Multiloop calculations: modern methods and applications.

Roman Lee

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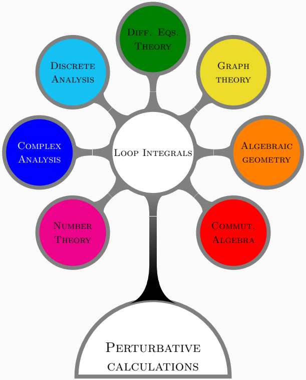
Budker Institute of Nuclear Physics, Novosibirsk

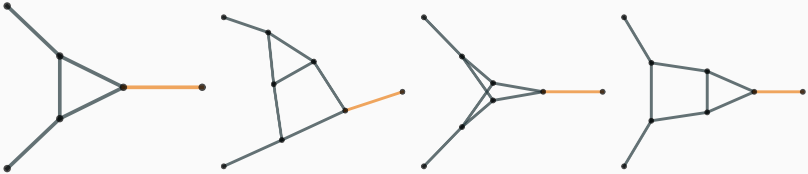
Feynman (multiloop) integrals: Motivation

- High-precision theoretical description of Standard Model processes is of crucial importance. In particular, the New Physics — new particles and interactions — is likely to appear as small deviations from SM and therefore can be detected only with high precision of theoretical predictions at hand.
- From the computational point of view, our ability to obtain high-precision results depends crucially on multiloop calculation techniques. Complexity grows both qualitatively and quantitatively in an explosive way with the number of loops and/or scales.
- Besides these practical purposes, multiloop calculations provide a perfect polygon for trying the methods from various mathematical fields: differential equations, complex analysis, number theory, algebraic geometry etc.



Example: form factors

2 loops:



- Dispersion relation
- Feynman parametrization

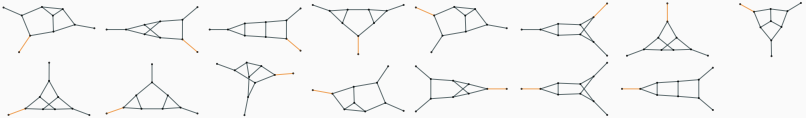
} [Matsuura, van der Marck, and van Neerven, 1989; Harlander, 2000]

- Mellin-Barnes parametrization
- ${}_pF_q$ expansion in indices, `HypExp`

} [Gehrmann, Huber, and Maitre, 2005]

3 loops:

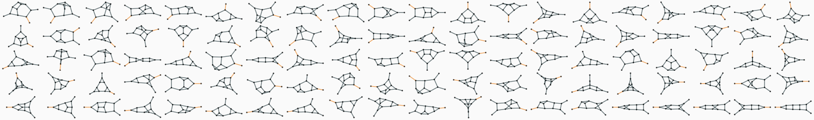
[Gehrmann, Heinrich, Huber, and Studerus, 2006; Heinrich, Huber, and Maître, 2008; RL, Smirnov, and Smirnov, 2010]



- Feynman parametrization
- Mellin-Barnes parametrization, `MB`, `AMBRE` [Czakon, 2006; Gluza et al., 2007]
- Recurrence+analyticity in d, [Tarasov, 1996; RL, 2010]
- PSLQ recognition [Ferguson et al., 1998]

4 loops:

[Henn, Smirnov, Smirnov, and Steinhauser, 2016; RL, Smirnov, Smirnov, and Steinhauser, 2019; RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2021b]



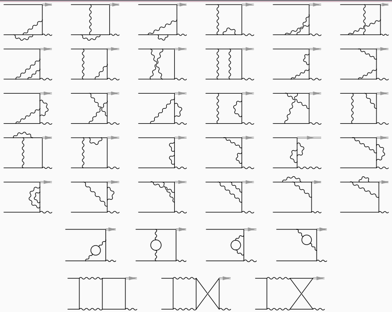
- $\sim$ 100 big topologies.
- Linear reducibility, `HyperInt` [Panzer, 2013]
- Parallelization for IBP reduction,
  finite fields reconstruction [von Manteuffel and Schabinger, 2015; Smirnov and Chuharev, 2020]
- Differential equations, reduction to ϵ-form [Henn, 2013; RL, 2015], `Libra` [RL, 2021]
- PSLQ recognition

