Effective Potential for general SO(N) scalar theory in LLA

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Effective potential in renormalizable case

- Lagrangian of SO(N)-model

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - g V_0(\phi_i \phi_i)$$

Φ^4-model

$$V_0 = \frac{(\phi_i \phi_i)^2}{4!}$$

- Coleman-Weinberg[CW'73] and Jackiw[Jackiw'75] LLA-results for ϕ^4-model and for SO(N)-model:

$$V(\phi) = \frac{g\phi^4/4!}{1 - \frac{3}{2}\frac{g\phi^2}{16\pi^2}\log(\phi^2/\mu^2)} \qquad V(\phi) = \frac{g(\phi^2)^2/4!}{1 - \frac{3}{2}(1 + \frac{N-1}{9})\frac{g\phi^2}{16\pi^2}\log(\phi^2/\mu^2)}$$

Effective potential in renormalizable case

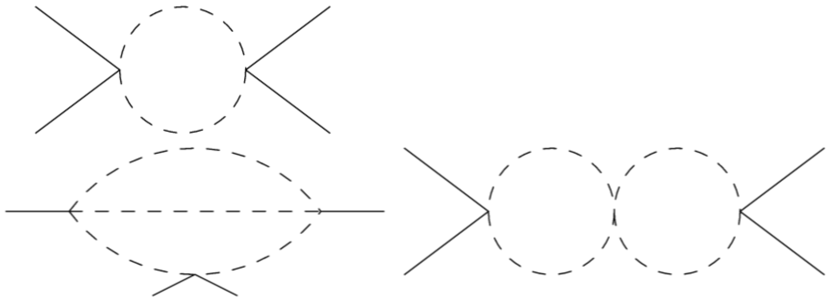
- Coleman-Weinberg [CW'73] and Jackiw [Jackiw'75] LLA-results for simple ϕ^4-model and for SO(N)-model:

$$V(\phi) = \frac{g\phi^4/4!}{1 - \frac{3}{2}\frac{g\phi^2}{16\pi^2}\log(\phi^2/\mu^2)} \qquad V(\phi) = \frac{g(\phi^2)^2/4!}{1 - \frac{3}{2}(1+\frac{N-1}{9})\frac{g\phi^2}{16\pi^2}\log(\phi^2/\mu^2)}$$

constants

$$V_{eff} = V_0 \begin{array}{c} \begin{array}{ccc} \text{LLA} & \text{NLA} & \text{N}^k\text{LA} \end{array} \\ \left[\begin{array}{ccc} \boxed{\begin{array}{c} 1 \\ a_1\ gL \\ a_2\ g^2L^2 \\ a_n\ g^nL^n \\ \ldots \end{array}} & \begin{array}{c} \\ b_1 g \\ b_2\ gL \\ b_n\ g^nL^{n-1} \\ \ldots\ldots\ldots\ldots \end{array} & \end{array}\right] \end{array} \begin{array}{l} \text{tree} \\ \text{1-loop} \\ \text{2-loop} \\ \text{n-loop} \\ \ldots \end{array}$$

1PI Feynman diagram topologies for Φ^4



Effective potential in general case: overlook

- Lagrangian of general SO(N)-model:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - gV_0(\phi_i\phi_i)$$

General potential

$$V_0 = \frac{(\phi^2)^{p/2}}{p!}$$

Exp|Φ|-model

$$V_0 = e^{|\phi|/m}$$

All PT-rules are applicable on non-renormalizable case

not necessarily constants

$$V_{eff} = V_0 \begin{array}{c} \begin{array}{ccc} \text{LLA} & \text{NLA} & \text{N}^k\text{LA} \end{array} \\ \begin{bmatrix} 1 & & \\ a_1\; gL & b_1 g & \\ a_2\; g^2L^2 & b_2\; gL & \\ a_n\; g^n L^n & b_n\; g^n L^{n-1} & \dots \\ \dots & \dots\dots\dots & \end{bmatrix} \end{array} \begin{array}{l} \\ \text{tree} \\ \text{1-loop} \\ \text{2-loop} \\ \text{n-loop} \\ \dots \end{array}$$

We will focus only on LLA's as for non-renormalizable interaction (NLA's are scheme-dependent)

In the case of non-renormalizable models, coefficients in front of the logarithms are no longer numbers, but depend on the field

