ON THE COALESCENCE MODEL FOR HIGH ENERGY NUCLEAR REACTIONS

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The coalescence model for composite particle spectra from high energy collisions is formulated in a way which clarifies the underlying assumptions and the meaning of the parameters in the model. A density matrix formalism is used to describe a highly excited part formed by a collision and the coalescence volume is found to be related to the internal wave function of the composite particle and the spatial distribution of nucleons in the highly excited part.

Composite particle spectra from high energy nucleon–nucleus and nucleus–nucleus collisions have been sucessfully explained by the coalescence model [1–3]. The model assumes that a group of nucleons whose momenta lie within a momentum sphere of radius p_0 coalesce to produce a composite particle. The probability for emission of a composite particle with Z protons and N neutrons can be expressed by the proton and neutron emission probabilities as

$$P(Z,N;\boldsymbol{k}) = (\tfrac{4}{3}\pi p_0^3)^{A-1}(Z!\,N!)^{-1}\,[P_\mathrm{p}(\boldsymbol{k})]^Z\,[P_\mathrm{n}(\boldsymbol{k})]^N\,, \tag{1}$$

where $\boldsymbol{k}$ is the momentum per nucleon and $A \equiv Z + N$. The probabilities are related to the corresponding cross sections as

$$P(Z,N;\boldsymbol{k}) \equiv \frac{\gamma}{\sigma_0}\frac{\mathrm{d}^3\sigma(Z,N)}{\mathrm{d}^3k}\,, \quad P_\mathrm{p}(\boldsymbol{k}) \equiv \frac{\gamma}{\sigma_0}\frac{\mathrm{d}^3\sigma_\mathrm{p}}{\mathrm{d}^3k}\,, \quad P_\mathrm{n}(\boldsymbol{k}) \equiv \frac{\gamma}{\sigma_0}\frac{\mathrm{d}^3\sigma_\mathrm{n}}{\mathrm{d}^3k}\,, \tag{2}$$

where $\gamma \equiv (1 + k^2/m^2)^{1/2}$ is the Lorentz factor with m denoting the nucleon mass. σ_0 is usually taken to be the reaction cross section but this choice is not always justified as will be discussed later. The coalescence radius p_0 is a parameter of the model to be determined through the analyses of data for each composite particle. Although the model is intuitively understandable, its theoretical foundation has not yet been well established.

In this note, we use a density matrix formalism [4] to describe the model and try to clarify the underlying assumptions and the meaning of the parameters. We assume that, after a fast process in a collision, a highly excited part (HX) is formed temporarily and decays by emitting various particles. The HX may be a fireball or a hot spot, but we need not specify it further. It is conveniently described by a density matrix [5] and we assume that the momentum distributions of emitted particles are approximately given by the density matrix at this stage. This assumption seems justified only for relatively fast particles for which various final state interactions after this stage can be neglected. The picture of the reaction process presented here looks similar to the one proposed by Mekjian

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[6] in the framework of a thermal model but neither thermal nor chemical equilibrium needs to be assumed in our picture ‡1.

The probability for proton emission at momentum $\boldsymbol{k}$ in the c.m. system of the HX is then given by

$$P_p(k) = \frac{1}{(2\pi)^3} \int dr\, dr'\, e^{-ikr} \rho_p(r, r') e^{ikr'} , \tag{3}$$

where $\rho_p(r, r')$ is the proton density matrix in the HX. We will, for the moment, restrict ourselves to non-relativistic particles and therefore the Lorentz factor, γ, will be omitted everywhere. The relativistic case will be briefly discussed later. σ_0, which relates the probability to the cross section as in eq. (2), is the formation cross section of the HX in this picture and is in general a fraction of the total reaction cross section. The neutron emission probability, $P_n(k)$, is similarly given by the neutron density matrix $\rho_n(r, r')$. The expression (3) can be generalized to the case of a composite particle and, for example, the probability for deuteron emission at momentum per nucleon $\boldsymbol{k}$ is given by

