Interacting spinor and electromagnetic fields in cosmology

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LIT, 21 November 2023

Plan

- Introductory remarks
- Basic Equations and their solutions
- Concluding remarks

Introduction

Maxwell-Dirac system is well studied in mathematical physics. The corresponding Maxwell-Dirac system is given by [Das1989]

$$4\pi\partial_\mu\partial^\mu\mathcal{A}^\nu = j^\nu, \tag{1a}$$

$$\left(\imath\bar{\gamma}^\mu\partial_\mu - \bar{\gamma}^\mu\mathcal{A}_\mu\right)\psi - m\psi = 0, \tag{1b}$$

with the Lorentz condition

$$\partial_\mu\mathcal{A}^\mu = 0. \tag{2}$$

In (1b) the electric current $j^\nu = \bar{\psi}\gamma^\nu\psi$ is constructed from Dirac spinors. Such system can be obtained from the Lagrangian with the interacting term

$$L_{int} = \mathcal{A}_\mu j^\mu. \tag{3}$$

Introduction

Though the present day universe is surprisingly homogeneous and isotropic, there are both theoretical arguments and observational data that support an anisotropic phase in the remote past. So in order to study the role of anisotropy in the evolution of the universe many authors consider Bianchi type-I (BI) cosmological model which is the straightforward generalization of FRW model.

Spinor is being widely used by many authors both in cosmology and astrophysics [Bronnikov-2020] . It was shown that the nonlinear spinor field can (i) accelerate the isotropization process of the initially anisotropic spacetime [BS-2001]; (ii) give rise to a singularity-free solution [BS-2004]; (iii) explain the late time accelerated mode of expansion of the Universe [BS-2006; Fabri-2012, Greene-2003, Popławski-2012,]

Introduction

The main advantages that spinor field gives are its sensitiveness to gravitational field and its ability to models different types of source field from perfect fluid to dark energy and dark matter, hence can describe the evolution of the universe at different stages.

But the presence of non-diagonal components of the energy-momentum tensor (EMT) of the spinor field imposes different types of restrictions both on space-time geometry and spinor field itself.

On the other hand electromagnetic field with induced nonlinearity does the same [Rybakov 2011].

In this repost we consider an interacting system of spinor and electromagnetic fields to soften those restrictions.

Basic Equations

The Einstein-Dirac-Maxwell system is given by the action

$$\mathcal{S}(g;\psi,\bar{\psi};\mathcal{A}_\mu) = \int \left(L_{\rm g} + L_{\rm sp} + L_{\rm em} + L_{\rm int}\right)\sqrt{-g}d^4x. \qquad (4)$$

Here $L_{\rm g}$ - gravitational field. The spinor field Lagrangian

$$L_{\rm sp} = \frac{\imath}{2}\left[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi\right] - m\bar{\psi}\psi - \lambda_1 Y(K) \qquad (5)$$

Here m - spinor field mass, λ_1 - self-coupling constant, and Y is some arbitrary functions of invariants generated from the real bilinear forms of a spinor field K:

$$K = \{I,\, J,\, I+J,\, I-J\}, \quad I = S^2 = (\bar{\psi}\psi)^2, \quad J = P^2 = (\imath\bar{\psi}\bar{\gamma}^5\psi)^2. \qquad (6)$$

Basic Equations

The electromagnetic field Lagrangian is taken in the conventional form

$$L_{\rm em} = -\frac{1}{16\pi} F_{\tau\eta} F^{\tau\eta}, \tag{7}$$

while the interaction Lagrangian is chosen as

$$L_{\rm int} = -\frac{\lambda_2}{16\pi} F_{\tau\eta} F^{\tau\eta} Z(K), \tag{8}$$

In (5) and (8) Y and Z are some functions of K. For convenience we further combine (9) and (8) to write

$$L_{\rm emint} = -\frac{1}{16\pi} F_{\tau\eta} F^{\tau\eta} X(K), \quad X(K) \equiv 1 + \lambda_2 Z(K). \tag{9}$$

