Aspects of divergences in six-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theories

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Goal

Brief review of methods and results on study of divergences in $6D$ extended supersymmetric gauge theories.

Based on a series of papers published during last years in collaboration with E.A. Ivanov, B.M. Merzlikin and K.V. Stepanayatz.

Two-loop divergences, JHEP 05 (2023) 089.

Study of higher dimensional supersymmetric field theories related to superstring theory.

Specific feature of the superstring theory is existence of so called D-branes which are the $D+1$ dimensional surfaces in the ten-dimensional space-time. In the low-energy limit the D-brane is associated with $D+1$-dimensional extended supersymmetric gauge theory. Therefore, study of low-energy limit of superstring theory can be related to extended supersymmetric field theory in various dimensions.

In this sense, $D3$-brane is associated with $D4$, $\mathcal{N}=4$ SYM theory. $D5$-brane is associated with $D6$, $\mathcal{N}=(1,1)$ SYM theory.

Basic Motivations. Study of quantum field models with large number of symmetries

- Explicit symmetries: gauge symmetry, global symmetries, supersymmetries.
- Quantization procedure with preservation of all explicit symmetries.
- Perturbation theory with preservation of all explicit symmetries.
- Hidden (on-shell) symmetries. Preservation of hidden symmetries.
- Divergences, renormalization and effective actions.
- Construction of the new extended supersymmetric invariants as the the quantum contributions to effective action

Some problems of higher dimensional supersymmetric gauge theories.

1. Describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings (initiated by N. Seiberg, E. Witten, 1996; N. Seiberg, 1997).

2. Description of the interacting multiple $M5$-branes.

- Hypothetic M-theory is characterized by two extended objects: $M2$-brane and $M5$-brane in eleven dimensional space.
- The field description of interacting multiple $M2$-branes is given by Bagger-Lambert-Gustavsson (J. Bagger, N. Lambert, 2007; 2008. A. Gustavsson, 2009) theory which is $3D$, $\mathcal{N} = 8$ supersymmetric gauge theory.
- Lagrangian description of the interacting multiple $M5$-branes is not constructed so far.

3. Problem of miraculous cancelation of some on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop (M.B. Green, J.H. Schwarz, L. Brink, 1982).
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions (P.S. Howe, K.S. Stelle, 1984, 2003; G. Bossard, P.S. Howe, K.S. Stelle, 2009).
- Direct one-loop and two-loop component calculations (mainly in on-shell and in bosonic sector (E.S. Fradkin, A.A. Tseytlin, 1983; N. Marcus, A. Sagnotti, 1984, 1985.)
- Direct calculations of scattering amplitudes in $6D, \mathcal{N} = (1,1)$ theory up to five loops and in $D8, 10$ theories up to four loops (L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, 2015).

Results: On-shell divergences in maximally extended $6D$ SYM theory start at three loops. One-shell divergences in $8D$ and $10D$ SYM theories start at one loop.

The problems we are dealing with is aimed at studying the off-shell divergence structure. To understand, what is a reason that the divergences at one and two loops are proportional to the classical equations of motion and why, starting from three loops, this is violated.

Preservation of manifest supersymmetry: off-shell superfield formulation. Best formulation for $6D$ supersymmetric gauge theories is a harmonic superspace approach. In the case of $\mathcal{N}=(1,1)$ theory it is provides explicit off-shell $\mathcal{N}=(1,0)$ supersymmetry and hidden on-shell $\mathcal{N}=(0,1)$ supersymmetry.

Preservation of classical gauge invariance in quantum theory: harmonic superfield background field method.

Preservation of explicit gauge invariance and $\mathcal{N}=(1,0)$ supersymmetry at all steps of loop calculations: superfield proper-time methods.

$6D$ superalgebra is described by two independent supercharges. The simplest representations corresponds to $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ supersymmetries. In this sense, the maximally extended rigid supergauge theory is the $\mathcal{N} = (1,1)$ SYM theory.

$\mathcal{N} = (1,0)$ harmonic superspace:
Bosonic (commuting) coordinates x^M, $(M = 0,1,2,3,4,5)$; $u^{\pm i}\,(i = 1,2)$.
Fermionic (anticommuting) coordinates θ^a_i, $(a = 1,2,3,4)$.
Analytic subspace $\zeta = ({x_A}^M,\, \theta^{\pm\, a},\, u^{\pm\, i})$.
$\theta^{\pm\, a} = \theta^a_i u^{\pm\, i}$.
Analytic subspace is closed under the $\mathcal{N} = (1,0)$ supersymmetry.