$$P(1,1;k) = \frac{2^3}{(2\pi)^3} \cdot \frac{3}{4} \int dr_1\, dr_2\, dr_1'\, dr_2'\, e^{-ik(r_1+r_2)} \psi_d^*(r_1 - r_2) \rho_{pn}(r_1, r_2; r_1'; r_2') \psi_d(r_1' - r_2') e^{ik(r_1'+r_2')} , \tag{4}$$

where ρ_{pn} is the proton–neutron two-particle density matrix and ψ_d is the deuteron internal wave function. The factor 2^3 is due to the fact that we are considering the momentum per nucleon and 3/4 is due to the spins. We then assume that ρ_{pn} is approximately given by the product of ρ_p and ρ_n, i.e.

$$\rho_{pn}(r_1, r_2; r_1', r_2') \approx \rho_p(r_1, r_1') \rho_n(r_2, r_2') , \tag{5}$$

which is equivalent to neglecting the p–n correlation in the HX ‡2. This seems justified for a randomized system in which specific correlations between nucleons are expected to be smeared out. We further assume that the one-particle density matrix, ρ_p for instance, can be written as

$$\rho_p(r, r') = D_p((r + r')/2) \int dk\, e^{ik(r-r')} P_p(k) , \tag{6}$$

where D_p describes the spatial distribution of protons in the HX and is normalized as

$$\int dr\, D_p(r) = 1 . \tag{7}$$

The factorized form (6) implies that the spatial and momentum distributions are not correlated with each other. With the assumptions (5) and (6), the expression (4) becomes

$$P(1, 1; k) = \int dp\, F_d(p) P_p(k + p) P_n(k - p) , \tag{8}$$

where F_d is defined by

$$F_d(p) \equiv 2^3 \cdot \frac{3}{4} \int \frac{dq}{(2\pi)^3}\, \tilde{\psi}_d^*(p + \tfrac{1}{2}q) \tilde{\psi}_d(p - \tfrac{1}{2}q) \tilde{D}_p(q) \tilde{D}_n(-q) , \tag{9}$$

with $\tilde{\psi}_d$, $\tilde{D}_p$ and $\tilde{D}_n$ denoting the Fourier transforms of ψ_d, D_p and D_n, respectively. Expression (8) indicates that a proton and a neutron whose momenta are $\boldsymbol{k} + \boldsymbol{p}$ and $\boldsymbol{k} - \boldsymbol{p}$, respectively, coalesce to produce a deuteron with a

‡1 A clear picture of a fireball explosion was given in ref. [5]. The present approach assumes that the transition from compressed hot matter to a dilute system of non-interacting particles is so fast that a sudden approximation can be used to calc e the emission probabilities of the particles.

‡2 It should be noted that, in the present scheme, a deuteron is not produced by a dynamical p–n correlation but by a localization of nucleons in phase space as described by the density matrices.

probability $F_d(\boldsymbol{p})$, which is determined by the deuteron internal wave function ψ_d and the spatial distribution functions D_p and D_n through eq. (9). If the $\boldsymbol{k}$-dependences of $P_p(\boldsymbol{k})$ and $P_n(\boldsymbol{k})$ are weak compared with the $\boldsymbol{p}$-dependence of $F_d(\boldsymbol{p})$, eq. (8) becomes equivalent to eq. (1) for $Z = N = 1$ with

$$\tfrac{4}{3}\pi p_0^3 = \int \mathrm{d}\boldsymbol{p}\, F_d(\boldsymbol{p})\,. \tag{10}$$

Using eq. (9), one can express the coalescence volume in terms of ψ_d, D_p and D_n as

$$\tfrac{4}{3}\pi p_0^3 = 2^3\cdot\tfrac{3}{4}\cdot(2\pi)^3 \int \mathrm{d}\boldsymbol{r}\,|\psi_d(\boldsymbol{r})|^2 D_2(\boldsymbol{r})\,,\quad D_2(\boldsymbol{r}) \equiv \int \mathrm{d}\boldsymbol{r}'\, D_p(\boldsymbol{r}-\boldsymbol{r}')D_n(\boldsymbol{r}')\,. \tag{11,12}$$