The spinor field equations corresponding to the Lagrangian (5) and (8) are

$$\imath\gamma^\mu\nabla_\mu\psi - m\psi - \mathcal{D}\psi - \imath\mathcal{G}\gamma^5\psi = 0, \tag{10a}$$

$$\imath\nabla_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} + \mathcal{D}\bar{\psi} + \imath\mathcal{G}\bar{\psi}\gamma^5 = 0, \tag{10b}$$

$$\mathcal{D} = 2S\left[\lambda_1 Y_K + F_{\tau\eta}F^{\tau\eta}\lambda_2 Z_K/(16\pi)\right],$$

$$\mathcal{G} = 2P\left[\lambda_1 Y_K + F_{\tau\eta}F^{\tau\eta}\lambda_2 Z_K/(16\pi)\right],$$

with $F_K = dF/dK$ and $Z_K = dZ/dK$. In view of (10) it can be shown that

$$L_{\rm sp} = \mathcal{D}S + \mathcal{G}P - \lambda_1 Y = \lambda_1(2KY_K - Y) + F_{\tau\eta}F^{\tau\eta}\lambda_2 KZ_K/(8\pi). \tag{11}$$

In the above expressions, $\nabla_\mu \psi = \partial_\mu \psi - \Omega_\mu \psi$ and $\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi}\Omega_\mu$, where Ω_μ is the spinor affine connection, defined by

$$\Omega_\mu = \frac{1}{4}\left(\bar{\gamma}_a \gamma^\beta \partial_\mu e^{(a)}_\beta - \gamma_\rho \gamma^\beta \Gamma^\rho_{\mu\beta}\right). \tag{12}$$

As one sees, the spinor affine connection is completely defined by the metric, hence spinor field becomes very sensitive to the gravitational one.

The electromagnetic field equations take the form

$$\partial_\eta \left(\sqrt{-g} F^{\tau\eta} X(K)\right) = 0. \tag{13}$$

The total energy-momentum tensor (EMT) of the interacting spinor and electromagnetic field has the form

$$
\begin{aligned}
T_{\mu}^{\rho} = {} & \frac{\imath}{4} g^{\rho\nu}\Big(\bar\psi\gamma_\mu\partial_\nu\psi + \bar\psi\gamma_\nu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\nu\psi - \partial_\nu\bar\psi\gamma_\mu\psi\Big) \\
& - \frac{\imath}{4} g^{\rho\nu}\bar\psi\Big(\gamma_\mu\Omega_\nu + \Omega_\nu\gamma_\mu + \gamma_\nu\Omega_\mu + \Omega_\mu\gamma_\nu\Big)\psi \\
& - \delta_\mu^\rho\lambda_1(2KY_K - Y(K)) \\
& - \frac{X(K)}{4\pi}\Big(F_{\mu\eta}F^{\rho\eta} - \frac{1}{4}\delta_\mu^\rho F_{\tau\eta}F^{\tau\eta}\Big) - \frac{\lambda_2 K Z_K}{4\pi}\delta_\mu^\rho F_{\tau\eta}F^{\tau\eta}.
\end{aligned}
\tag{14}
$$

The second term marked in red plays crucial role in generating nontrivial non-diagonal components in the EMT.

Basic Equations

The gravitational field we choose in the form

$$ds^2 = dt^2 - a_1^2 dx_1{}^2 - a_2^2 dx_2{}^2 - a_3^2 dx_3{}^2, \tag{15}$$

where the metric functions are the function of t only, i.e., $a_i = a_i(t)$. We will consider the case when electromagnetic 4-potential as $\mathcal{A}_\mu = (0,\, \mathcal{A}_1,\, \mathcal{A}_2,\, \mathcal{A}_3)$, and the spinor and the electromagnetic fields depend on t only, i.e., $\psi = \psi(t),\ \bar\psi = \bar\psi(t)$ and $\mathcal{A}_i = \mathcal{A}_i(t), \quad i = 1,\, 2,\, 3$. In this case $F^{\mu\nu}$ has only the following non-zero components : F^{01}, F^{02}, F^{03}. In this case for the electromagnetic field we have

$$F^{0i} = \frac{q_i}{VX(K)}, \quad F_{0i} = \dot{\mathcal{A}}_i = -\frac{a_i^2 q_i}{VX(K)}, \quad V = a_1 a_2 a_3$$
$$q_i = \text{const.}, \quad i = 1,\, 2,\, 3. \tag{16}$$

