



**ОИЯИ**  
ОБЪЕДИНЁННЫЙ ИНСТИТУТ  
ЯДЕРНЫХ ИССЛЕДОВАНИЙ

'24  
April

Dmitry Kazakov



Bogoliubov Laboratory of Theoretical Physics

В ожидании\_e

# Taking non-renormalizable interactions under control

Dubna

# Use of equations of motion

## Field transformation

$$(*) \quad \Phi(x) \rightarrow \Phi(x) + \Delta\Phi(x)$$

$$\Delta\Phi \sim \lambda \quad \leftarrow \text{Coupling constant}$$

## Lagrangian transformation

$$\mathcal{L}[\Phi] \rightarrow \mathcal{L}[\Phi] + \mathcal{L}'[\Phi]\Delta\Phi + \mathcal{O}(\lambda^2)$$

- The S-matrix elements with the proper external lines renormalization factors are not influenced by the replacement of the fields (\*)
- From this it follows that for any change in  $\Delta\mathcal{L}$  which is proportional to  $\mathcal{L}'[\Phi]$  does not influence the S-matrix. In other words one can use equations of motion

$$\mathcal{L}'[\Phi^{cl}] = 0$$

to simplify expressions for  $\Delta\mathcal{L}$



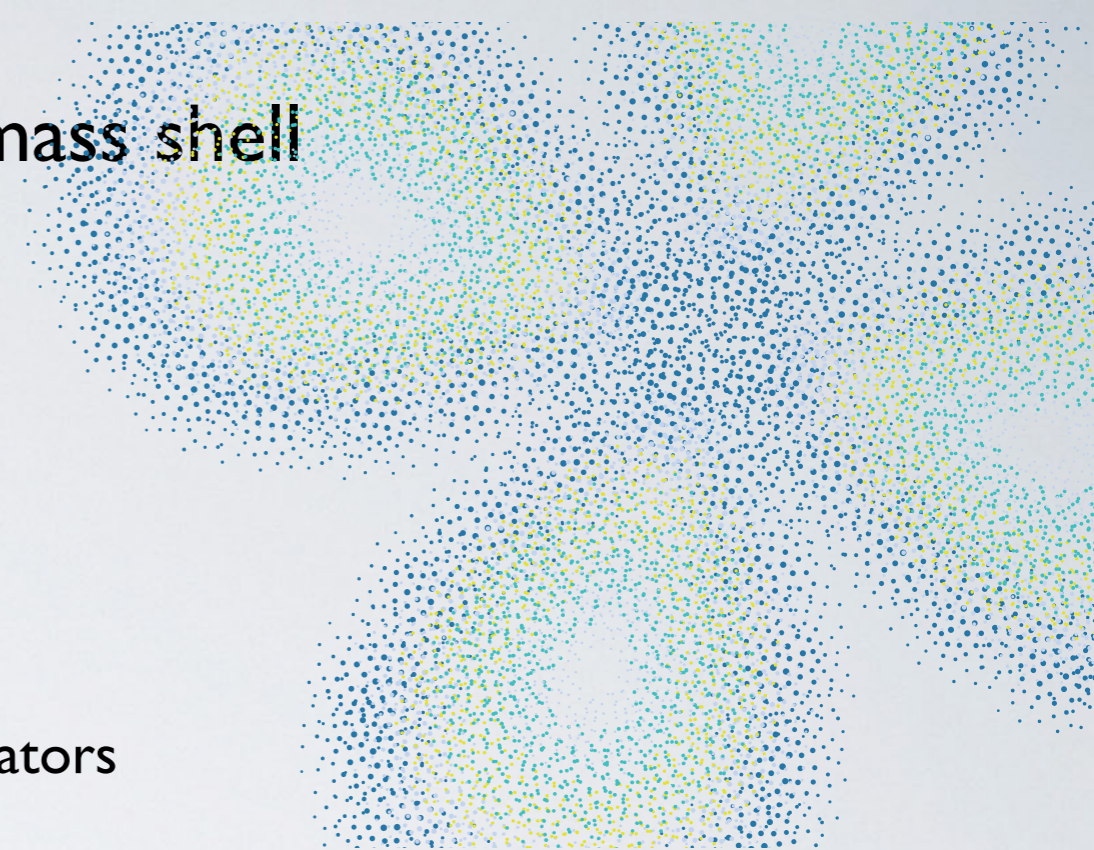
# Local counter terms on mass shell

Counter terms

$$\Delta\mathcal{L} = \sum_{i=1}^N z_i O_i(\Phi)$$



Local operators



Equations of motion

$$R_j(\Phi) = 0$$

Off shell

$$\Delta\mathcal{L} = \sum_{i=1}^N \sum_{j=1}^{M=N-K} c_{ij} z_i O_j(\Phi) + \sum_{j=1}^K z_j \Phi_j R_j(\Phi)$$

