ОИЯИ
ОБЪЕДИНЁННЫЙ ИНСТИТУТ
ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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В ожидании_е

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Taking non-renormalizable interactions under control

Dubna

Use of equations of motion

Field transformation

$$(*) \quad \Phi(x) \to \Phi(x) + \Delta\Phi(x)$$

$\Delta\Phi \sim \lambda$ ← Coupling constant

Lagrangian transformation

$$\mathcal{L}[\Phi] \to \mathcal{L}[\Phi] + \mathcal{L}'[\Phi]\Delta\Phi + +\mathcal{O}(\lambda^2)$$

- The S-matrix elements with the proper external lines renormalization factors are not influenced by the replacement of the fields $(*)$
- From this it follows that for any change in $\Delta\mathcal{L}$ which is proportional to $\mathcal{L}'[\Phi]$ does not influence the S-matrix. In other words one can use equations of motion

$$\mathcal{L}'[\Phi^{cl}] = 0$$

to simplify expressions for $\Delta\mathcal{L}$

Local counter terms on mass shell

Counter terms $$\Delta\mathcal{L} = \sum_{i=1}^{N} z_i O_i(\Phi)$$

Local operators

Equations of motion $$R_j(\Phi) = 0$$

Off shell $$\Delta\mathcal{L} = \sum_{i=1}^{N} \sum_{j=1}^{M=N-K} c_{ij} z_i O_j(\Phi) + \sum_{j=1}^{K} z_j \Phi_j R_j(\Phi)$$

On shell $$\Delta\tilde{\mathcal{L}} = \sum_{i=1}^{N} \sum_{j=1}^{M=N-K} c_{ij} z_i \tilde{O}_j(\Phi_{cl})$$

Local counter terms on mass shell

Example: Renormalizable theories ϕ^4_4

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

Equation of motion $R_1 = \partial^2 \phi + m^2 \phi + \lambda/3! \phi^3 = 0$

Off shell
$$\Delta\mathcal{L} = -z_1 \frac{1}{2} \Phi \partial^2 \Phi - \frac{1}{2} z_2 m^2 \Phi^2 - z_4 \frac{\lambda}{4!}\Phi^4,$$

$$\Delta\mathcal{L} = -\frac{1}{2}(z_2 - z_1) m^2 \Phi^2 - (z_4 - 2z_1)\frac{\lambda}{4!}\Phi^4 - \frac{z_1}{2}\Phi(\partial^2 \Phi + m^2 \Phi + \frac{\lambda}{6}\Phi^3),$$

On shell
$$\Delta\tilde{\mathcal{L}} = -\frac{1}{2}(z_2 - z_1) m^2 \Phi^2 - (z_4 - 2z_1)\frac{\lambda}{4!}\Phi^4,$$

Renormalization of the couplings $z_m = z_2 - z_1, z_\lambda = z_4 - 2z_1$

ϕ_6^4

UV divergences within dim reg

Loop Expansion (non-renormalizable case)

$$\Delta\mathcal{L}_1 \sim \lambda^2(s+t+u)\Phi^4(\frac{1}{\epsilon}+c_{11})$$

Momentum space

$$\Delta\mathcal{L}_1 \sim \lambda^2\partial^2\Phi^2\Phi^2(\frac{1}{\epsilon}+c_{11}),$$

Coordinate space

Off shell

$$\Delta\mathcal{L} = \underset{(\frac{1}{\epsilon}+c_{11})}{\lambda^2\partial^2\Phi^2\Phi^2} + \lambda^3[\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{12})}{\partial^4\Phi^2\Phi^2} + \underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{13})}{\partial^2\Phi^2\partial^2\Phi^2})] + \lambda^4[...] + \lambda^5[...]$$

$$\searrow \qquad \searrow \qquad \searrow$$

$$\underset{(\frac{1}{\epsilon}+c_{21})}{\lambda^3\Phi^6} + \lambda^4[\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{22})}{\partial^2\Phi^4\Phi^2} + \underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{23})}{\partial^2\Phi^2\Phi^4}]$$

$$\searrow \qquad \searrow$$

$$\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{32}),}{\lambda^5\Phi^8}$$