Effective potential: general formalism

- Generating functional

$$Z(J) = \int D\phi \; e^{i \int d^4x \mathcal{L} + J\phi}$$

- 1PI generating functional

$$W(J) = -i \log(Z(J))$$

- Legendre transformation

$$\Gamma(\phi) = W(J) - \int d^4x \; J\phi$$

- Shifted action:

$$e^{i\Gamma(\hat{\phi})} = \int D\phi \; e^{i(S[\phi+\hat{\phi}] - \phi S'[\hat{\phi}])}$$

$$S[\phi + \hat{\phi}] = S[\phi] + \cancel{\phi S'[\hat{\phi}]} + \frac{1}{2}\phi^2 S''[\hat{\phi}] + \text{interaction terms}$$

effective mass

Feynman rules

Efficient way to find effective potential is to sum <u>1PI vacuum diagrams</u>

Effective mass from shifted action

$$m^2_{ab} = g\frac{\partial^2 V_0}{\partial\phi_a \partial\phi_b} = \boxed{g\hat{v}_2\left(\delta_{ab} - \frac{\phi_a\phi_b}{\phi^2}\right) + g v_2 \frac{\phi_a\phi_b}{\phi^2}}$$

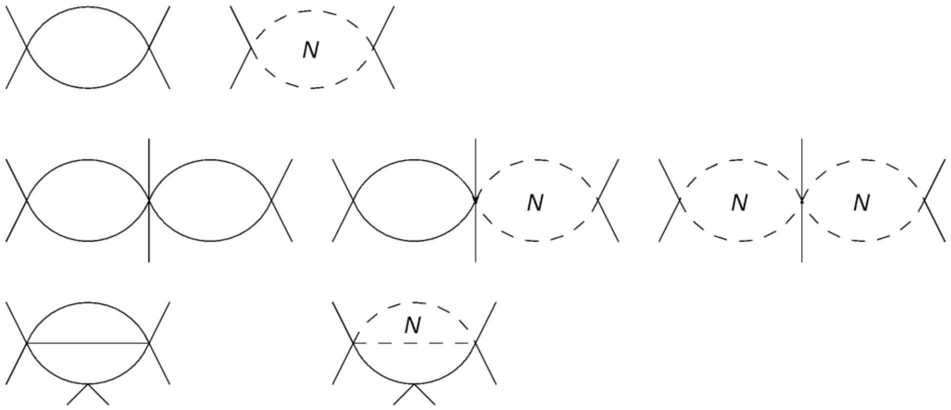
$$\hat{v}_2 = 2\frac{\partial}{\partial(\phi^2)}V \qquad v_2 = \frac{\partial^2 V}{\partial\phi^2}$$

Propagators:

$$G'_{ab}(p) = \frac{1}{p^2 - g\hat{v}_2}\left(\delta_{ab} - \frac{\phi_a\phi_b}{\phi^2}\right)$$

$$G_{ab}(p) = \frac{1}{p^2 - g v_2}\left(\frac{\phi_a\phi_b}{\phi^2}\right)$$

Vertices are derivatives of V(ϕ) and symm. combination of δ_{ab}



For example

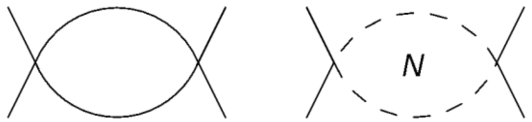
$$v_n = \frac{\partial^n V}{\partial\phi^n} \qquad \hat{v}_4 = 4\frac{\partial^2 V}{\partial(\phi^2)^2} \qquad V_{eff} = g\sum_{k=0}^{\infty}(-g)^k V_k$$

$$t_{abcd} = \delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$$

etc

One-loop result

- One-loop diagrams:



- Φ^4-model:

<u>singular</u> part

$$\Delta V_1 = g\frac{\phi^4}{4}\frac{1}{4\epsilon} + g(N-1)\frac{\phi^4}{36}\frac{1}{4\epsilon}$$

leading logs

$$\to \frac{g}{64\pi^2}\frac{\phi^4}{4}\log\left(\frac{g\phi^2}{2\mu^2}\right) + (N-1)\frac{g}{64\pi^2}\frac{\phi^4}{36}\log\left(\frac{g\phi^2}{6\mu^2}\right)$$

- Φ^6-model

<u>singular</u> part

$$\Delta V_1 = g\left(\frac{\phi^4}{4!}\right)^2\frac{1}{4\epsilon} + g(N-1)\left(\frac{\phi^4}{5!}\right)^2\frac{1}{4\epsilon}$$

leading logs

$$\to \frac{g}{64\pi^2}\left(\frac{\phi^4}{4!}\right)^2\log\left(\frac{g}{\mu^2}\frac{\phi^4}{4!}\right) + \frac{g}{64\pi^2}(N-1)\left(\frac{\phi^4}{5!}\right)^2\log\left(\frac{g}{\mu^2}\frac{\phi^4}{5!}\right)$$