On shell

$$\Delta\tilde{\mathcal{L}} = \sum_{i=1}^N \sum_{j=1}^{M=N-K} c_{ij} z_i \tilde{O}_j(\Phi_{cl})$$



# Local counter terms on mass shell

Example: Renormalizable theories

$$\phi_4^4$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Equation of motion

$$R_1 = \partial^2\phi + m^2\phi + \lambda/3!\phi^3 = 0$$

Off shell

$$\Delta\mathcal{L} = -z_1\frac{1}{2}\Phi\partial^2\Phi - \frac{1}{2}z_2m^2\Phi^2 - z_4\frac{\lambda}{4!}\Phi^4,$$

$$\Delta\mathcal{L} = -\frac{1}{2}(z_2 - z_1)m^2\Phi^2 - (z_4 - 2z_1)\frac{\lambda}{4!}\Phi^4 - \frac{z_1}{2}\Phi(\partial^2\Phi + m^2\Phi + \frac{\lambda}{6}\Phi^3),$$

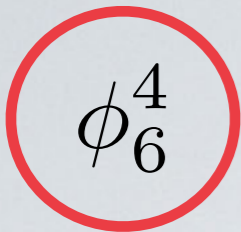
On shell

$$\Delta\tilde{\mathcal{L}} = -\frac{1}{2}(z_2 - z_1)m^2\Phi^2 - (z_4 - 2z_1)\frac{\lambda}{4!}\Phi^4,$$

Renormalization of the couplings

$$z_m = z_2 - z_1, z_\lambda = z_4 - 2z_1$$





# Loop Expansion (non-renormalizable case)

UV divergences  
within dim reg

$$\Delta\mathcal{L}_1 \sim \lambda^2 (s + t + u) \Phi^4 \left( \frac{1}{\epsilon} + c_{11} \right)$$

$$\Delta\mathcal{L}_1 \sim \lambda^2 \partial^2 \Phi^2 \Phi^2 \left( \frac{1}{\epsilon} + c_{11} \right),$$

Momentum space

Coordinate space

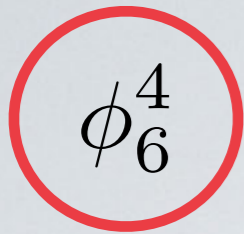
Off shell

$$\Delta\mathcal{L} = \lambda^2 \partial^2 \Phi^2 \Phi^2 \left( \frac{1}{\epsilon} + c_{11} \right) + \lambda^3 \left[ \partial^4 \Phi^2 \Phi^2 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{12} \right) + \partial^2 \Phi^2 \partial^2 \Phi^2 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{13} \right) \right] + \lambda^4 [\dots] + \lambda^5 [\dots]$$

$$\lambda^3 \Phi^6 \left( \frac{1}{\epsilon} + c_{21} \right) + \lambda^4 \left[ \partial^2 \Phi^4 \Phi^2 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{22} \right) + \partial^2 \Phi^2 \Phi^4 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{23} \right) \right]$$

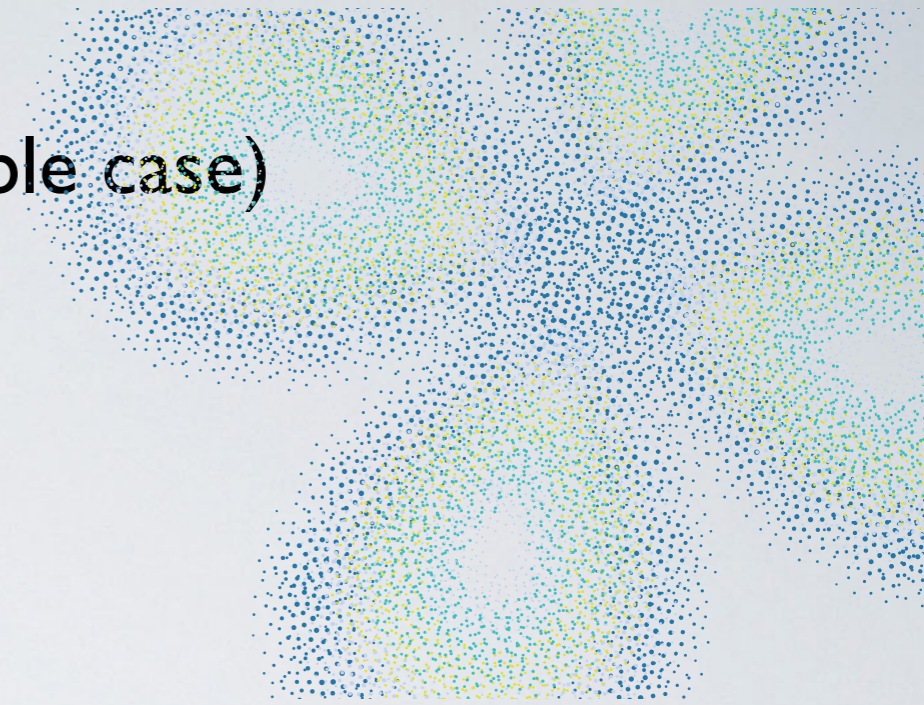
$$\lambda^5 \Phi^8 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{32} \right),$$