Basic $\mathcal{N} = (1,0)$ harmonic superfields:

Hypermultiplet is described by analytic superfield $q^+(\zeta)$.
On-shell field contents: scalar field $f^i(x)$ and the spinor field $\psi_a(x)$)

Vector multiplet is described by analytic superfield V^{++}.
On-shell field contents: vector field and spinor field.

Theory of $\mathcal{N} = (1,0)$ non-Abelian vector multiplet coupled to hypermultiplet, (E.I. Ivanov, A.V. Smilga, B.M. Zupnik, Nucl.Phys. B (2005)).

- Action

$$S[V^{++}, q^+] = \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \mathrm{tr} \int d^{14}z\, du_1 \dots du_n \, \frac{V^{++}(z,u_1) \dots V^{++}(z,u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)}$$

$$- \int d\zeta^{-4} du \tilde{q}^+ \nabla^{++} q^+$$

- Harmonic covariant derivative

$$\nabla^{++} = D^{++} + iV^{++}$$

- Equations of motion

$$\frac{1}{2f^2} F^{++} - i\tilde{q}^+ q^+ = 0 \,, \qquad \nabla^{++} q^+ = 0 \,.$$

$$F^{++} = (D^+)^4 V^{--} ,\ D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0$$

$\mathcal{N} = (1,1)$ SYM theory

$\mathcal{N} = (1,1)$ SYM theory can be formulated in terms of $\mathcal{N} = (1,0)$ harmonic superfields as the $\mathcal{N} = (1,0)$ vector multiplet coupled to hypermultiplet in adjoint representation. The theory is manifestly $\mathcal{N} = (1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N} = (0,1)$ supersymmetry.

- Action

$$S[V^{++}, q^+] = S_{SYM}[V^{++}] + S_{HYPER}[q^+, V^{++}]$$

- The action is manifestly $\mathcal{N} = (1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N} = (0,1)$ supersymmetry

$$\delta V^{++} = \epsilon^+ q^+, \qquad \delta q^+ = -(D^+)^4(\epsilon^- V^{--})$$

where the transformation parameter $\epsilon^{\pm}_A = \epsilon_{aA}\theta^{\pm A}$.

General strategy

- We start with harmonic superfield formulations of vector multiplet coupled to hypermultiplet.
- Effective action is formulated in the framework of the harmonic superfield background field method. It provides manifest $\mathcal{N} = (1, 0)$ supersymmetry and gauge invariance of effective action under the classical gauge transformations.
- Effective action can be calculated on the base of superfield proper-time technique. It provides preservation of manifest $\mathcal{N} = (1, 0)$ supersymmetry and manifest gauge invariance at all steps of calculations.
- The effective action can also be calculated perturbatively on the base of Feynman diagrams in superspace (supergraph technique).
- One-loop analysis. We study the model, where the $\mathcal{N} = (1, 0)$ vector multiplet interacts with hypermultiplet in the arbitrary representation of the gauge group. Then, we assume in the final result for one-loop divergences, that this representation is adjoint what corresponds to $\mathcal{N} = (1, 1)$ SYM theory. Finite one-loop effective action without renormalization.
- Two-loop analysis. All the possible divergences can be listed, using the the superfield power counting and then they can be calculated in the framework of the background field method.

- The superfields V^{++}, q^{+} are splitting into the sum of the background superfields V^{++}, Q^{+} and the quantum superfields v^{++}, q^{+}

$$V^{++} \to V^{++} + f v^{++}, \qquad q^{+} \to Q^{+} + q^{+}$$

- The action is expending in a power series in quantum fields. As a result, we obtain the initial action $S[V^{++}, q^{+}]$ as a functional $\tilde{S}[v^{++}, q^{+}; V^{++}, Q^{+}]$ of background superfields and quantum superfields.
- The gauge-fixing function are imposed only on quantum superfiled

$$\mathcal{F}^{(+4)}_{\tau} = D^{++} v^{++}_{\tau} = e^{-ib}(\nabla^{++} v^{++}) e^{ib} = e^{-ib} \mathcal{F}^{(+4)} e^{ib} \ ,$$

where $b(z)$ is a background-dependent gauge bridge superfield and τ means τ-frame. In the non-Abelian gauge theory, the gauge-fixing function is background-dependent.
- Faddev-Popov procedure is used. One obtains the effective action $\Gamma[V^{++}, Q^{+}]$ which is gauge invariant under the classical gauge transformations.