Two loop results

- Φ^4 model

$$\Delta V_2 = \frac{3g^2\phi^4}{32\epsilon^2} + (N-1)\frac{g^2\phi^4}{48\epsilon^2} + (N-1)^2\frac{g^2\phi^4}{864\epsilon^2}$$

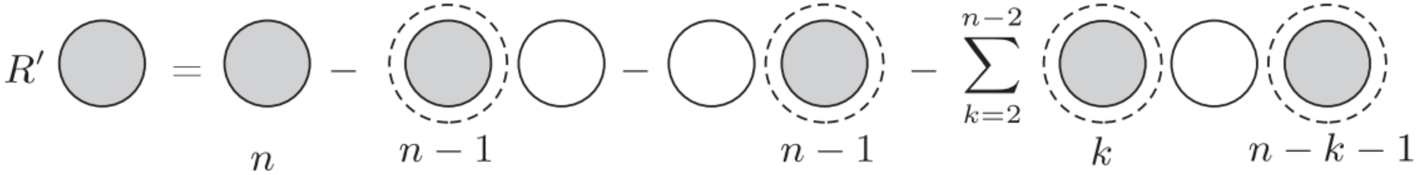
Coincidence with the results of [CC'98, Kastening'96] (even on 3-loop level)

- Φ^6 model

$$\Delta V_2 = (N-1)^2\frac{g^3\phi^{10}}{4^2 5!4!\epsilon^2} + (N-1)\frac{19g^3\phi^{10}}{(5!)^3\epsilon^2} + \frac{7g^3\phi^{10}}{2(4!)^3\epsilon^2}$$

BPHZ-procedure

R'-operation for n-loop graph



n-loop divergence **always** is **local** due to Bogoliubov-Parasiuk theorem [BP'57, Hepp'66,Zimmerman'69], result of R'(G) must not contain terms like $\sim \log(\mu^2)/\epsilon$

Consequence:

$$A_n^{(n)} = (-1)^{n+1} \frac{A_1^{(n)}}{n}$$

Higher order leading divergences are governed by one-loop divergence

Now we have all the needed information to obtain the **recurrence relations**

Recurrence relation

- Based on calculated diagrams we can write recurrence relation which generate leading poles:

$$n\Delta V_n = \frac{N-1}{4}\sum_{k=0}^{n-1}\bar{D}_2\Delta V_k\bar{D}_2\Delta V_{n-k-1}+\frac{1}{4}\sum_{k=0}^{n-1}D_2\Delta V_k D_2\Delta V_{n-k-1} \qquad D_2=\frac{\partial^2}{\partial\phi^2}$$

$$\bar{D}_2 = 2\frac{\partial}{\partial(\phi^2)}$$

- Or, shortly

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1}D_{ab}\Delta V_k D_{ab}V_{n-k-1} \qquad D_{ab}=\frac{\partial^2}{\partial\phi_a\partial\phi_a}$$

As the coefficient of the leading logarithm is always equal to the one of the leading pole now we know short way to find exact leading log behaviour

N=1 limit

- Generalized RG-equation from [Kazakov, I.R, Tolkachev'23] is restored

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1} D_2 V_k D_2 \Delta V_{n-1-k}$$

- Introducing function

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty}(-z)^n \Delta V_n(\phi) \qquad z = \frac{g}{\epsilon}$$

$$g\phi^{p-4} < 16\pi^2$$

$$\log(m^2(\phi)/\mu^2) > 1$$

- Exact generalized RG-equation and effective potential

$$\frac{\partial}{\partial z}\Sigma = -\frac{1}{4}(D_2\Sigma)^2 \quad V_{eff} = g\Sigma(z,\phi)\Big|_{z\to\frac{g}{16\pi^2}\log(gv_2/\mu^2)} \qquad f(0) = 1$$

Power-like potential

$$\Sigma(z,\phi) = \frac{\phi^p}{p!}f(z\phi^{p-4})$$

In the case of power-like potential

$$-\frac{1}{4p!}\left[p(p-1)f(z) + (p-4)(3p-5)zf'(z) + (p-4)^2y^2f''(z)\right]^2 = f'(z)$$