## Loop Expansion (non-renormalizable case)

UV divergences within dim reg



On shell

Coordinate space

$$\partial^2 \Phi + \frac{\lambda}{3!} \Phi^3 = 0 \quad \partial_\mu \partial^2 \Phi + \frac{\lambda}{2} \partial_\mu \Phi \Phi^2 = 0$$

$$\begin{aligned} \Delta \tilde{\mathcal{L}} = & \lambda^2 \partial^2 \Phi^2 \Phi^2 \left( \frac{1}{\epsilon} + c_{11} + c_{21} \right) \\ & + \lambda^3 \left[ \partial^4 \Phi^2 \Phi^2 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{12} + c_{22} + c_{32} \right) + \partial^2 \Phi^2 \partial^2 \Phi^2 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + c_{13} + c_{23} \right) \right] + \dots \end{aligned}$$





# Loop Expansion (non-renormalizable case)

UV divergences  
within dim reg

Coordinate space

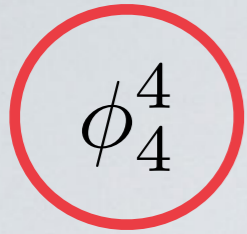
Off shell

$$\begin{aligned} \Delta \mathcal{L} &= \lambda \Phi^6 + \lambda^2 \Phi^8 + \lambda^3 \Phi^{10} + \lambda^4 \Phi^{12} + \dots \\ &\quad \searrow \quad \swarrow \quad \nearrow \\ &\quad + \lambda^2 \partial^2 \Phi^6 + \lambda^3 \partial^2 \Phi^8 + \lambda^4 \partial^2 \Phi^{10} + \dots \\ &\quad \quad \quad \swarrow \quad \swarrow \\ &\quad \quad \quad + \lambda^2 \partial^4 \Phi^4 + \lambda^3 \partial^4 \Phi^6 + \lambda^4 \partial^4 \Phi^8 + \dots, \end{aligned}$$