Background field method

- The effective action $\Gamma[V^{++},Q^+] = S[V^{++},Q^+] + \bar{\Gamma}[V^{++},Q^+]$ is written in terms of path integral

$$e^{i\bar{\Gamma}[V^{++},Q^+]} = Det^{1/2}\,\widehat{\Box} \int \mathcal{D}v^{++}\,\mathcal{D}q^+\,\mathcal{D}\mathbf{b}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\varphi\; e^{iS_{quant}[v^{++},q^+,\mathbf{b},\mathbf{c},\varphi,V^{++},Q^+]}$$

- The quantum action S_{quant} has the structure

$$S_{quant} = S[V^{++}+fv^{++},Q^+ +q^+] - S[V^{++},Q^+] - S'[V^{++},Q^+](fv^{++},q^+) +$$

$$+S_{GF}[v^{++},V^{++}] + S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}] + S_{NK}[\varphi,V^{++}].$$

- Gauge fixing term $S_{GF}[v^{++},V^{++}]$, Faddeed-Popov ghost action $S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}]$, Nielsen-Kalosh ghost action $S_{NK}[\varphi,V^{++}]$
- Operator $\widehat{\Box}$

$$\widehat{\Box} = \eta^{MN}\nabla_M\nabla_N + W^{+a}\nabla^-_a + F^{++}\nabla^{--} - \frac{1}{2}(\nabla^{--}F^{++})$$

- All ghosts are the analytic superfields

One-loop approximation. Only quadratic in quantum fields and ghosts terms are taken into account in the path integral for effective action. It gives after some transformation the one-loop contribution $\Gamma^{(1)}[V^{++},Q^{+}]$ to effective action in terms of formal functional determinants in analytic subspace of harmonic superspace

$$\Gamma^{(1)}[V^{++},Q] = \frac{i}{2} Tr_{(2,2)} \ln[\delta^{(2,2)}\, \widehat{\Box}^{AB} - 2f^2 Q^{+m}(T^A G_{(1,1)} T^B)_m{}^n Q^+_n] -$$

$$-\frac{i}{2} Tr_{(4,0)} \ln \widehat{\Box} - iTr \ln(\nabla^{++})^2_{\rm Adj} + \frac{i}{2} Tr \ln(\nabla^{++})^2_{\rm Adj} + iTr \ln \nabla^{++}_{\rm R}$$

As usual, $TrlnO = lnDetO$, Tr means the functional trace in analytic subspace and matrix trace.
$(T^A)_m{}^n$ are are generators of the representation for the hypermultiplet.
The $G_{(1,1)}$ is the Green function for the operator ∇^{++}.
Index A numerates the generators, $V^{++} = V^{++A}T^A$. Operator $\widehat{\Box}$ acts on the components V^{++A} as $(\widehat{\Box} V^{++})^A = \widehat{\Box}^{AB} V^{++B}$
$\rm Adj$ and $\rm R$ mean that the corresponding operators are taken in the adjoint representation and in the representation for hypermultiplet.

Calculating the one-loop divergences of superfield functional determinants is carried out in the framework of proper-time technique (superfield version of Schwinger-De Witt technique). Such technique allows us to preserve the manifest gauge invariance and manifest $\mathcal{N}=(1,0)$ supersymmetry at all steps of calculations.

General scheme of calculations

- Proper-time representation

$$Tr ln O \sim Tr \int_0^\infty \frac{d(is)}{(is)^{1+\varepsilon}} e^{isO_1} \delta(1,2)|_{2=1}$$

- Here s is the proper-time parameter and ε is a parameter of dimensional regularization.
- Typically the $\delta(1,2)$ contains $\delta^8(\theta_1-\theta_2)$, which vanishes at $\theta_1=\theta_2$
- Typically the operator O contains some number of spinor derivatives D^+_a, D^-_a which act on the Grassmann delta-functions $\delta^8(\theta_1-\theta_2)$ and can kill them. Non-zero result will be only if all these δ-functions are killed.
- Only these terms are taking into account which have the pole $\frac{1}{\varepsilon}$ after integration over proper-time.