This ODE is too difficult to solve analytically

p=4

$$f'(z) = -\frac{3}{2}f(z)^2$$

N=1 limit

- Exact generalized RG-equation and effective potential

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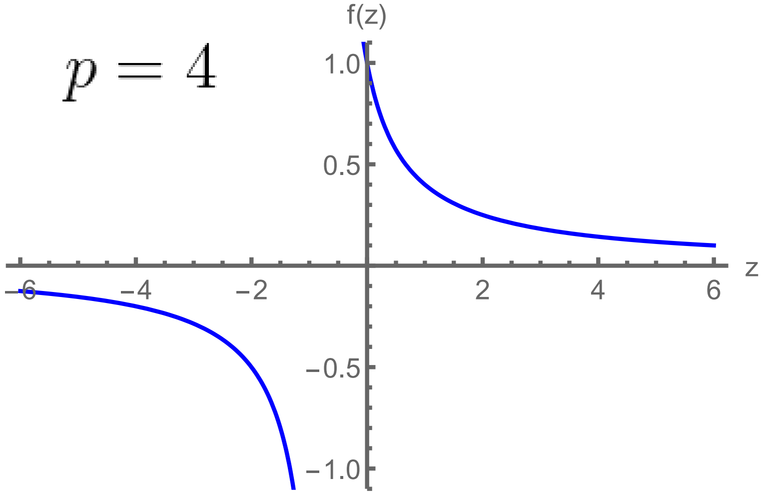
$$\boxed{-\frac{1}{4p!}\left[p(p-1)f(z)+(p-4)(3p-5)zf'(z)+(p-4)^2y^2f''(z)\right]^2 = f'(z)}$$

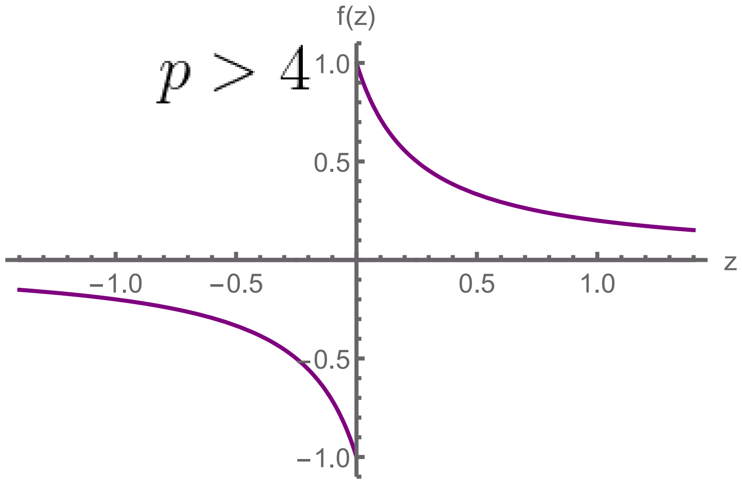
$$f(0)=1$$

$$f'(0) = -\frac{1}{4}\frac{p(p-1)}{(p-2)!}$$

$p=4$



$p>4$



$$g\phi^{p-4} < 16\pi^2$$

$$\log(m^2(\phi)/\mu^2) > 1$$

Large N limit

- In this limit we can find

$$n\Delta V_n = \frac{N}{4}\sum_{k=0}^{n-1}\bar{D}_2\Delta V_k\bar{D}_2\Delta V_{n-k-1}$$

Again we introduce the function summing all poles (effective potential)

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty}(-z)^n\Delta V_n(\phi) \qquad z = \frac{g}{\epsilon} \qquad V_{eff} = g\Sigma(z,\phi)\Big|_{z\to\frac{g}{16\pi^2}\log(g\hat{v}_2/\mu^2)}$$

$g\left(\phi^2\right)^{p/2-2} < 16\pi^2$
$\log(m(\phi)^2/\mu^2) > 1$

- Generalized RG-equation is given by

$$\frac{\partial}{\partial z}\Sigma(z,\phi) = -\frac{N}{4}\left(\bar{D}_2\Sigma\right)^2$$

RG-equation for power like potential:

$$-\frac{N}{4p!}\left((p-4)zf'(z)+pf(z)\right)^2 = f'(z) \quad f(0)=1$$

The ODE is the first order so we can solve it analytically (and numerically)