$D_2(\boldsymbol{r})$ gives the distribution of the p–n relative coordinate in the HX and is closely related to the interaction volume introduced by Mekjian [6]. In fact, if the spatial size of the internal wave function ψ_d is much smaller than that of the HX, then eq. (11) gives

$$\tfrac{4}{3}\pi p_0^3 \simeq 2^3\cdot\tfrac{3}{4}\cdot(2\pi)^3 D_2(0)\,. \tag{13}$$

$D_2(0)$ thus corresponds to the inverse of the interaction volume. In the actual situation, however, the size of the deuteron is comparable to that of the HX and therefore one has to use eq. (11) to relate the coalescence volume with the spatial size of the HX.

Expressions analogous to eqs. (8–10) can be obtained for the other composite particles such as ^{3}H, ^{3}He and ^{4}He. In the case of triton (^{3}H) one gets

$$P(1,2;\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\, F_t(\boldsymbol{p}_1,\boldsymbol{p}_2)P_p(\boldsymbol{k}+\boldsymbol{p}_1)P_n(\boldsymbol{k}-\tfrac{1}{2}\boldsymbol{p}_1+\boldsymbol{p}_2)P_n(\boldsymbol{k}-\tfrac{1}{2}\boldsymbol{p}_1-\boldsymbol{p}_2)\,, \tag{14}$$

$$F_t(\boldsymbol{p}_1,\boldsymbol{p}_2) = 3^3\cdot\frac{1}{4}\int\frac{\mathrm{d}\boldsymbol{q}_1\mathrm{d}\boldsymbol{q}_2}{(2\pi)^6}\,\widetilde{\psi}_t^*(\boldsymbol{p}_1+\tfrac{1}{2}\boldsymbol{q}_1,\boldsymbol{p}_2+\tfrac{1}{2}\boldsymbol{q}_2)\widetilde{\psi}_t(\boldsymbol{p}_1-\tfrac{1}{2}\boldsymbol{q}_1,\boldsymbol{p}_2-\tfrac{1}{2}\boldsymbol{q}_2)$$
$$\times\widetilde{D}_p(\boldsymbol{q}_1)\widetilde{D}_n(-\tfrac{1}{2}\boldsymbol{q}_1+\boldsymbol{q}_2)\widetilde{D}_n(-\tfrac{1}{2}\boldsymbol{q}_1-\boldsymbol{q}_2)\,, \tag{15}$$

where $\widetilde{\psi}_t$ is the Fourier transform of the triton internal wave function ψ_t. The coalescence volume is related to F_t as

$$\tfrac{1}{2}(\tfrac{4}{3}\pi p_0^3)^2 = \int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\, F_t(\boldsymbol{p}_1,\boldsymbol{p}_2)\,. \tag{16}$$

Trivial modifications give the expression for ^{3}He. For alpha particles (^{4}He), one gets

$$P(2,2;\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\, F_\alpha(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)P_p(\boldsymbol{k}+\boldsymbol{p}_1)P_p(\boldsymbol{k}-\tfrac{1}{3}\boldsymbol{p}_1+\boldsymbol{p}_2)$$
$$\times P_n(\boldsymbol{k}-\tfrac{1}{3}\boldsymbol{p}_1-\tfrac{1}{2}\boldsymbol{p}_2+\boldsymbol{p}_3)P_n(\boldsymbol{k}-\tfrac{1}{3}\boldsymbol{p}_1-\tfrac{1}{2}\boldsymbol{p}_2-\boldsymbol{p}_3)\,, \tag{17}$$