NNLO cross sections

- Massless form factors represent a traditional topic of the multiloop calculations where the “world records” are fixed. But from the experimental point of view less loops and more scales are more important.
- In particular, only very recently multiloop methods have grown to **NNLO differential cross section** calculations of $2 \to 2$ processes with massive particles. NNLO corrections to differential cross sections are not even known for basic QED process: $e^+e^- \to \gamma\gamma$, $e^+e^- \to \mu^+\mu^-$, etc. Some results start to appear [Duhr, Smirnov, and Tancredi, 2021; Banerjee et al., 2020; Fadin and RL, 2023].
- The complexity of NNLO calculations with massive internal lines is partly connected with appearance of **non-polylogarithmic integrals**. Effective approach to the calculation of such integrals is, probably, the most hot topic in multiloop calculations.

Example: process $e^+e^- \to \gamma\gamma^*$ at two loops.

- Diagrams: 37 distinct diagrams.
- IBP reduction
  - $m = 0$: reduction is easy, leads to 60 masters.
  - $m \neq 0$: reduction is hardly doable, leads to $\gtrsim 400$ masters.
- DE reduction
  - $m = 0$: easily reducible to ϵ-form. All masters are polylogarithmic.
  - $m \neq 0$: irreducible to ϵ-form, many masters are non-polylogarithmic.



It is easy to calculate the amplitude at $m = 0$, but almost impossible for $m \neq 0$.

How to account for small $m \neq 0$?

1. Physical approach: use factorization formula [Fadin and RL, 2023] relating massive to massless amplitude (Andrey Arbuzov's talk) .
2. Perform IBP reduction exactly in m, solve DE via Frobenius expansion.
3. Use expansion by regions technique and IBP reduction in parametric representation.

Calculation path

1. Diagram generation

✓Generate diagrams contributing to the chosen order of perturbation theory.

Tools: `qgraf` [Nogueira, 1993], `FeynArts` [Hahn, 2001], `tapir` [Gerlach et al., 2022],. . .

2. IBP reduction

Setup IBP reduction, derive differential system for master integrals.

Tools: `FIRE6` [Smirnov and Chuharev, 2020], `Kira2` [Klappert et al., 2021], `LiteRed` [RL, 2012], . . .

3. DE Solution

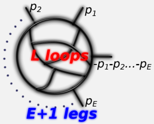
Reduce the system to ϵ-form, write down solution in terms of polylogarithms. Fix boundary conditions by auxiliary methods.

Tools: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], `Libra` [RL, 2021]

Given a Feynman diagram, consider a family

$$j(\boldsymbol{n}) = \int d\mu_L \prod_{k=1}^{N} D_k^{-n_k}, \quad d\mu_L = \prod_{k=i}^{L} d^d l_i$$

$D_1, \ldots, D_M$ — denominators of the diagram,
$D_{M+1}, \ldots, D_N$ — irreducible numerators, such that $N = L(L+1)/2 + L \cdot E$.



From $0 = \int d\mu_L \frac{\partial}{\partial l_i} \cdot q_m \prod_{k=1}^{N} D_k^{-n_k}$ one obtains

IBP identities

$$[c_{kl} B_k A_l + c_l A_l]\, j(\boldsymbol{n}) = 0.$$

Here c_{kl}, c_l are some coefficients.

$$A_l j(n_l) = n_l j(n_l + 1),$$
$$B_l j(n_l) = j(n_l - 1)$$

IBP identities allow one to express any integral in the family via a finite number of master integrals. They also allow to construct differential and difference equations for the latter.

Differential equations and dimensional recurrences

As a result of IBP reduction we express amplitudes via a finite set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. What is more important, we obtain equations for them:

Differential equations

[Kotikov, 1991; Remiddi, 1997]

$$\partial_x \boldsymbol{j} = M(x, d)\boldsymbol{j}$$

Dimensional recurrences

[Tarasov, 1996; Derkachov et al., 1990]

$$\boldsymbol{j}(d-2) = R(x, d)\boldsymbol{j}(d)$$

Dimensional recurrence relations are especially useful for one-scale integrals, when the differential equations can not help. Using the analytical properties wrt d, [RL, 2010] to fix the arbitrary periodic functions, one can obtain the solution in the form of convergent sums. High-precision evaluation of these sums can be done with `SummerTime` package. Using PSLQ algorithm, one can turn the obtained numerical results into analytical expressions. This is the approach which was successfully applied to the calculation of the 3-loop form factors.