Power-like potential

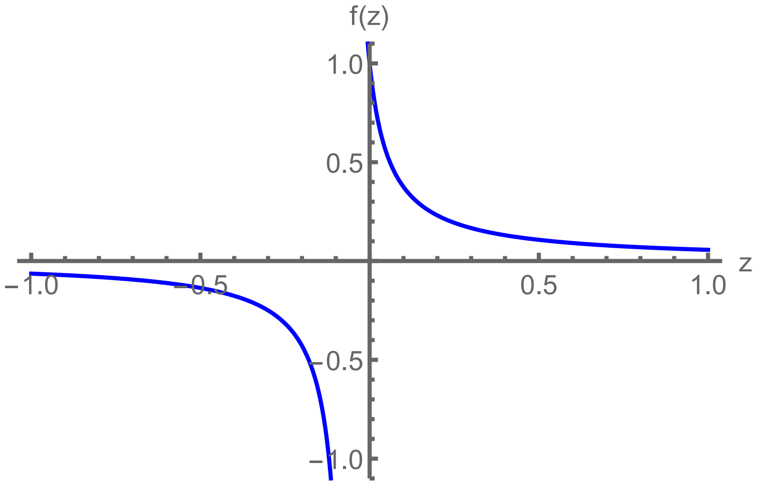
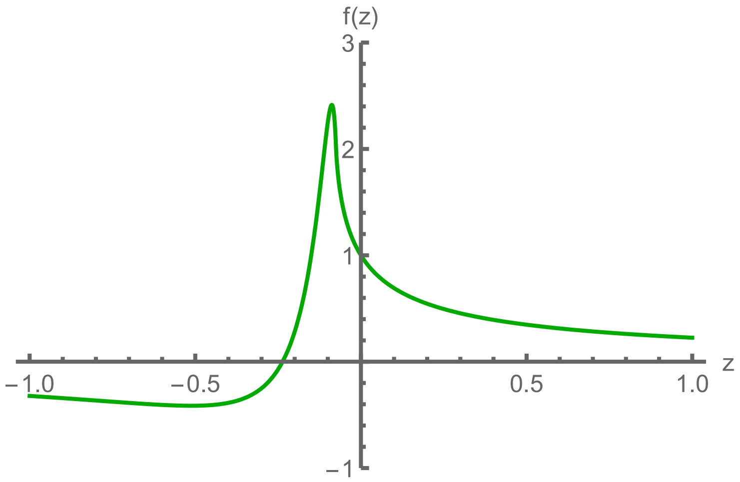
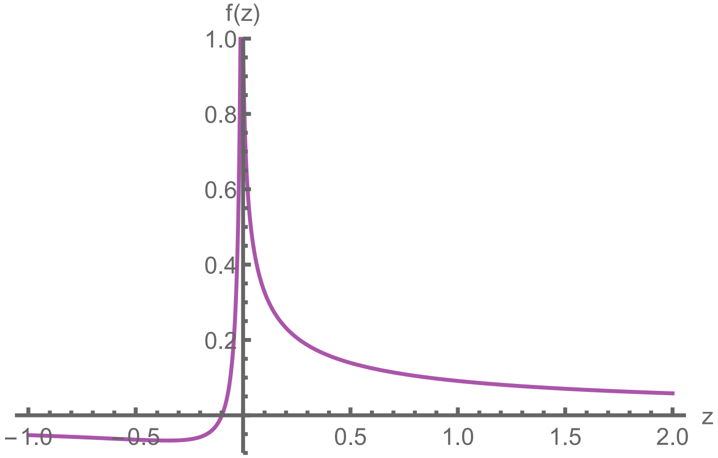
$$\Sigma(z,\phi) = \frac{(\phi^2)^{p/2}}{p!}f(z(\phi^2)^{p/2-2})$$

p=4

$$f'(z) = -\frac{N}{6}f(z)^2$$

Large N limit

$N = 100$

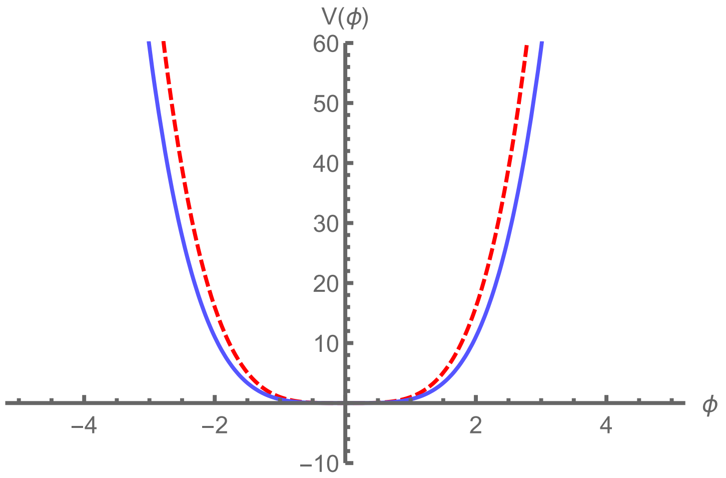
Φ^4 model	Φ^6 model	$\exp(\|\Phi\|)$ model
$f'(z) = -\frac{N}{6} f(z)^2$	$\frac{N}{180}\left(zf'(z) + 3f(z)\right)^2 = -f'(z)$	$N(zf'(z) + f(z))^2 = -f'(z)$
		
