for every $\ell < D$ coprime with D, either $\{\ell a\} < \{\ell c\} < \{\ell b\}$ or $\{\ell b\} < \{\ell c\} < \{\ell a\}$. (Here $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$ of x.)

The last condition is equivalent to the fact that, for every $\ell < D$ coprime with D, the two sets $\{e^{2\pi i \ell a}, e^{2\pi i \ell b}\}$ and $\{e^{2\pi i \ell c}, 1\}$ are interlaced on the unit circle. '

$$\mathscr{L} = \partial_x + a(x) \quad (13) \qquad\qquad \mathscr{L}_p = \partial_x + a(x) \bmod p \quad (14)$$

Proposition 3.1. *The monic first order differential operator* (13) *has a nonzero rational (resp. algebraic) solution if and only if its constant coefficient* $a(x)$ *has at most a simple pole with integral (resp. rational) residue at each point of* $\overline{\mathbb{Q}}$ *and vanishes at* ∞.

Proposition 3.3. *Consider* $b(x) \in \mathbb{F}_p(x)$. *The differential equation*

$$y' + b(x)y = 0$$

has a nonzero rational solution if and only if $b(x)$ *has at most a simple pole with residue in* $\mathbb{F}_p$ *at each point of* $\overline{\mathbb{F}_p}$ *and vanishes at* ∞.

Theorem 3.5. *If* (13) *has a nonzero algebraic solution, then, for almost all primes* p, (14) *has a nonzero rational solution.*

Theorem 3.6 (Honda [43]). *The converse of Theorem 3.5 holds true, i.e., if, for almost all primes* p, (14) *has a nonzero rational solution, then* (13) *has a nonzero algebraic solution.*

$$y' + b(x)y = 0 \qquad \text{with } b(x) \in \mathbb{F}_p(x). \tag{18}$$

Consider the $\mathbb{F}_p(x^p)$-linear map

$$\begin{aligned} \Delta : \mathbb{F}_p(x) &\to \mathbb{F}_p(x) \\ f &\mapsto f' + b(x)f. \end{aligned}$$

Definition 3.8. The map

$$\Delta^p : \mathbb{F}_p(x) \to \mathbb{F}_p(x)$$

is called the p-curvature of (18).

Proposition 3.9. *The differential equation (18) has a nonzero rational solution if and only if* $\Delta^p = 0$.

$$\mathscr{L} = \partial_x^n + b_{n-1}(x)\cdot\partial_x^{n-1} + \cdots + b_1(x)\cdot\partial_x + b_0(x) \tag{22}$$

Let $Y' + B(x)Y = 0$ be the differential system associated to (22), where

$$B = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \end{pmatrix} \in M_n(\mathbb{F}_p(x)).$$

$$\begin{aligned} \Delta : \mathbb{F}_p(x)^n &\rightarrow \mathbb{F}_p(x)^n \\ F &\mapsto F' + B(x)F. \end{aligned}$$

Definition 3.17. The map

$$\Delta^p : \mathbb{F}_p(x)^n \rightarrow \mathbb{F}_p(x)^n$$

is called the p-curvature of (22).

Conjecture 3.16 (Grothendieck's conjecture)**.** *For a differential operator* $\mathscr{L} \in \mathbb{Q}(x)\langle\partial_x\rangle$ *as in Eq.* (21)*, the following properties are equivalent:*

(1) $\mathscr{L}$ *has a full basis of algebraic solutions;*

(2) for almost all primes p*,* $\mathscr{L}_p$ *has a full basis of rational solutions.*

Conjecture 3.21 (Grothendieck's conjecture in terms of p-curvature)**.** *For a differential operator* $\mathscr{L}$ *as in Eq.* (21)*, the following properties are equivalent:*

(1) $\mathscr{L}$ *has a full basis of algebraic solutions;*

(2) for almost all primes p*, the* p*-curvature of* $\mathscr{L}_p$ *vanishes;*

(3) for almost all primes p*,* $\mathscr{L}_p$ *divides* ∂_x^p *in the ring of differential operators* $\mathbb{F}_p(x)\langle\partial_x\rangle$*.*

$$ {}_pF_q\left(\begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \middle| z\right) = \sum_{n=0}^{\infty} c_n z^n, \qquad \frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\dots(n+a_p)}{(n+b_1)\dots(n+b_1)(n+1)}. $$

$$ \mathscr{H}(\mathbf{a}, \mathbf{b}) \coloneqq (x\partial_x + b_1 - 1) \cdots (x\partial_x + b_s - 1)x\partial_x - x(x\partial_x + a_1) \cdots (x\partial_x + a_{s+1}). $$

Theorem 3.22 ("interlacing criterion", Beukers-Heckman, [8]). *Given two sets of rational numbers $\mathbf{a} = \{a_1, \dots, a_{s+1}\}$ and $\mathbf{b} = \{b_1, \dots, b_s, b_{s+1} = 1\}$, assumed to be disjoint modulo $\mathbb{Z}$, let D be the common denominator of their elements. Then, the following assertions are equivalent:*

1. *the hypergeometric function ${}_{s+1}F_s([a_1, \dots, a_{s+1}], [b_1, \dots, b_s]; x)$ is algebraic;*
2. *the operator $\mathscr{H}(\mathbf{a}, \mathbf{b})$ admits a full basis of algebraic solutions;*
3. *for all $1 \le \ell < D$ with $\gcd(\ell, D) = 1$ the sets $\{e^{2\pi i \ell a_j}, j \le s+1\}$ and $\{e^{2\pi i \ell b_j}, j \le s+1\}$ interlace on the unit circle.*

Thank you for your attention!