Recent application

Recently the four-loop HQET propagators have been calculated with this method, [RL and Pikelner, 2023]. The required analytical properties (with large abundance) have been fixed via the reduction of sets of integrals finite in specific point of the basic stripe.

Differential equations for master integrals

- Differential equations for master integrals have the form

$$\partial_x \boldsymbol{j} = M(x,\epsilon)\boldsymbol{j}$$

- One can try to simplify the equation by transformation $\boldsymbol{j} = T\tilde{\boldsymbol{j}}$, so that

$$\partial_x \tilde{\boldsymbol{j}} = \widetilde{M}\tilde{\boldsymbol{j}}, \qquad \widetilde{M} = T^{-1}\left[MT - \partial_x T\right]$$

- [Henn, 2013]: there is often a "canonical" basis $\boldsymbol{J} = T^{-1}\boldsymbol{j}$ such that

$$\partial_x \boldsymbol{J} = \epsilon S(x)\boldsymbol{J} \qquad (\epsilon\text{-form})$$

- General solution is easily expanded in ϵ:

$$U(x,x_0) = \mathrm{Pexp}\left[\epsilon\int_{x_0}^{x} dx S(x)\right] = \sum_n \epsilon^n \iiint\limits_{x>x_n>\ldots>x_0} dx_n \ldots dx_1 S(x_n)\ldots S(x_1)$$

- We usually want to send the lower limit x_0 to a singular point (say, to 0), so we have to consider the regularized operator $U(x,\underline{0}) = \lim_{x_0\to 0} U(x,x_0)x_0^{\epsilon S_0}$.
- Algorithm of finding transformation to ϵ-form: [RL, 2015]. Implemented in 3 publicly available codes: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], and recently in `Libra` [RL, 2021].

Some “new” ideas

Frobenius method

Path-ordered exponent

$$U(x,\underline{0}) = \text{Pexp}\left[\int_{x_0}^{x} M(x)dx\right] x_0^{M_0}, \quad M_0 = \text{res}_{x=0} M(x)$$

can also be expanded in generalized power series when x is small enough.

$$U(x,\underline{0}) = \sum_{\lambda\in S} x^{\lambda} \sum_{n=0}^{\infty}\sum_{k=0}^{K_\lambda} \frac{1}{k!} C\left(n+\lambda, k\right) x^n \ln^k x\,.$$

Note that for expansion around singular point (which we usually want) non-integer powers x^λ and $\log x$ might appear.

The convergence radius is the distance to the nearest singularity. However, it is easy to perform analytical continuation to the whole complex plane by matching expansions at different points. Let $x=1$ is also the singular point, then the continuation of $U(x,\underline{0})$ beyond $x=1$ is simply

$$U(x>1,\underline{0}) = U(x,\underline{1})U^{-1}(1/2,\underline{1})U(1/2,\underline{0})$$

Frobenius expansion provides a systematic way to obtain numerical results for any family of multiloop integrals, including non-polylogarithmic ones.

ϵ-regular basis

The coefficients of Frobenius expansion are functions of ϵ, therefore the ϵ-expansion leads to double series. The typical problems of this is the loss of numerical precision and also of orders of ϵ-expansion. Despite of these complications, this approach allowed the authors of Ref. [Fael et al., 2022] to calculate numerically the massive quark and lepton form factors.

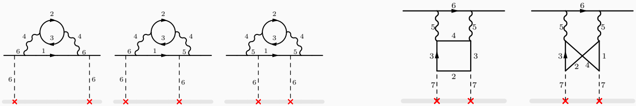
A much better approach is to pass to ϵ-regular basis introduced in Ref. [RL and Onishchenko, 2019].

Properties of ϵ-regular basis

- Expansion of each "master-integral" starts from ϵ^0.
- Leading terms constitute a basis, i.e., there is no vanishing linear combination of these terms with nonzero rational (in kinematic invariants) coefficients.

Once the ϵ-regular basis is found one can simply put $\epsilon = 0$.

- Total Born cross section of $Z + e^- \to Z + e^- + e^- + e^+$, [RL et al., 2024].
- Two-loop contributions to LS and HFS in hydrogen [Krachkov and RL, 2023].



