# Правило сумм Бьёркена с аналитической связью при малых значениях $Q^2$

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in collaboration with

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# Analytic Perturbation Theory ("Minimal Approach")

- $Q^2$ analyticity
- preserve RG-invariance
- $\surd$ Landau singularity removed

## Power series is replaced by non-power functional series[1,2]

$$\sum_k c_k \alpha_s^k \to \sum_k c_k \mathcal{A}_k$$

$$\mathcal{A}_n^{(i)}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{\rho_n^{(i)}(\sigma)}{\sigma + Q^2} d\sigma, \quad \rho_n^{(i)}(\sigma) = \mathrm{Im}[\alpha_{(i)}^n(-\sigma)] \quad (1)$$

Later generalized to the non-integer powers of $\alpha_s$ - Fractional APT[3,4,5]

[1] I. L. Solovtsov and D. V. Shirkov. In: *Theor. Math. Phys.* 120 (1999). [Teor. Mat. Fiz.120,482(1999)], pp. 1220–1244. arXiv: hep-ph/9909305 [hep-ph].

[2] K. A. Milton, I. L. Solovtsov, and O. P. Solovtsova. In: *Phys. Lett. B* 415 (1997), pp. 104–110. arXiv: hep-ph/9706409.

[3] A. P. Bakulev, S. V. Mikhailov, and N. G. Stefanis. In: *Phys. Rev. D* 72 (2005), p. 074014. arXiv: hep-ph/0506311.

[4] A. P. Bakulev, S. V. Mikhailov, and N. G. Stefanis. In: *Phys. Rev. D* 75 (2007). [Erratum: Phys.Rev.D 77, 079901 (2008)], p. 056005. arXiv: hep-ph/0607040.

[5] A. P. Bakulev, S. V. Mikhailov, and N. G. Stefanis. In: *JHEP* 06 (2010), p. 085. arXiv: 1004.4125 [hep-ph].



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# FAPT $1/L$ expansion ($L = \ln(\frac{Q^2}{\Lambda^2})$)

$$\tilde{a}^{(k)}_{n+1}(Q^2) = \frac{(-1)^n}{n!}\frac{d^n a_s^{(k)}(Q^2)}{(dL)^n}, \quad a_s^{(k)}(Q^2) = \frac{\beta_0 \alpha_s^{(k)}(Q^2)}{4\pi} = \beta_0\, \bar{a}_s^{(k)}(Q^2), \tag{2}$$

The series of derivatives $\tilde{a}_n(Q^2)$ can successfully replace the corresponding series of $a_s$-powers (see, e.g. (Kotikov, Zemlyakov:2022)). Indeed, each derivative reduces the $a_s$ power but is accompanied by an additional $\beta$-function $\tilde{a}_s^2$. Thus, each application of a derivative yields an additional as, and thus it is indeed possible to use a series of derivatives instead of a series of $a_s$-powers. In the LO, the series of derivatives $\tilde{a}_n = a_s^n$.

Beyond LO, the relationship between $\tilde{a}_n$ and $a_s^n$ was established in (Cvetic, Valenzuela: 2006), (Cvetic, Kogerler, Valenzuela: 2010) and extended to the fractional case, where n is a non-integer $\nu$, in (Cvetic, Kotikov: 2012)

$$\tilde{a}^{(i+1)}_{\nu}(Q^2) = \tilde{a}^{(1)}_{\nu}(Q^2) + \sum_{m=1}^{(i)} \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)}\left(\hat{R}_m \tilde{a}^{(1)}_{\nu+m}(Q^2)\right) \tag{3}$$

$$\tilde{A}^{(i+1)}_{MA,\nu}(Q^2) = \tilde{A}^{(1)}_{MA,\nu}(Q^2) + \sum_{m=1}^{i} \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)}\left(\hat{R}_m \tilde{A}^{(1)}_{MA,\nu+m}(Q^2)\right) \tag{4}$$

# Bjorken polarized sum rule

Bjorken sum rule is defined as the 1st Mellin moment of the $g_1$ by the Bjorken variable of the difference for proton and neutron for the fixed $Q^2$:

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 \left[ g_1^p(x, Q^2) - g_1^n(x, Q^2) \right] dx. \tag{5}$$

BSR in the OPE form (twist-2 + twist-4):

$$\Gamma_1^{p-n}(Q^2) = \frac{|g_{A/V}|}{6} \left[ 1 - D_{Bj}(Q^2) \right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}} , \tag{6}$$

where $|g_{A/V}| = 1.2762 \pm 0.0005$

# Higher twist resummation

## Non-perturbative HT resummation via spectral function[a]:

$$\sum_{n=1}^{\infty} \mu_{2n+2}^{p-n} \left( \frac{M_{HT}^2}{Q^2} \right)^n = \int_{-\infty}^{\infty} \frac{f(x) M^2}{Q^2 - x M^2} dx, \quad (7)$$

[a] O. Teryaev. In: *Nucl. Phys. Proc. Suppl.* 245 (2013), pp. 195–198. arXiv: 1309.1985 [hep-ph].