ϕ_6^4

Loop Expansion (non-renormalizable case)

UV divergences within dim reg

On shell

Coordinate space

$$\partial^2\Phi + \frac{\lambda}{3!}\Phi^3 = 0 \qquad \partial_\mu\partial^2\Phi + \frac{\lambda}{2}\partial_\mu\Phi\Phi^2 = 0$$

$$\Delta\tilde{\mathcal{L}} = \lambda^2\partial^2\Phi^2\Phi^2(\frac{1}{\epsilon} + c_{11} + c_{21})$$

$$+\lambda^3[\partial^4\Phi^2\Phi^2(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{12} + c_{22} + c_{32}) + \partial^2\Phi^2\partial^2\Phi^2(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{13} + c_{23})] + \dots$$

ϕ^6_4

Loop Expansion (non-renormalizable case)

UV divergences within dim reg

Coordinate space

Off shell

$$
\begin{array}{rcl}
\Delta\mathcal{L} & = & \lambda\Phi^6 + \lambda^2\Phi^8 \quad + \quad \lambda^3\Phi^{10} \quad + \quad \lambda^4\Phi^{12} + \ldots \\
 & & \qquad\quad \searrow \qquad\quad \searrow\!\!\!\!\nearrow \qquad\quad \nearrow \\
 & & \qquad\quad + \lambda^2\partial^2\Phi^6 + \lambda^3\partial^2\Phi^8 \; + \; \lambda^4\partial^2\Phi^{10} + \ldots \\
 & & \qquad\qquad\qquad \searrow\!\!\!\!\nearrow \qquad\quad \searrow\!\!\!\!\nearrow \\
 & & \qquad\quad + \lambda^2\partial^4\Phi^4 + \lambda^3\partial^4\Phi^6 + \lambda^4\partial^4\Phi^8 + \ldots,
\end{array}
$$

On shell

$$
\begin{aligned}
\Delta\tilde{\mathcal{L}} = \lambda\Phi^6 + \lambda^2\partial^2\Phi^6 + \lambda^3\partial^4\Phi^6 + \ldots \\
+\lambda^2\Phi^8 + \lambda^3\partial^2\Phi^8 + \ldots
\end{aligned}
$$

ϕ^4_4

The Amplitudes

Renormalizable case

$$A_4 = \lambda + A\lambda^2(\frac{3}{\epsilon} - \log(s/\mu) - \log(t/\mu) - \log(u/\mu)) + \dots$$ One loop

$$\Delta\mathcal{L} = A\frac{\lambda^2}{4!}\Phi^4(-\frac{3}{\epsilon} - c)$$ ← Arbitrary constant

$$A_4^{finite} = \lambda + A\lambda^2(-\log(s/\mu) - \log(t/\mu) - \log(u/\mu) - c) + \dots$$

$$A_4^0 = \lambda + A\lambda^2(\log(s_0/\mu) - \log(t_0/\mu) - \log(u_0/\mu) - c).$$ Normalization

$$\lambda = A_4^0 + (A_4^0)^2 A(\log(s_0/\mu) - \log(t_0/\mu) - \log(u_0/\mu) - c)$$

$$A_4^{finite} = A_4^0 + (A_4^0)^2 A(-\log(s/s_0) - \log(t/t_0) - \log(u/u_0)) + \dots$$

- Summary: To fix the arbitrariness it is enough to put <u>one</u> condition on <u>one</u> scattering amplitude. All the rest is calculated unambiguously.