On shell

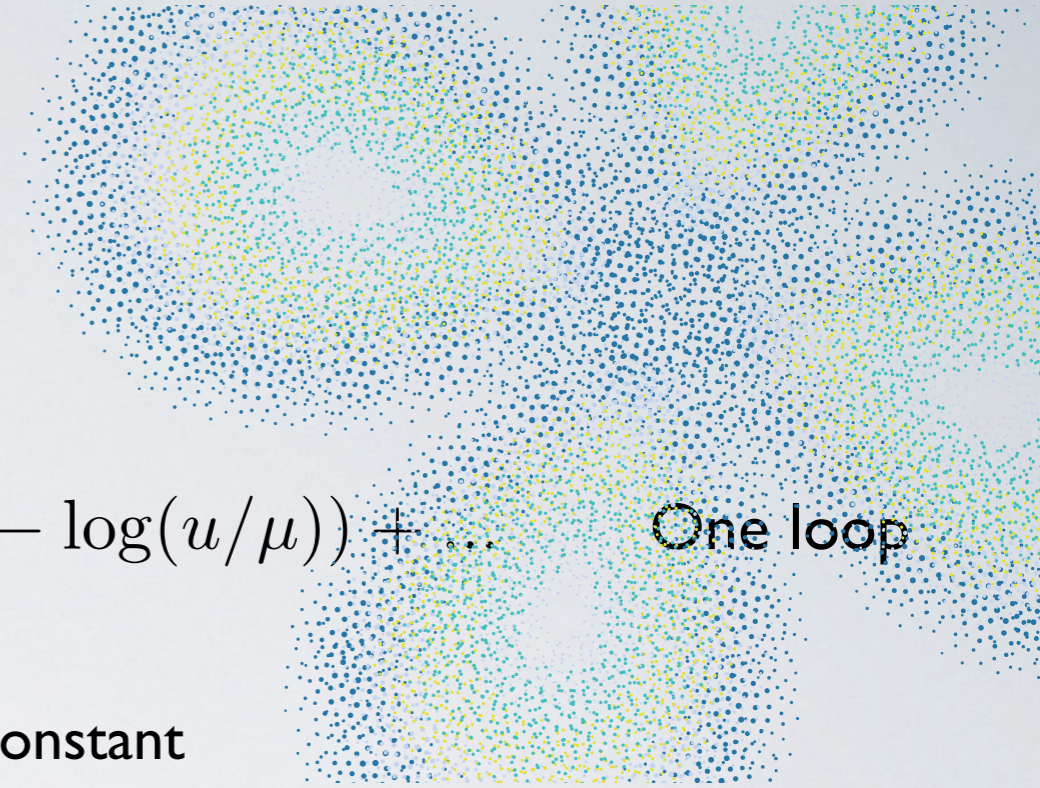
$$\begin{aligned} \Delta \tilde{\mathcal{L}} &= \lambda \Phi^6 + \lambda^2 \partial^2 \Phi^6 + \lambda^3 \partial^4 \Phi^6 + \dots \\ &\quad + \lambda^2 \Phi^8 + \lambda^3 \partial^2 \Phi^8 + \dots \end{aligned}$$





## The Amplitudes

Renormalizable case



$$A_4 = \lambda + A\lambda^2\left(\frac{3}{\epsilon} - \log(s/\mu) - \log(t/\mu) - \log(u/\mu)\right) + \dots$$

$$\Delta\mathcal{L} = A\frac{\lambda^2}{4!}\Phi^4\left(-\frac{3}{\epsilon} - c\right)$$

← Arbitrary constant

$$A_4^{finite} = \lambda + A\lambda^2(-\log(s/\mu) - \log(t/\mu) - \log(u/\mu) - c) + \dots$$

$$A_4^0 = \lambda + A\lambda^2(\log(s_0/\mu) - \log(t_0/\mu) - \log(u_0/\mu) - c).$$

Normalization

$$\lambda = A_4^0 + (A_4^0)^2 A(\log(s_0/\mu) - \log(t_0/\mu) - \log(u_0/\mu) - c)$$

$$A_4^{finite} = A_4^0 + (A_4^0)^2 A(-\log(s/s_0) - \log(t/t_0) - \log(u/u_0)) + \dots$$

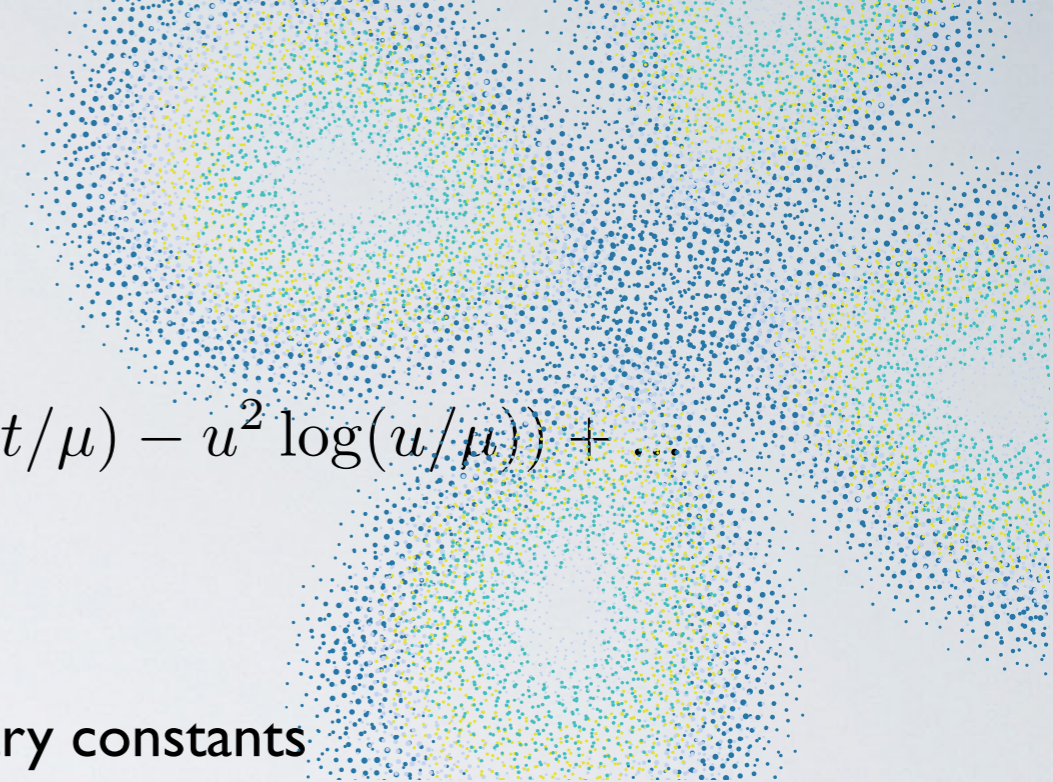
- Summary: To fix the arbitrariness it is enough to put one condition on one scattering amplitude. All the rest is calculated unambiguously.