## $\delta$-like like spectral density[a]

$$\Delta_{HT}(Q^2) = \mu_4 \frac{M_{HT}^2}{Q^2 + M_{HT}^2} \quad (8)$$

[a] V. L. Khandramai, O. V. Teryaev, and I. R. Gabdrakhmanov. In: *J. Phys. Conf. Ser.* 678.1 (2016), p. 012018.

## Breit-Wigner like spectral density[a]

$$\Delta_{HT}(Q^2) = \mu_4 \frac{M_{HT}^2 (M_{HT}^2 + Q^2)}{(M_{HT}^2 + Q^2)^2 + M_{HT}^2\ \sigma^2} \quad (9)$$

[a] I. R. Gabdrakhmanov, O. V. Teryaev, and V. L. Khandramai. In: *J. Phys. Conf. Ser.* 938.1 (2017), p. 012046.

Both using Massive APT[a] with $\ln(Q^2) \to \ln(Q^2 + M_{gl}^2)$ modified coupling $A(Q^2, M_{gl}^2)$

[a] D. V. Shirkov. In: *Phys. Part. Nucl. Lett.* 10 (2013), pp. 186–192. arXiv: 1208.2103 [hep-th].

# $1/L$ FAPT expansion

**BSR twist-2 perturbative series with couplants from[a] (we assume $f = 3$ active quark flavors)**

$$D_{BS}^{(k)}(Q^2) = \frac{4}{\beta_0}\left(\tilde{a}_1^{(k)} + \sum_{m=2}^{k} \tilde{d}_{m-1}\tilde{a}_m^{(k)}\right), \quad (10)$$

$$D_{MA,BS}^{(k)}(Q^2) = \frac{4}{\beta_0}\left(\tilde{A}_{MA}^{(k)} + \sum_{m=2}^{k} \tilde{d}_{m-1}\tilde{A}_{MA,\nu=m}^{(k)}\right), \quad (11)$$

[a] A. V. Kotikov and I. A. Zemlyakov. In: *J. Phys. G* 50.1 (2023), p. 015001. arXiv: 2203.09307 [hep-ph].

# $1/L$ expansion with the massive twist-4 HT representations[6]

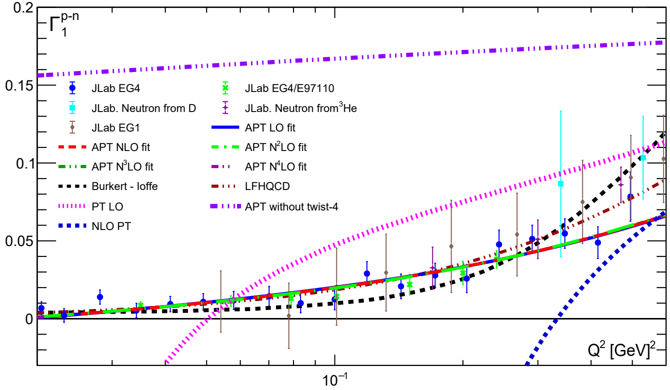


Figure: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of APT with $\sigma = \sigma_\rho$.

[6] I. R. Gabdrakhmanov, N. A. Gramotkov, A. V. Kotikov, D. A. Volkova, and I. A. Zemlyakov. In: *JETP Letters* 118 (2023). arXiv: 2307.16225 [hep-ph].

# Low $Q^2$ behavior

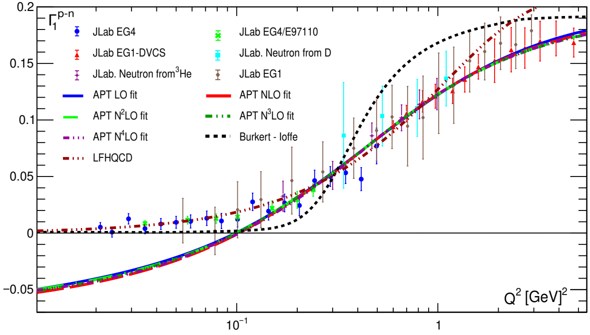


Figure: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of APT for $Q2 \leq 0.6$ GeV$^2$.

$\Gamma_{MA,1}^{p-n}(Q^2)$ takes negative unphysical values for low $Q^2 < 0.02\mathrm{GeV}^2$

# The Gerasimov-Drell-Hearn sum rule

The GDH[7],[8] sum rule for the nucleon connects it's anomalous magnetic moment and the difference of the photoabsorption cross sections for parallel - $\sigma_P$ and antiparallel $\sigma_A$ spins of the photon and the nucleon

$$\frac{2\pi e^2 \mu^2}{M^2} = \int_0^{\infty} \frac{d\nu}{\nu} (\sigma_P(\nu) - \sigma_A(\nu)), \quad (12)$$

[7] S. B. Gerasimov. In: *Sov. J. Nucl. Phys.* 2 (1966). [Yad. Fiz.2,598(1965)], pp. 430–433.