Basic Equations

Diagonal components of EMT of the system:

$$T_0^0 = P_2 + \frac{1 + \lambda_2(Z + 4KZ_K)}{X} Q\bar{Q}, \tag{17a}$$

$$T_1^1 = P_1 + Q\left[q_1^2 a_1^2 - q_2^2 a_2^2 - q_3^2 a_3^2 + Q_2\right], \tag{17b}$$

$$T_2^2 = P_1 + Q\left[-q_1^2 a_1^2 + q_2^2 a_2^2 - q_3^2 a_3^2 + Q_2\right], \tag{17c}$$

$$T_3^3 = P_1 + Q\left[-q_1^2 a_1^2 - q_2^2 a_2^2 + q_3^2 a_3^2 + Q_2\right], \tag{17d}$$

where, where we denote, $P_1 = \lambda_1 \left(Y - 2KY_K\right)$, $P_2 = m_{\rm sp} S + \lambda_1 Y$, $Q = 1/(8\pi V^2 X)$, $\bar{Q} = q_1^2 a_1^2 + q_2^2 a_2^2 + q_3^2 a_3^2$, $Q_2 = 4\lambda_2 KZ_K \bar{Q}/X$

whereas the non-diagonal components are

$$T_2^1 = \frac{a_2 a_3}{4a_1}\left(\frac{\dot a_1}{a_1} - \frac{\dot a_2}{a_2}\right) A^3 - 2 q_1 q_2 a_2 Q, \tag{18a}$$

$$T_1^3 = \frac{a_1 a_2}{4a_3}\left(\frac{\dot a_3}{a_3} - \frac{\dot a_1}{a_1}\right) A^2 - 2 q_3 q_1 a_1 Q, \tag{18b}$$

$$T_3^2 = \frac{a_3 a_1}{4a_2}\left(\frac{\dot a_2}{a_2} - \frac{\dot a_3}{a_3}\right) A^1 - 2 q_2 q_3 a_3 Q, \tag{18c}$$

Here $A^\mu = \bar\psi \gamma^5 \gamma^\mu \psi$ is the pseudovector constructed from Dirac spinor. It can be shown that

$$S = C/V, \quad K = C^2/V^2, \quad V = a_1 a_2 a_3, \quad C = \text{const.} \tag{19}$$

If $K = \{J,\ I + J,\ I - J\}$ (19) holds for a massless spinor field, while for $K = I$ it is valid for both massive and massless spinors.

Basic Equations

From the equations for the invariants of the spinor field

$$\dot{S}_0^0 + 2\mathcal{G} A_0^0 = 0, \tag{20a}$$

$$\dot{P}_0^0 - 2(m_{\rm sp} + \mathcal{D}) A_0^0 = 0, \tag{20b}$$

$$\dot{A}_0^0 + 2(m_{\rm sp} + \mathcal{D}) P_0 - 2\mathcal{G} S_0 = 0, \tag{20c}$$

$$\dot{A}_0^1 = 0, \tag{20d}$$

$$\dot{A}_0^2 = 0, \tag{20e}$$

$$\dot{A}_0^3 = 0, \tag{20f}$$

one dully finds

$$S^2 + P^2 + {A^0}^2 = \frac{c_0^2}{V^2}, \quad A^1 = \frac{c_1}{V}, \quad A^2 = \frac{c_2}{V}, \quad A^3 = \frac{c_3}{V}, \tag{21}$$

Basic Equations

The nondiagonal components of EMT on account of (20) gives

$$\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} = \frac{8 q_1 q_2 a_1 VQ}{c_3 a_3}, \tag{22a}$$

$$\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} = \frac{8 q_2 q_3 a_2 VQ}{c_1 a_1}, \tag{22b}$$

$$\frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} = \frac{8 q_3 q_1 a_3 VQ}{c_2 a_2}, \tag{22c}$$

which leads to the following relation between the electromagnetic, spinor and gravitational fields:

$$c_1 c_2 q_1 q_2 a_1^2 a_2 + c_2 c_3 q_2 q_3 a_2^2 a_3 + c_3 c_1 q_3 q_1 a_3^2 a_1 = 0. \tag{23}$$

With a_i being positive c_i and q_i should be of different signs.

Basic Equations

It should be noted that in absence of electromagnetic field field the non-diagonal components of the EMT leads to

$$\left(\frac{\dot a_1}{a_1} - \frac{\dot a_2}{a_2}\right) A^3 = 0, \tag{24a}$$

$$\left(\frac{\dot a_3}{a_3} - \frac{\dot a_1}{a_1}\right) A^2 = 0, \tag{24b}$$

$$\left(\frac{\dot a_2}{a_2} - \frac{\dot a_3}{a_3}\right) A^1 = 0,, \tag{24c}$$

which gives raise to three possibilities [BS-2018]:
(i)$A^1 = A^2 = A^3 = 0 \to$ linear and massless spinor field;
(ii) $A^2 = A^3 = 0$, and $a_2 = a_3$ - LRSBI spacetime;
(iii)$a_1 = a_2 = a_3 = a$ - FLRW model.