IBP reduction in parametric representation

Motivation: Note that $N = L(L+1)/2 + L \cdot E$ grows quadratically with L, while M, the # of lines in the diagram, grows only linearly. Parametric representation: only M indices. Besides, IBP reduction in parametric representation extends the applicability of the method (non-standard propagators, reduction after asymptotic expansion).

Parametric representation [RL and Pomeransky, 2013]

$$\tilde{j}^{(d)}(n_1, \dots n_M) = \int \frac{\prod_{k=1}^{M} dx_k x_k^{n_k - 1}}{G(\boldsymbol{x})^{d/2}}$$

$G = U + F$, where U and F are Feynman graph polynomials.

IBP identities relating integrals with the same d require constructing *syzygy module* for ideal generated by $\langle G, \partial_1 G, \dots, \partial_M G \rangle$.

IBP identities from syzygies [RL, 2014]. Baikov rep.: [Zhang, 2014]

Syzygy $QG + Q_1\partial_1 G + \dots + Q_M \partial_M G = 0$ leads to IBP identity

$$\int \prod_{k=1}^{M} dx_k \frac{\partial}{\partial x_n} \left(Q_n \frac{\prod_{k=1}^{M} x_k^{n_k - 1}}{G(\boldsymbol{x})^{d/2}} \right) = [\tfrac{d}{2} Q(A) + Q_k(A) B_k] \tilde{j}(\boldsymbol{n}) = 0$$

Simplifications with symbol map

There is a standard approach to the simplification of the polylogarithmic expressions using symbol map. One might think of symbols as a cleaner way to represent iterated (or path-ordered) integrals with logarithmic weights (with some reservations, though):

$$I = \int\!\!\cdots\!\!\int\limits_{1>\tau_n>\ldots>\tau_1>0} d\ln p_n(\tau_n)\ldots d\ln p_1(\tau_1) \xrightarrow{\quad\mathcal{S}\quad} p_n\otimes\ldots\otimes p_1$$

Formal symbol manipulation rules then easily follow, e.g.

$$d\ln(pq) = d\ln p + d\ln q \qquad \Longrightarrow \qquad (\ldots\otimes pq\otimes\ldots) = (\ldots\otimes p\otimes\ldots) + (\ldots\otimes q\otimes\ldots)$$

Similarly, by ordering the integration variables in the product of integrals, we get $\mathcal{S}(I_1 I_2) = \mathcal{S}(I_1) ⧢ \mathcal{S}(I_2)$, where ⧢ denotes a shuffle product, e.g.

$$(a\otimes b)⧢(c\otimes d) = a\otimes b\otimes c\otimes d + a\otimes c\otimes b\otimes d + a\otimes c\otimes d\otimes b + c\otimes a\otimes b\otimes d + c\otimes a\otimes d\otimes b + c\otimes d\otimes a\otimes b$$

We have, in particular, symbols for classical polylogarithms

$$\mathcal{S}(\mathrm{Li_n}(x)) = \underbrace{x\otimes\ldots\otimes x}_{n-1}\otimes(x-1)$$

Simplifications with symbol map

Symbols are good for checking the identities, e.g., using $\mathcal{S}$ it is easy to establish[1]

$$7\mathrm{Li}_2\left(\frac{1+\varepsilon/z}{1-i\varepsilon}\right)-7\mathrm{Li}_2\left(\frac{1+\bar{\varepsilon}/z}{1+i\bar{\varepsilon}}\right)+7\mathrm{Li}_2\left(\frac{z+\bar{\varepsilon}}{\bar{\varepsilon}-i}\right)-7\mathrm{Li}_2\left(\frac{z+\varepsilon}{\varepsilon+i}\right)+11\mathrm{Li}_2\left(\frac{z+\varepsilon}{\varepsilon-i}\right)-11\mathrm{Li}_2\left(\frac{z+\bar{\varepsilon}}{\bar{\varepsilon}+i}\right)$$
$$+4\mathrm{Li}_2(1+z\varepsilon)-4\mathrm{Li}_2(1+z\bar{\varepsilon})+18\mathrm{Li}_2(-iz)-18\mathrm{Li}_2(iz)+11\mathrm{Li}_2\left(\frac{1+\bar{\varepsilon}/z}{1-i\bar{\varepsilon}}\right)-11\mathrm{Li}_2\left(\frac{1+\varepsilon/z}{1+i\varepsilon}\right)$$
$$=\frac{2i\pi^2}{5\sqrt{3}}-\frac{23}{3}i\pi\ln z+6i\pi\ln\left(2-\sqrt{3}\right)-\frac{i\psi'\left(\frac{1}{6}\right)}{5\sqrt{3}}-24iG,\qquad \text{where } \varepsilon=1/\bar{\varepsilon}=e^{2\pi i/3}.$$