$$F_\alpha(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3) = 4^3\cdot\frac{1}{16}\int\frac{\mathrm{d}\boldsymbol{q}_1\mathrm{d}\boldsymbol{q}_2\mathrm{d}\boldsymbol{q}_3}{(2\pi)^9}\,\widetilde{\psi}_\alpha^*(\boldsymbol{p}_1+\tfrac{1}{2}\boldsymbol{q}_1,\boldsymbol{p}_2+\tfrac{1}{2}\boldsymbol{q}_2,\boldsymbol{p}_3+\tfrac{1}{3}\boldsymbol{q}_3)\widetilde{\psi}_\alpha(\boldsymbol{p}_1-\tfrac{1}{2}\boldsymbol{q}_1,\boldsymbol{p}_2-\tfrac{1}{2}\boldsymbol{q}_2,\boldsymbol{p}_3-\tfrac{1}{2}\boldsymbol{q}_3)$$
$$\times\widetilde{D}_p(\boldsymbol{q}_1)\widetilde{D}_p(-\tfrac{1}{3}\boldsymbol{q}_1+\boldsymbol{q}_2)\widetilde{D}_n(-\tfrac{1}{3}\boldsymbol{q}_1-\tfrac{1}{2}\boldsymbol{q}_2+\boldsymbol{q}_3)\widetilde{D}_n(-\tfrac{1}{3}\boldsymbol{q}_1-\tfrac{1}{2}\boldsymbol{q}_2-\boldsymbol{q}_3)\,, \tag{18}$$

$$\tfrac{1}{4}(\tfrac{4}{3}\pi p_0^3)^3 = \int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\, F_\alpha(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)\,, \tag{19}$$

where $\tilde{\psi}_\alpha$ is the Fourier transform of the internal wave function for ^{4}He.

Since the functions F defined by eqs. (9), (15) and (18) are the crucial quantities in the present description, it is instructive to have their explicit forms for some simplified choice of the wave functions ψ and the function D. Assuming gaussian forms, i.e.

$$\psi_d(r) = (\nu_d/2\pi)^{3/4}\exp(-\tfrac{1}{4}\nu_d r^2)\,,\quad \psi_t(r_1,r_2) = (\nu_t^2/3\pi^2)^{3/4}\exp(-\tfrac{1}{3}\nu_t r_1^2 - \tfrac{1}{4}\nu_t r_2^2)\,,$$
$$\psi_\alpha(r_1,r_2,r_3) = (\nu_\alpha^3/4\pi^3)^{3/4}\exp(-\tfrac{3}{8}\nu_\alpha r_1^2 - \tfrac{1}{3}\nu_\alpha r_2^2 - \tfrac{1}{4}\nu_\alpha r_3^2),\quad D_p(r) = D_n(r) = (\nu/\pi)^{3/2}\exp(-\nu r^2)\,,$$

one obtains

$$F_d(p) = 2^3\cdot\tfrac{3}{4}[4\nu/(\nu_d+\nu)]^{3/2}\exp(-2p^2/\nu_d)\,,\quad F_t(p_1,p_2) = 3^3\cdot\tfrac{1}{4}[4\nu/(\nu_t+\nu)]^3\exp(-3p_1^2/2\nu_t - 2p_2^2/\nu_t)\,,$$
$$F_\alpha(p_1,p_2,p_3) = 4^3\cdot\tfrac{1}{16}[4\nu/(\nu_\alpha+\nu)]^{9/2}\exp(-4p_1^2/3\nu_\alpha - 3p_2^2/2\nu_\alpha - 2p_3^2/\nu_\alpha)\,.$$