Basic Equations

Let us consider the diagonal components of the Einstein system:

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} = \kappa\left[P_1 + Q\left[q_1^2 a_1^2 - q_2^2 a_2^2 - q_3^2 a_3^2 + Q_2\right]\right], \tag{25a}$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa\left[P_1 + Q\left[q_2^2 a_2^2 - q_3^2 a_3^2 - q_1^2 a_1^2 + Q_2\right]\right], \tag{25b}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} = \kappa\left[P_1 + Q\left[q_3^2 a_3^2 - q_1^2 a_1^2 - q_2^2 a_2^2 + Q_2\right]\right], \tag{25c}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} = \kappa\left[P_2 + \frac{1 + \lambda_2(Z + 4KZ_K)}{X} Q\bar{Q}\right]. \tag{25d}$$

Eq. (25d) is the consequence of other three.

Introducing directional Hubble parameters $H_i = \dot{a}_i/a_i$ we rewrite (25a), (25b) and (25c) as follows:

$$
\begin{aligned}
\dot{a}_1 &= H_1 a_1, && \text{(26a)}\\
\dot{a}_2 &= H_2 a_2, && \text{(26b)}\\
\dot{a}_3 &= H_3 a_3, && \text{(26c)}\\
\dot{H}_1 &= (\kappa/2)\left[P_1 + Q\left[q_3^2 a_3^2 + q_2^2 a_2^2 - 3 q_1^2 a_1^2 + Q_2/2\right]\right] \\
&\quad + (1/2)\left[2H_1^2 - H_1 H_2 - H_3 H_1 + H_2 H_3\right], && \text{(26d)}\\
\dot{H}_2 &= (\kappa/2)\left[P_1 + Q\left[q_3^2 a_3^2 + q_1^2 a_1^2 - 3 q_2^2 a_2^2 + Q_2/2\right]\right] \\
&\quad + (1/2)\left[2H_2^2 - H_1 H_2 - H_2 H_3 + H_3 H_1\right], && \text{(26e)}\\
\dot{H}_3 &= (\kappa/2)\left[P_1 + Q\left[q_2^2 a_2^2 + q_1^2 a_1^2 - 3 q_3^2 a_3^2 + Q_2/2\right]\right] \\
&\quad + (1/2)\left[2H_3^2 - H_3 H_1 - H_2 H_3 + H_1 H_2\right]. && \text{(26f)}
\end{aligned}
$$

The system (26) is a multiparametric problem with c_1, c_2, c_3, q_1, q_2, q_3, λ_1, λ_2. We also have the freedom to choose the initial values for metric functions and Hubble parameters and the nonlinear terms $Y(K)$ and $Z(K)$. Hence depending on the choice of these parameters and nonlinearities one can simulate different kinds of solutions. Our aim is to give some qualitative picture. We solve the foregoing system numerically. Taking this into account we set: $c_1 = c_2 = c_2 = 1$, $a_1(0) = a_2(0) = a_3(0) = 1$, $H_1(0) = 0.45$, $H_2(0) = H_3(0) = 0.5$. Further setting $q_1 = 0.02$, $q_2 = 0.05$ from (23) we find $q_3 = -0.014$. Further we choose the self-coupling constant $\lambda_1 = 0.7$and the coupling constant $\lambda_2 = 0.3$. Setting $Y(K) = K^{n_1}$ and $Z(K) = K^{n_2}$ for $n_1 = 3$ and $n_2 = 2$ we obtain the solutions for metric functions $a_i(t)$ and Hubble parameters $H_i(t)$, numerically.

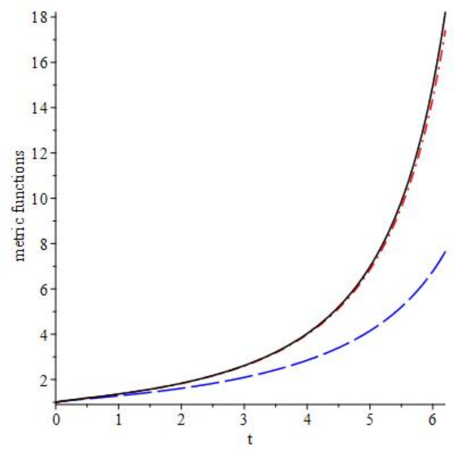

Рис.: Evolution of the metric functions a_1 (blue long dash line), a_2 (red dash dot line) and a_3 (black solid line).