[8] S. D. Drell and A. C. Hearn. In: *Phys. Rev. Lett.* 16 (1966), pp. 908–911.

# Photoproduction limit

Following[a] and analogically to[b] we take into account Gerasimov-Drell-Hearn sum rule and get the equation

$$\frac{d}{dQ^2}\Gamma_1^{p-n}(Q^2=0)=G, \quad G=\frac{-(\mu_p-1)^2+\mu_n^2}{8M^2}=0.0631, \tag{13}$$

also finiteness of cross section for real photons implies:

$$\Gamma_1^{p-n}(Q^2=0)=0, \tag{14}$$

[a] J. Soffer and O. Teryaev. In: *Phys. Rev. Lett.* 70 (1993), pp. 3373–3375.

[b] Khandramai, Teryaev, and Gabdrakhmanov, “Infrared modified QCD couplings and Bjorken sum rule”; Gabdrakhmanov, Teryaev, and Khandramai, “Infrared models for the Bjorken sum rule in the APT approach”.

# Photoproduction limit

For MA coupling:

$$A^{(k)}(Q^2=0)=\tilde{A}^{(k)}_{m=1}(Q^2=0)=1,\ \tilde{A}^{(k)}_{m>1}(Q^2=0)=0, \qquad (15)$$

By definition $Q^2\frac{d}{dQ^2}\tilde{A}_n(Q^2)\sim\tilde{A}_{n+1}(Q^2)$, so

$$\frac{d}{dQ^2}\tilde{A}_n(Q^2\to 0)\to\infty\,. \qquad (16)$$

thus every term in $D_{MA,BS}(Q^2)$ becomes to be divergent at $Q^2\to 0$.
So we assume the relation between twist-2 and twist-4 terms, that leads to the appearance of a new contribution.
Taking into account GDH sum rule and (15) possible modification to improve low $Q^2$ behavior is:

$$\Gamma^{p-n}_{MA,1}(Q^2)=\frac{g_A}{6}\left(1-D_{MA,BS}(Q^2)\cdot\frac{Q^2}{Q^2+M^2}\right)+\frac{\hat{\mu}_{MA,4}M^2}{Q^2+M^2}+\frac{\hat{\mu}_{MA,6}M^4}{(Q^2+M^2)^2}, \qquad (17)$$



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# Photoproduction limit

Thus our modification (17) leads to:

$$\begin{cases} \hat{\mu}_{MA,6} = -G\,M^2 + \frac{5 g_{A/V}}{54} \approx -G\,M^2 + 0.1182, \\ \hat{\mu}_{MA,4} = -\frac{g_{A/V}}{6} - \hat{\mu}_{MA,6} = G\,M^2 - \frac{7 g_{A/V}}{27} \approx G\,M^2 - 0.3309\,, \end{cases} \quad (18)$$



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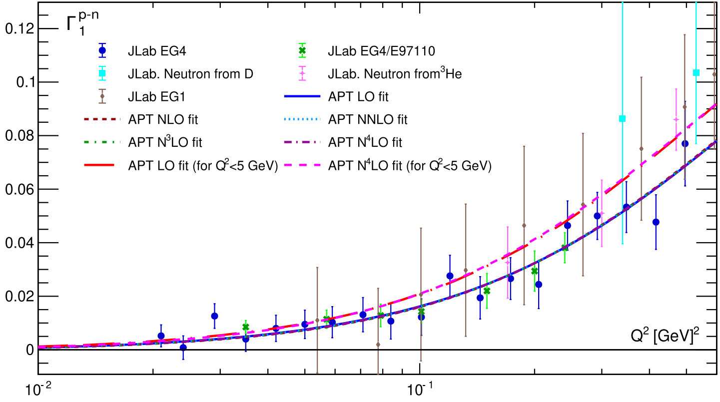


Figure: The results for $\Gamma_1^{p-n}(Q^2)$ (17) in the first four orders of APT for $Q^2 <$0.6 GeV$^2$[a]

[a] I. R. Gabdrakhmanov, N. A. Gramotkov, A. V. Kotikov, O. V. Teryaev, D. A. Volkova, and I. A. Zemlyakov. In: (2024). arXiv: 2404.01873 [hep-ph].

# Conclusion

- As previous studies showed: while conventional PT fails to describe BSR data, APT with the massive twist-4 HT term gives a good agreement with the data
- However the fits extended to very low $Q^2$ values produce negative results
- We successfully solved the issue modifying of the OPE and found a good agreement in the whole experimental region with the help of the GDH sum rule
- The distinctive feature of the solution is continuation of the BSR down to the photoproduction limit with a finite derivative $\frac{d}{dQ^2}\Gamma_{MA,1}^{p-n}$