Eqs. (10), (16) and (19) then relate the coalescence radii p_0 to the size parameter ν of the HX as

$$\tfrac{4}{3}\pi p_0^3 = \tfrac{3}{4}\cdot 2^{3/2}(4\pi)^{3/2}[\nu_d\nu/(\nu_d+\nu)]^{3/2}\,,\qquad \text{for the deuteron}\,,$$
$$\tfrac{1}{2}(\tfrac{4}{3}\pi p_0^3)^2 = \tfrac{1}{4}\cdot 3^{3/2}(4\pi)^3[\nu_t\nu/(\nu_t+\nu)]^3\,,\qquad \text{for the triton}\,,$$
$$\tfrac{1}{4}(\tfrac{4}{3}\pi p_0^3)^3 = \tfrac{1}{16}\cdot 4^{3/2}(4\pi)^{9/2}[\nu_\alpha\nu/(\nu_\alpha+\nu)]^{9/2}\,,\qquad \text{for the alpha particle}\,.$$

While the k-dependence of $P_p(k)$ and $P_n(k)$ has been neglected in the above relations, its effect can be taken into account in the case of gaussian P_n and P_n, i.e.

$$P_p(k) \propto P_n(k) \propto \exp(-\beta k^2)\,.$$

The modified relations result in multiplying p_0 in the above by the factor $(1+\beta\nu_c)^{1/2}$ ($\nu_c = \nu_d, \nu_t$ or ν_α).

We take, as an example, the case of a Ne + U collision at the incident energy of 400 MeV/nucleon and use the coalescence radii obtained through the analyses of the data [1] to deduce the values of the parameter ν. The wave function parameters are [7]

$$\nu_d = 0.20\ \text{fm}^{-2}\,,\quad \nu_t = \nu_{^3\text{He}} = 0.36\ \text{fm}^{-2}\,,\quad \nu_\alpha = 0.58\ \text{fm}^{-2}\,.$$

β is taken to be either zero (weak k-dependence of P_n and P_n) or 0.52 fm^2 corresponding to the temperature of 40 MeV. The results are shown in table 1. The size parameter R_{th} of the interaction volume in Mekjian's thermal model [6] is also given and $\sqrt{3/5}\,R_{th}$ is compared with the rms radius $R_{rms} \equiv \sqrt{3/2\nu}$ of the HX in the present model. Due to the finite size effect of the composite particles, R_{rms} is always smaller than $\sqrt{3/5}\,R_{th}$ and also its dependence on the composite particles is different from that of R_{th}. The k-dependence of P_p and P_n further reduces the values of R_{rms}. Although the table is useful in studying the relations between the models in a semi-quantitative way, the values of R_{rms} should not be taken too seriously, since the coalescence radius p_0 contains an ambiguity due to our lack of knowledge on the formation cross section σ_0 of the HX.

A fully relativistic extension of the present description is not trivial and requires a relativistic theory of com-

Table 1
The size parameter of the highly excited part (HX) for a Ne + U collision at 400 MeV/nucleon.

Composite particles	p_0 a) (MeV/c)	$\sqrt{3/5}\,R_{th}$ b) (fm)	R_{rms} (β = 0) (fm)	R_{rms} (β = 0.52 fm^2) (fm)
d	129	5.30	4.53	4.23
t, ^{3}He	129	4.54	4.38	3.94
^{4}He	142	3.81	3.71	3.16

a) Ref. [1]. b) Ref. [6].

posite particles. However, minimal kinematical modifications can be included so long as the internal motion of the composite particles is non-relativistic. These are the Lorentz factor γ in the expression of the emission probability in terms of the density matrix which gives the same factor in eq. (2) and a slight modification of the momentum variables in the definition of the function F.

The density matrix formalism seems appropriate for the quantal description of the coalescence model and clarifies the roles of various factors characterizing the composite particles and the particle emission source (HX). The thermal model proposed by Mekjian seems to correspond to a limiting case where the sizes of the composite particles are much smaller than that of the HX. It should be noted here that the density matrix formalism also seems appropriate for the description of the two-particle correlation which has been discussed in connection with the Hanbury–Brown–Twiss effect, and leads to an expression very similar to that obtained by Koonin in the wave packet formalism [8]. Such an application will be discussed in a separate paper.

Finally, we note that the present approach is applicable to the analyses of hadron spectra from high energy collisions where the highly excited part is now described by a density matrix for quarks [9].

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