$f(z) = \frac{1}{1 + \frac{N}{6} z}$	$f(z) = \frac{15}{N^3 z^3}\left(90Nz\left(\frac{Nz}{30}+1\right) - 450\left(\frac{2Nz}{15}+1\right)^{3/2} + 450\right)$	$f(z) = \frac{W(Nz)(W(Nz)+2)}{4Nz}$

Large N limit

$N = 100$

Φ⁴ model

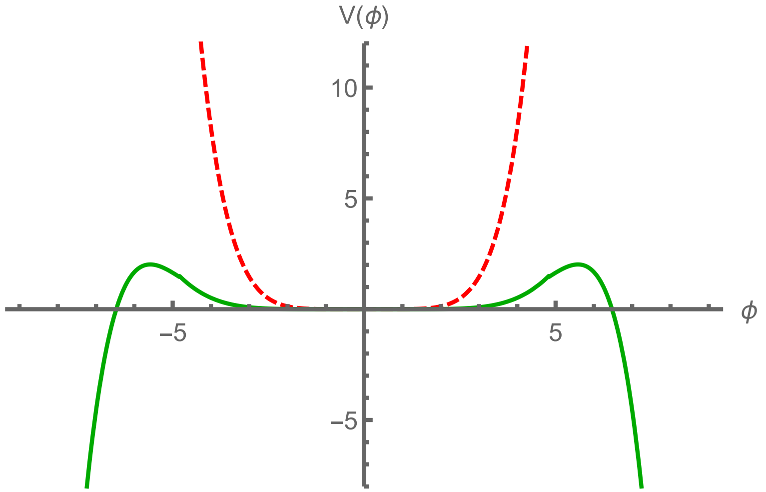
$$f'(z) = -\frac{N}{6} f(z)^2$$



$$f(z) = \frac{1}{1 + \frac{N}{6} z}$$

Φ⁶ model

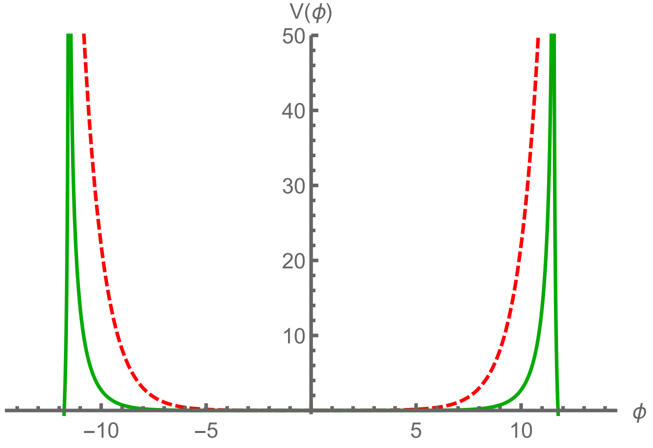
$$\frac{N}{180}\left(z f'(z) + 3 f(z)\right)^2 = -f'(z)$$



$$f(z) = \frac{15}{N^3 z^3}\left(90 N z\left(\frac{Nz}{30} + 1\right) - 450\left(\frac{2Nz}{15} + 1\right)^{3/2} + 450\right)$$

exp(|Φ|) model

$$N\left(z f'(z) + f(z)\right)^2 = -f'(z)$$



$$f(z) = \frac{W(Nz)(W(Nz) + 2)}{4Nz}$$

Conclusions and prospects

- A recurrence relation for SO(N) scalar model with general power-like potential was found
- The resulting recurrence relations recovers the known theories within its limits
- Analytical evaluation were provided in large N limit

- Subleading orders and scheme dependence in scalar models have to be investigated in details
- Generalized RG-equation for leading logs in curved space-times and higher dimensions also should be studied
- EP in matrix models? SUSY?..

Thanks for attention!