# The Amplitudes



$$A_4 = \lambda + A\lambda^2 \left( \frac{s^2 + t^2 + u^2}{\epsilon} - s^2 \log(s/\mu) - t^2 \log(t/\mu) - u^2 \log(u/\mu) \right) + \dots$$

$$A_6 = B\lambda^3 Q^2 \left( \frac{1}{\epsilon} - \log Q/\mu \right) + \dots$$

$$A_8 = E\lambda^4 \left( \frac{1}{\epsilon} - \log P/\mu \right) + \dots$$

Arbitrary constants

$$\Delta\mathcal{L} = \partial^4\Phi^2\Phi^2 A\lambda^2 \left( -\frac{1}{\epsilon} - c_1 \right) + B\lambda^3 \partial^2\Phi^2\Phi^4 \left( -\frac{1}{\epsilon} - c_2 \right) + E\lambda^4 \Phi^8 \left( -\frac{1}{\epsilon} - c_3 \right) \quad \text{Off shell}$$

$$\Delta\tilde{\mathcal{L}} = \partial^4\Phi^2\Phi^2 A\lambda^2 \left( -\frac{A+B+E}{\epsilon} - c_1 - c_2 - c_3 \right) = \partial^4\Phi^2\Phi^2 A\lambda^2 \left( -\frac{A+B+E}{\epsilon} - c \right), \quad \text{On shell}$$

$$A_4^{finite} = \lambda + A\lambda^2 \left( -s^2 \log(s/\mu) - t^2 \log(t/\mu) - u^2 \log(u/\mu) - c(s^2 + t^2 + u^2) \right) + \dots$$

$$A_6^{finite} = B\lambda^3 Q^2 \left( -\log Q/\mu \right) + \dots,$$

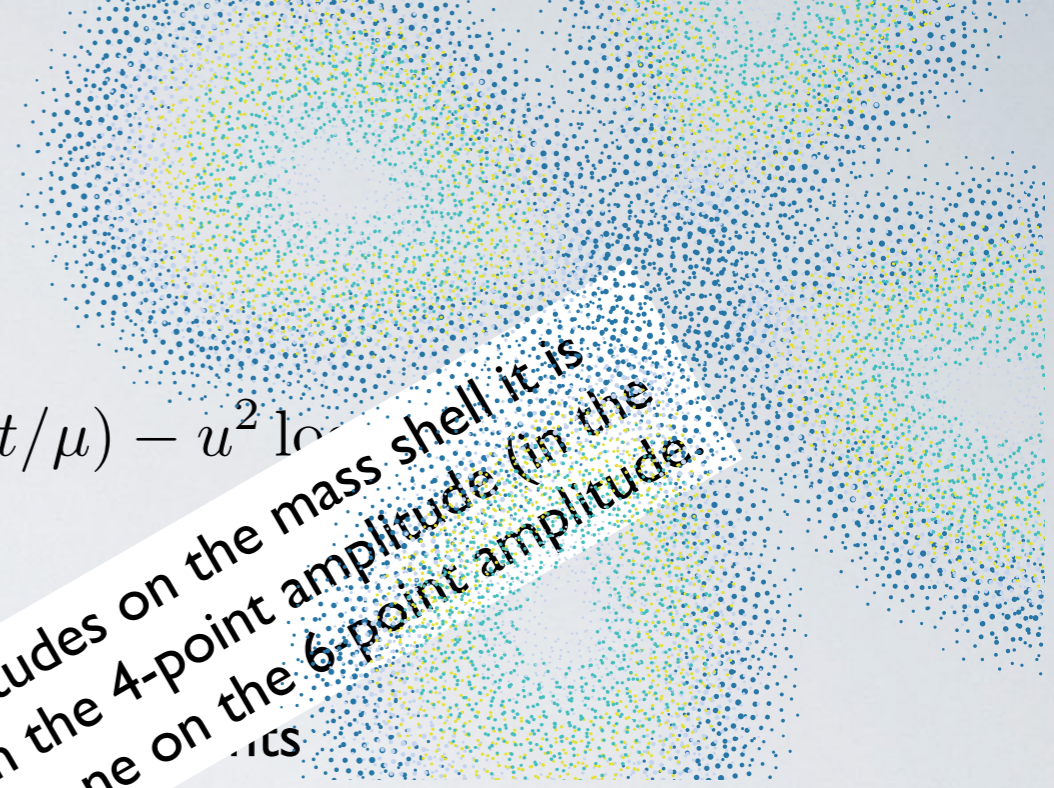
$$A_8^{finite} = E\lambda^4 \left( -\log P/\mu \right) + \dots$$