One-loop divergences

Results of calculations

$$\Gamma^{(1)}_{div}[V^{++},Q^{+}] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} tr\int d\zeta^{(-4)} du (F^{++})^2 -$$

$$-\frac{2if^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)} du \widetilde{Q}^{+m}(C_2\delta_m{}^n - C(R)_m{}^n)F^{++}Q^{+}{}_n.$$

- The quantities $C_2, T(R), C(R)$ are defined as follows

$$tr(T^A T^B) = T(R)\delta^{AB}$$

$$tr(T^A_{Adj}T^B_{Adj}) = f^{ACD}f^{BCD} = C_2\delta^{AB}$$

$$(T^A T^A)_m{}^n = C(R)_m{}^n.$$

- In $\mathcal{N}=(1,1)$ SYM theory, the hypermultiplet is in the same representation as the vector multiplet. Then $C_2 = T(R) = C(R)$. Then $\Gamma^{(1)}_{div}[V^{++},Q^{+}] = 0!$

Procedure of calculations: gauge multiplet sector

- Two-loop divergences are calculated within background field method and proper-time technique like in one-loop case.
- We begin with only gauge multiplet background.
- Power counting shows that the only possible two-loop divergent contribution in the gauge superfield sector has the following structure

$$\Gamma^{(2)}_{\rm div}[V^{++}] = a \int d\zeta^{(-4)}\, du\, {\rm tr}\left(F^{++}\, \widehat{\Box}\, F^{++}\right)$$

  with some constant a, which diverges after removing a regularization.
- Within background field method, the two-loop contributions to superfield effective action are given by two-loop vacuum harmonic supergraphs with background field dependent lines.
- The background field dependent propagators (lines) are represented by proper-time integrals.
- Constant a in principle should have the following structure $a = \frac{d_1}{\varepsilon} + \frac{d_2}{\varepsilon^2}$ with arbitrary real parameters $d_1\, d_2$.

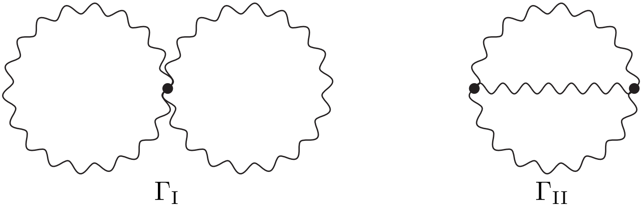

Figure: Two-loop Feynman supergraphs with gauge self-interactions vertices.

Figure: Two-loop Feynman supergraphs with hypermultiplet and ghosts vertices.

Two-loop divergences

Procedure of calculations

- One can prove that the in the case under consideration the only two-loop divergent contribution comes from the '∞' supergraph.
- Contribution of this supergraph contains the product of two Green functions $G^{(2,2)}(z_1, u_1; z_2, u_2)$ at $z_1 = z_2$.
- Divergent part of such Green function can be calculated and has the form $\sim \frac{1}{\varepsilon} F^{++}$. Therefore $G^{(2,2)}(z_1, u_1; z_2, u_2)|_{z_1=z_2} \sim \frac{1}{\varepsilon} F^{++} + g^{++}$ where g^{++} is some finite functional.
- It means that full two loop contribution of the '∞' supergraph looks like

$$b \int d\zeta^{(-4)}\, du \, (\frac{1}{\varepsilon} F^{++} + g^{++}) \, \widehat{\Box} \, (\frac{1}{\varepsilon} F^{++} + g^{++}).$$

  with some constant b. Therefore there are two types of contributions, one containing $\frac{1}{\varepsilon}$ and another one containing $\frac{1}{\varepsilon^2}$.
- The terms with simple pole $\frac{1}{\varepsilon}$ has the form $\sim \frac{1}{\varepsilon} F^{++} \, \widehat{\Box} \, g^{++}$.
- However, the power counting tells us that the two loop divergence has the form $\sim F^{++} \, \widehat{\Box} \, F^{++}$. Therefore, we must assume that $g^{++} = 0$ or $g^{++} \sim F^{++}$.