However, strictly speaking, they are much less powerful in simplifying expressions. E.g., if we omit in the left-hand side a couple of dilogs with not so simple arguments, we could have failed to recognize in the symbol of the resulting expression that of the sum of the omitted dilogs.

Simplification algorithm idea

For a given expression:

1. find all possible arguments of Li_n which might enter the simplified form.
2. find equivalent form with the minimal number of polylogs.

[1] NB: This identity was used in real life (as well as some yet more complicated identities) for the simplification of the total cross section of Compton scattering @NLO [RL, Schwartz, and Zhang, 2021a].

Summary

- Each step towards increasing the # of loops and/or # of scales requires new methods. Those involve both technological advances (e.g. massive parallelization) and new algorithms coming various fields of mathematics.
- IBP reduction still remains a bottleneck for some calculations. IBP reduction in parametric representation can extend the applicability of method.
- Differential equations method is already in a very good shape. However, there is still no regular approach to the computation of non-polylogarithmic integrals.
- From the practical point of view, there is always a method based on Frobenius expansion and ϵ-regular basis which might be used to obtain numerical high-precision results.
- Simplification of the results is notoriously difficult. The use of symbol map may help, but need to be augmented by additional algorithms.

Thank you!

References

Pulak Banerjee et al. Theory for muon-electron scattering @ 10 ppm: A report of the MUonE theory initiative. *Eur. Phys. J. C*, 80(6):591, 2020. doi: 10.1140/epjc/s10052-020-8138-9.

M. A. Bezuglov and A. I. Onishchenko. Non-planar elliptic vertex. *JHEP*, 04:045, 2022. doi: 10.1007/JHEP04(2022)045.

M. A. Bezuglov, A. V. Kotikov, and A. I. Onishchenko. On series and integral representations of some NRQCD master integrals. 5 2022.

Johannes Broedel, Claude Duhr, Falko Dulat, Brenda Penante, and Lorenzo Tancredi. Elliptic polylogarithms and Feynman parameter integrals. *JHEP*, 05:120, 2019. doi: 10.1007/JHEP05(2019)120.

Johannes Broedel, Claude Duhr, and Nils Matthes. Meromorphic modular forms and the three-loop equal-mass banana integral. *JHEP*, 02:184, 2022. doi: 10.1007/JHEP02(2022)184.

Robin Brüser, Christoph Dlapa, Johannes M. Henn, and Kai Yan. The full angle-dependence of the four-loop cusp anomalous dimension in QED. 2020.

K. G. Chetyrkin and F. V. Tkachov. Integration by parts: The algorithm to calculate β-functions in 4 loops. *Nucl. Phys. B*, 192:159, 1981.

M. Czakon. Automatized analytic continuation of Mellin-Barnes integrals. *Comput. Phys. Commun.*, 175:559–571, 2006. doi: 10.1016/j.cpc.2006.07.002.

S. Derkachov, J. Honkonen, and Y. Pis' mak. Three-loop calculation of the random walk problem: an application of dimensional transformation and the uniqueness method. *Journal of Physics A: Mathematical and General*, 23(23):5563, 1990.

Claude Duhr, Vladimir A. Smirnov, and Lorenzo Tancredi. Analytic results for two-loop planar master integrals for Bhabha scattering. *JHEP*, 09:120, 2021. doi: 10.1007/JHEP09(2021)120.

V. S. Fadin and RL. Two-loop radiative corrections to $e^+e^- \to \gamma\gamma^*$ cross section. *JHEP*, 11:148, 2023. doi: 10.1007/JHEP11(2023)148.

Matteo Fael, Fabian Lange, Kay Schönwald, and Matthias Steinhauser. Massive Vector Form Factors to Three Loops. *Phys. Rev. Lett.*, 128(17):172003, 2022. doi: 10.1103/PhysRevLett.128.172003.