Basic Equations

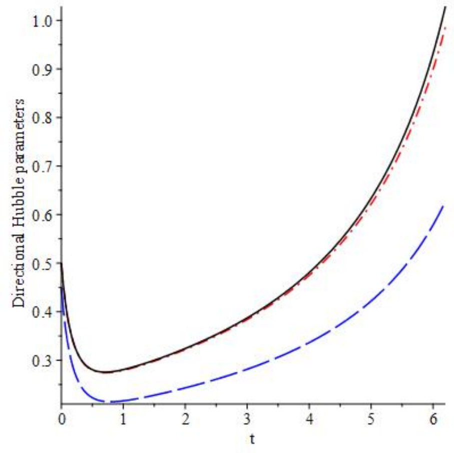

Рис.: Evolution of the directional Hubble parameters H_1 (blue long dash line), H_2 (red dash dot line) and H_3 black solid line.

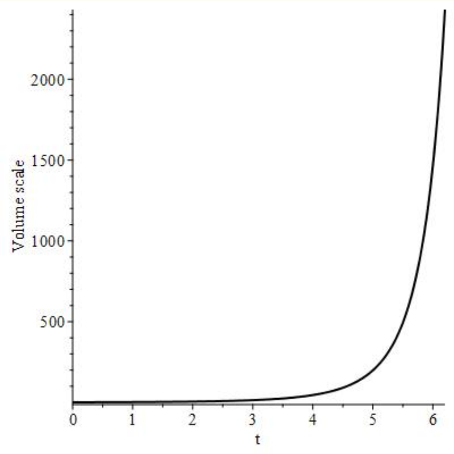

Рис.: Evolution of the volume scale V

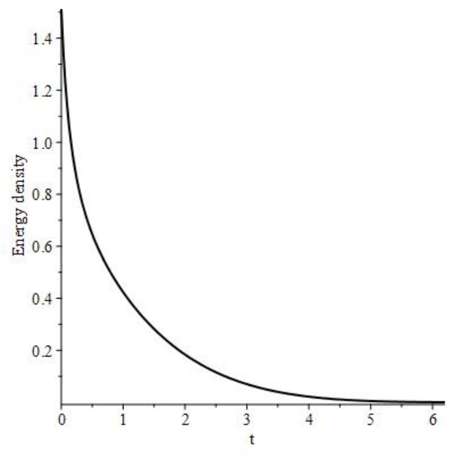

Рис.: Evolution of the energy density T_0^0

Basic Equations

To define whether there is a space-time singularity we study the following invariants: $I_1 = R$ - Riccie Scalar, $I_2 = R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann Scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. For the BI space-time case we have

$$I_1 = -2\left(\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\right), \tag{27a}$$

$$I_2 = \left[\left(R_0^0\right)^2 + \left(R_1^1\right)^2 + \left(R_2^2\right)^2 + \left(R_3^3\right)^2\right], \tag{27b}$$

$$I_3 = 4\left[\left(\frac{\ddot{a}_1}{a_1}\right)^2 + \left(\frac{\ddot{a}_2}{a_2}\right)^2 + \left(\frac{\ddot{a}_3}{a_3}\right)^2 + \left(\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}\right)^2 + \left(\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}\right)^2 + \left(\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\right)^2\right], \tag{27c}$$

Basic Equations

$$R_0^0 = -\left(\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3}\right), \tag{28a}$$

$$R_1^1 = -\left(\frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\right), \tag{28b}$$

$$R_2^2 = -\left(\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}\right), \tag{28c}$$

$$R_3^3 = -\left(\frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}\right). \tag{28d}$$

From the invariants one sees that only at a space-time point, where any of the scale factors a_1, a_2, a_3 becomes zero, the invariants I_1, I_2, I_3 become infinity, hence the space-time becomes singular at this point. In our case the scale factors are nontrivial, hence there is no singularity as such. Different choice of spinor field nonlinearity allows to simulate different types of universe, which we plan to study later.

Conclusions

Within the scope of a Bianchi type-I (BI) cosmological model we study the interacting system of spinor and electromagnetic fields and its role in the evolution of the Universe. In some earlier studies it was found that in case of a pure spinor field presence of nontrivial non-diagonal components of EMT leads to the following results: (i) spinor field becomes massless and linear; (ii) the BI space-time transforms into a locally rotationally symmetric (LRS - BI) or (iii) it evolves into a FLRW space-time from the very beginning. In case of electromagnetic field with induced nonlinearity spacetime becomes isotropic. In case of interacting spinor and electromagnetic fields restrictions are not as severe as in other cases.

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THANK YOU!