Results of calculations in gauge multiplet sector

- Further we consider only the case $g^{++} = 0$.
- In this case, the divergent part of two-loop effective action has the form

$$\Gamma^{(2)}_{div} = \frac{8f^2}{(4\pi)^6 \varepsilon^2} (C_2)^2 \mathrm{tr} \int d\zeta^{(-4)}\, du\, F^{++}\, \widehat{\Box}\, F^{++},$$

where $F^{++} = 0$ is the classical equation of motion in the case when the hypermultiplet is absent.

- Coefficient c_2 looks like

$$c_2 = \frac{8f^2}{(4\pi)^6 \varepsilon^2} (C_2)^2.$$

Field redefinition and cancelation of divergences

- Consider the off-shell transformation of the superfield V^{++} in the classical action $V^{++} \to V^{++} - a\,\widehat{\Box}\,F^{++}$.
- The corresponding transformation of the classical action is $\delta S = -a \int d\zeta^{(-4)}\, du \,\mathrm{tr}\, F^{++}\,\widehat{\Box}\,F^{++}$. That allows to cancel completely off-shell the two-loop divergence of the effective action in the gauge multiplet sector.
- Thus, one can state that the theory under consideration is off-shell finite at one- and two-loops (at least in gauge multiplet sector).

Hypermultiplet dependence of the two-loop divergences: indirect analysis.

- The hypermultiplet-dependent contribution to two-loop divergences can be obtained by the straightforward quantum computations of the two-loop effective action taking into account the hypermultiplet background.
- The general form of hypermultiplet dependent divergences can in principle be found without direct calculations, assuming the invariance of the effective action under the hidden $\mathcal{N}=(0,1)$ supersymmetry.
- The result has an extremely simple form

$$\Gamma^{(2)}_{\rm div}[V^{++},q^{+}] = a\int d\zeta^{(-4)}\,du\,{\rm tr}\,E^{++}\,\widehat{\Box}\,E^{++},$$

  where $E^{++}=F^{++}+\frac{i}{2}[q^{+A},q^{+}_{A}]$ is the left hand side of classical equation of motion for vector multiplet superfield coupled to hypermultiplet.
- Two-loop divergences vanish on-shell as expected.

Hypermultiplet dependence of the two-loop divergences: direct calculations:

$$\Gamma^{(2)}_{\rm div}[V^{++},q^{+}] = \frac{f^2(C_2)^2}{8(2\pi)^6\epsilon^2}\int d\zeta^{(-4)}\,du\,\mathrm{tr}\,E^{++}\,\widehat{\Box}\,E^{++}$$

$+$terms proportional to e.o.m for hypermultiplet.

Summary

- The six-dimensional $\mathcal{N} = (1, 0)$ supersymmetric theory of the non-Abelian vector multiplet coupled to hypermultiplet in the $6D$, $\mathcal{N} = (1, 0)$ harmonic superspace was considered.
- Background field method in harmonic superspace was constructed .
- Manifestly supersymmetric and gauge invariant effective action, depending both on vector multiplet and hypermultiplet superfields, was formulated.
- Superficial degree of divergence is evaluated and structure of one- and two-loop counterterms was studied.
- An efficient manifestly gauge invariant and $\mathcal{N} = (1, 0)$ supersymmetric technique to calculate the one- and two-loop contributions to effective action was developed. As an application of this technique, we found the one- and two-loop divergences of the theory under consideration.
- The same one-loop divergences have been calculated independently with help of $\mathcal{N} = (1, 0)$ supergraphs.
- It is proved that $\mathcal{N} = (1, 1)$ SYM theory is one-loop off-shell finite. There is no need to use the equations of motion to prove this property.

Summary

- Two-loop divergences of the $6D, \mathcal{N} = (1,1)$ SYM theory were calculated in gauge multiplet sector. The hypermultiplet dependence of two-loop divergences was restore on the base of hidden $\mathcal{N} = (0,1)$ supersymmetry.
- The hypermultiplet dependence of two-loop divergences was explicitly calculated.

Some open problems

- Calculations of the two-loop $\frac{1}{\epsilon}$ divergences.
- Study of the three-loop divergences.

THANK YOU VERY MUCH!