The Amplitudes

ϕ^4_8

$$A_4 = \quad \lambda + A\lambda^2(\frac{s^2+t^2+u^2}{\epsilon} - s^2\log(s/\mu) - t^2\log(t/\mu) - u^2\log(u/\mu)) + ...$$

$$A_6 = B\lambda^3 Q^2(\frac{1}{\epsilon} - \log Q/\mu) + ...$$

$$A_8 = E\lambda^4(\frac{1}{\epsilon} - \log P/\mu) + ...$$

Arbitrary constants

$$\Delta\mathcal{L} = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{1}{\epsilon} - c_1) + B\lambda^3\partial^2\Phi^2\Phi^4(-\frac{1}{\epsilon} - c_2) + E\lambda^4\Phi^8(-\frac{1}{\epsilon} - c_3)$$ Off shell

$$\Delta\tilde{\mathcal{L}} = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{A+B+E}{\epsilon} - c_1 - c_2 - c_3) = \partial^4\Phi^2\Phi^2 A\lambda^2(-\frac{A+B+E}{\epsilon} - c),$$ On shell

$$A_4^{finite} = \lambda + A\lambda^2(-s^2\log(s/\mu) - t^2\log(t/\mu) - u^2\log(u/\mu) - c(s^2+t^2+u^2)) + ...$$

$$A_6^{finite} = B\lambda^3 Q^2(-\log Q/\mu) + ...,$$

$$A_8^{finite} = E\lambda^4(-\log P/\mu) + ...$$

Now, to fix arbitrariness, we will impose two conditions on the 4-point amplitude and one condition on the 6-point amplitude. The first fixes λ and c (μ will fall out), and the second - μ

ϕ^4_8

The Amplitudes

$$A_4 = \lambda + A\lambda^2\left(\frac{s^2+t^2+u^2}{\epsilon} - s^2\log(s/\mu) - t^2\log(t/\mu) - u^2 \log \ldots\right.$$

$$A_6 = B\lambda^3 Q^2\left(\frac{1}{\epsilon} - \log Q/\mu\right) + \ldots$$

$$A_8 = E\lambda^4\left(\frac{1}{\epsilon} - \log P/\mu\right) + \ldots$$

$$\Delta\mathcal{L} = \partial^4\Phi^2\Phi^2 A\lambda^2\left(-\frac{1}{\epsilon} - c_1\right) + B\lambda^3 \ldots\, c_2) + E\lambda^4\Phi^8\left(-\frac{1}{\epsilon} - c_3\right)$$ Off shell

$$\Delta\tilde{\mathcal{L}} = \partial^4\Phi^2\Phi^2 A\lambda^2\left(-\frac{A+B+E}{\ldots}\right. \ldots = \partial^4\Phi^2\Phi^2 A\lambda^2\left(-\frac{A+B+E}{\epsilon} - c\right),$$ On shell

Summary: to fix the complete arbitrariness in the amplitudes on the mass shell it is necessary to impose an infinite number of conditions on the 4-point amplitude (in the finite order of PT, the number of conditions is finite) and one on the 6-point amplitude.

$$A_4^{finite} = \lambda \ldots (s/\mu) - t^2\log(t/\mu) - u^2\log(u/\mu) - c(s^2+t^2+u^2)) + \ldots$$

$$A_6^{finite} \ldots (-\log Q/\mu) + \ldots,$$

$$A_8^{finite} = E\lambda^4(-\log P/\mu) + \ldots$$

Now, to fix arbitrariness, we will impose two conditions on the 4-point amplitude and one condition on the 6-point amplitude.The first fixes λ and c (μ will fall out), and the second - μ

Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.

- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are <u>unambiguously</u> fixed by the normalization conditions.

- The procedure for obtaining predictions for observables - scattering amplitudes on a mass shell - is as follows: One has to calculate the amplitudes directly on the mass shall, subtract all divergences with arbitrary subtraction constants and then fix these constants by imposing conditions on the scattering amplitudes.

- In any theory, it is possible to fix the arbitrariness normalizing the amplitude of 2->2 scattering (plus one condition on 3->3 scattering). Then all other amplitudes will be determined unambiguously.