Now, to fix arbitrariness, we will impose two conditions on the 4-point amplitude and one condition on the 6-point amplitude. The first fixes  $\lambda$  and  $c$  ( $\mu$  will fall out), and the second -  $\mu$





# The Amplitudes



$$A_4 = \lambda + A\lambda^2 \left( \frac{s^2 + t^2 + u^2}{\epsilon} - s^2 \log(s/\mu) - t^2 \log(t/\mu) - u^2 \log(u/\mu) - c(s^2 + t^2 + u^2) \right) + \dots$$

$$A_6 = B\lambda^3 Q^2 \left( \frac{1}{\epsilon} - \log Q/\mu \right) + \dots$$

$$A_8 = E\lambda^4 \left( \frac{1}{\epsilon} - \log P/\mu \right) + \dots$$

$$\Delta\mathcal{L} = \partial^4\Phi^2\Phi^2 A\lambda^2 \left( -\frac{1}{\epsilon} - c_1 \right) + B\lambda^3 Q^2 \left( -\frac{1}{\epsilon} - c_2 \right) + E\lambda^4 \Phi^8 \left( -\frac{1}{\epsilon} - c_3 \right) \quad \text{Off shell}$$

$$\Delta\tilde{\mathcal{L}} = \partial^4\Phi^2\Phi^2 A\lambda^2 \left( -\frac{A+B+E}{\epsilon} - c \right) = \partial^4\Phi^2\Phi^2 A\lambda^2 \left( -\frac{A+B+E}{\epsilon} - c \right), \quad \text{On shell}$$

$$A_4^{finite} = \lambda + A\lambda^2 \left( \frac{s^2 + t^2 + u^2}{\epsilon} - s^2 \log(s/\mu) - t^2 \log(t/\mu) - u^2 \log(u/\mu) - c(s^2 + t^2 + u^2) \right) + \dots$$

$$A_6^{finite} = B\lambda^3 Q^2 \left( -\log Q/\mu \right) + \dots,$$

$$A_8^{finite} = E\lambda^4 \left( -\log P/\mu \right) + \dots$$

Summary: to fix the complete arbitrariness in the amplitudes on the mass shell it is necessary to impose an infinite number of conditions on the 4-point amplitude (in the finite order of PT, the number of conditions is finite) and one on the 6-point amplitude.

Now, to fix arbitrariness, we will impose two conditions on the 4-point amplitude and one condition on the 6-point amplitude. The first fixes  $\lambda$  and  $c$  ( $\mu$  will fall out), and the second -  $\mu$



# Conclusions

- The use of equations of motion makes it possible to reduce the number of independent operators and limit it to a set of operators with a fixed number of external lines.
- This opens up the possibility to work with non-renormalizable theories and obtain unambiguous predictions. The initial Lagrangian acquires an infinite number of new structures, which are unambiguously fixed by the normalization conditions.
- The procedure for obtaining predictions for observables - scattering amplitudes on a mass shell - is as follows: One has to calculate the amplitudes directly on the mass shell, subtract all divergences with arbitrary subtraction constants and then fix these constants by imposing conditions on the scattering amplitudes.
- In any theory, it is possible to fix the arbitrariness normalizing the amplitude of  $2 \rightarrow 2$  scattering (plus one condition on  $3 \rightarrow 3$  scattering). Then all other amplitudes will be determined unambiguously.