Helaman RP Ferguson, Daivd H Bailey, and Paul Kutler. A polynomial time, numerically stable integer relation algorithm. Technical report, 1998.

T. Gehrmann, T. Huber, and D. Maitre. Two-loop quark and gluon form-factors in dimensional regularisation. *Phys. Lett. B*, 622: 295–302, 2005. doi: 10.1016/j.physletb.2005.07.019.

T. Gehrmann, G. Heinrich, T. Huber, and C. Studerus. Master integrals for massless three-loop form-factors: One-loop and two-loop insertions. *Phys. Lett. B*, 640:252–259, 2006. doi: 10.1016/j.physletb.2006.08.008.

Marvin Gerlach, Florian Herren, and Martin Lang. `tapir` a tool for topologies, amplitudes, partial fraction decomposition and input for reductions. 1 2022.

Oleksandr Gituliar and Vitaly Magerya. Fuchsia: a tool for reducing differential equations for Feynman master integrals to epsilon form. *Comput. Phys. Commun.*, 219, 2017.

J. Gluza, K. Kajda, and T. Riemann. AMBRE: A Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals. *Comput. Phys. Commun.*, 177:879–893, 2007. doi: 10.1016/j.cpc.2007.07.001.

Thomas Hahn. Generating Feynman diagrams and amplitudes with FeynArts 3. *Comput. Phys. Commun.*, 140:418–431, 2001. doi: 10.1016/S0010-4655(01)00290-9.

Robert V. Harlander. Virtual corrections to g g —> H to two loops in the heavy top limit. *Phys. Lett. B*, 492:74–80, 2000. doi: 10.1016/S0370-2693(00)01042-X.

G. Heinrich, T. Huber, and D. Maître. Master integrals for fermionic contributions to massless three-loop form factors. *Physics Letters B*, 662(4):344–352, 2008. ISSN 0370-2693. doi: https://doi.org/10.1016/j.physletb.2008.03.028. URL https://www.sciencedirect.com/science/article/pii/S0370269308003341.

Johannes M. Henn. Multiloop integrals in dimensional regularization made simple. *Phys.Rev.Lett.*, 110(25):251601, 2013. doi: 10.1103/PhysRevLett.110.251601.

Johannes M. Henn, Alexander V. Smirnov, Vladimir A. Smirnov, and Matthias Steinhauser. A planar four-loop form factor and cusp anomalous dimension in QCD. *JHEP*, 05:066, 2016. doi: 10.1007/JHEP05(2016)066.

Jonas Klappert, Fabian Lange, Philipp Maierhöfer, and Johann Usovitsch. Integral reduction with Kira 2.0 and finite field methods. *Comput. Phys. Commun.*, 266:108024, 2021. doi: 10.1016/j.cpc.2021.108024.

A. V. Kotikov. Differential equations method: New technique for massive Feynman diagrams calculation. *Phys. Lett.*, B254:158–164, 1991. doi: 10.1016/0370-2693(91)90413-K.

Petr A. Krachkov and RL. Two-loop corrections to lamb shift and hyperfine splitting in hydrogen via multi-loop methods. *JHEP*, 07:211, 2023. doi: 10.1007/JHEP07(2023)211.

T. Matsuura, S.C. van der Marck, and W.L. van Neerven. The calculation of the second order soft and virtual contributions to the drell-yan cross section. *Nuclear Physics B*, 319(3):570–622, 1989. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(89)90620-2. URL https://www.sciencedirect.com/science/article/pii/0550321389906202.

Bernhard Mistlberger. Higgs boson production at hadron colliders at N^3LO in QCD. *JHEP*, 05:028, 2018. doi: 10.1007/JHEP05(2018)028.

Paulo Nogueira. Automatic Feynman graph generation. *J. Comput. Phys.*, 105:279–289, 1993. doi: 10.1006/jcph.1993.1074.

Erik Panzer. On the analytic computation of massless propagators in dimensional regularization. *Nuclear Physics, Section B 874 (2013), pp. 567-593*, May 2013. doi: 10.1016/j.nuclphysb.2013.05.025.

Mario Prausa. epsilon: A tool to find a canonical basis of master integrals. *Comput. Phys. Commun.*, 219:361–376, 2017. doi: 10.1016/j.cpc.2017.05.026.

Ettore Remiddi. Differential equations for Feynman graph amplitudes. *Nuovo Cim.*, A110:1435–1452, 1997.

RL. Space-time dimensionality d as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to d. *Nucl. Phys. B*, 830:474, 2010. ISSN 0550-3213. doi: DOI:10.1016/j.nuclphysb.2009.12.025. URL http://www.sciencedirect.com/science/article/B6TVC-4Y34PW6-2/2/bd2b4965b69dc349aa8f5f9040fc5d30.

RL. Presenting litered: a tool for the loop integrals reduction, 2012.

RL. Modern techniques of multiloop calculations. In Etienne Augé and Jacques Dumarchez, editors, *Proceedings, 49th Rencontres de Moriond on QCD and High Energy Interactions*, pages 297–300, Paris, France, 2014. Moriond, Moriond.

RL. Reducing differential equations for multiloop master integrals. *J. High Energy Phys.*, 1504:108, 2015. doi: 10.1007/JHEP04(2015)108.

RL. Libra: A package for transformation of differential systems for multiloop integrals. *Computer Physics Communications*, 267:108058, 2021. ISSN 0010-4655. doi: https://doi.org/10.1016/j.cpc.2021.108058. URL https://www.sciencedirect.com/science/article/pii/S0010465521001703.

RL and Kirill T. Mingulov. Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method. *Comput. Phys. Commun.*, 203:255–267, jun 2016. doi: 10.1016/j.cpc.2016.02.018. URL http://dx.doi.org/10.1016/j.cpc.2016.02.018.

RL and A. I. Onishchenko. ϵ-regular basis for non-polylogarithmic multiloop integrals and total cross section of the process $e^+e^- \to 2(Q\bar{Q})$. *JHEP*, 12:084, 2019. doi: 10.1007/JHEP12(2019)084.

RL and Andrey F. Pikelner. Four-loop hqet propagators from the dra method. *JHEP*, 02:097, 2023. doi: 10.1007/JHEP02(2023)097.

RL and Andrei A. Pomeransky. Critical points and number of master integrals. *J. High Energy Phys.*, 1311:165, 2013. doi: 10.1007/JHEP11(2013)165.

RL, V. Smirnov, and A. Smirnov. Analytic results for massless three-loop form factors. *Journal of High Energy Physics*, 2010(4):1–12, 2010.

RL, Alexander V. Smirnov, Vladimir A. Smirnov, and Matthias Steinhauser. Four-loop quark form factor with quartic fundamental colour factor. *JHEP*, 02:172, 2019. doi: 10.1007/JHEP02(2019)172.

RL, Schwartz, and Zhang. Compton Scattering Total Cross Section at Next-to-Leading Order. *Phys. Rev. Lett.*, 126(21):211801, 2021a. doi: 10.1103/PhysRevLett.126.211801.

RL, Andreas von Manteuffel, Robert M. Schabinger, Alexander V. Smirnov, Vladimir A. Smirnov, and Matthias Steinhauser. The Four-Loop $\mathcal{N} = 4$ SYM Sudakov Form Factor. 10 2021b.

RL, Alexey A. Lyubyakin, and Vladimir A. Smirnov. Total born cross section of e+e−-pair production by an electron in the coulomb field of a nucleus. *Phys. Lett. B*, 848:138408, 2024. doi: 10.1016/j.physletb.2023.138408.

A. V. Smirnov and F. S. Chuharev. FIRE6: Feynman Integral REduction with Modular Arithmetic. *Comput. Phys. Commun.*, 247:106877, 2020. doi: 10.1016/j.cpc.2019.106877.

O. V. Tarasov. Connection between feynman integrals having different values of the space-time dimension. *Phys. Rev. D*, 54:6479, 1996. doi: 10.1103/PhysRevD.54.6479.

Andreas von Manteuffel and Robert M. Schabinger. A novel approach to integration by parts reduction. *Phys. Lett. B*, 744:101–104, 2015. doi: 10.1016/j.physletb.2015.03.029.

Andreas von Manteuffel and Lorenzo Tancredi. A non-planar two-loop three-point function beyond multiple polylogarithms. *JHEP*, 06: 127, 2017. doi: 10.1007/JHEP06(2017)127.

Yang Zhang. Integration-by-parts identities from the viewpoint of differential geometry. In *19th Itzykson Meeting on Amplitudes 2014*, 8 2014.