Multiloop Baxter equations and Quantum Spectral Curve

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Outline

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ABJM is $\mathcal{N}=6$ superconformal Chern-Simons-matter theory with gauge group $U(N)\times U(N)$ on $\mathbb{R}^{1,2}$ and Chern-Simons levels k and $-k$. The field content is given by gauge fields A_μ and $\hat{A}_\mu$, four complex scalars Y^A and four Weyl spinors ψ_A.

In planar limit $k, N \to \infty, \quad \lambda \equiv \frac{N}{k} = \text{fixed}$ it is dual to superstring theory on $AdS_4 \times \mathbb{CP}^3$

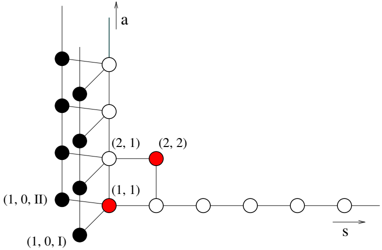
We will be interested in anomalous dimensions of operators

$$\mathrm{tr}\left[D_+^S (Y^1 Y_4^\dagger)^L\right]$$

with Dynkin labels $[L+S, S; L, 0, L]$ under $\mathrm{OSp}(6|4)$

Aharony, Bergman, Jafferis, Maldacena 2008

ABJM theory and **Y**, **T** systems



ABJM Y-system

$$Y_{a,s}(u+i/h)Y_{a,s}(u-i/h) = \frac{(1+Y_{a,s+1}(u))(1+Y_{a,s-1}(u))}{(1+1/Y_{a+1,s}(u))(1+1/Y_{a-1,s}(u))}, \quad s>1, (a,s)\neq(2,2)$$

$$Y_{a,1}(u+i/h)Y_{a,1}(u-i/h) = \frac{(1+Y_{a,2}(u))(1+Y^{I}_{a,0}(u))(1+Y^{II}_{a,0}(u))}{(1+1/Y_{a+1,1}(u))(1+1/Y_{a-1,1}(u))},$$

$$Y^{\alpha}_{a,0}(u+i/h)Y^{\beta}_{a,0}(u-i/h) = \frac{(1+Y_{a,1}(u))}{(1+1/Y^{\alpha}_{a+1,0}(u))(1+1/Y^{\beta}_{a-1,0}(u))}, \quad \alpha,\beta\in\{I,II\}$$

Cavaglia, Fioravanti, Tateo, 2013
Gromov, Levkovich-Maslyuk 2013

ABJM T-system

$$Y_{a,s}(u) = \frac{T_{a,s+1}(u)\,T_{a,s-1}(u)}{T_{a+1,s}(u)\,T_{a-1,s}(u)}, \qquad \text{for } s \geq 2, a \geq 1,$$

$$Y_{a,1}(u) = \frac{T_{a,2}(u)\,T^{I}_{a,0}(u)\,T^{II}_{a,0}(u)}{T_{a+1,1}(u)\,T_{a-1,1}(u)}, \qquad \text{for } a \geq 1,$$

$$Y^{\alpha}_{a,0}(u) = \frac{T_{a,1}(u)\,T^{\beta}_{a,-1}(u)}{T^{\alpha}_{a+1,0}(u)\,T^{\beta}_{a-1,0}(u)}, \qquad \text{for } a \geq 1, \quad \alpha,\beta \in \{I, II\}, \beta \neq \alpha.$$

Discrete Hirota equation for T-functions:

$$T^{[+1]}_{a,s}\,T^{[-1]}_{a,s} = \prod_{(a'\sim a)_{\updownarrow}} T_{a',s} + \prod_{(s'\sim s)_{\leftrightarrow}} T_{a,s'},$$

where the products are over horizontal ($\leftrightarrow$) and vertical ($\updownarrow$) neighbouring nodes

T-functions are gauge dependent!

There are two special gauges **T** and $\mathbb{T}$, so that

$$\begin{aligned}\mathbb{T}_{1,s} &= \mathbf{P}_1^{[+s]}\mathbf{P}_2^{[-s]} - \mathbf{P}_2^{[+s]}\mathbf{P}_1^{[-s]}, \qquad \mathbb{T}_{0,s} = 1,\\ \mathbb{T}_{2,s} &= \mathbb{T}_{1,1}^{[+s]}\mathbb{T}_{1,1}^{[-s]}, \qquad \mathbb{T}_{3,2}/\mathbb{T}_{2,3} = \mu_{12}, \quad s \geq a\end{aligned}$$

and

$$\begin{aligned}\mathbf{T}_{n,s} &= (-1)^{n(s+1)}\mathbb{T}_{n,s}\left(\mu_{12}{}^{[n+s-1]}\right)^{2-n}, \qquad s \geq 1\\ \mathbf{T}_{n,0}^{\alpha} &= (-1)^n\mathbb{T}_{n,0}^{\alpha}\left(\sqrt{\mu_{12}{}^{[n-1]}}\right)^{2-n},\\ \mathbf{T}_{n,-1}^{\alpha} &= \mathbb{T}_{n,-1}^{\alpha} = 1, \qquad \alpha = I, II,\end{aligned}$$

The $\mathbf{T}_{n,s}$ functions are required to satisfy:

$$\begin{aligned}\mathbf{T}_{n,0}^{\alpha} &\in \mathcal{A}_{n+1}, \qquad \alpha = I, II, \quad n \geq 0\\ \mathbf{T}_{n,1} &\in \mathcal{A}_n, \quad n \geq 1,\end{aligned}$$

$\mathcal{A}_n$ is the class of functions free of branch cuts for $|\mathrm{Im}(u)| < \frac{n}{2}$.

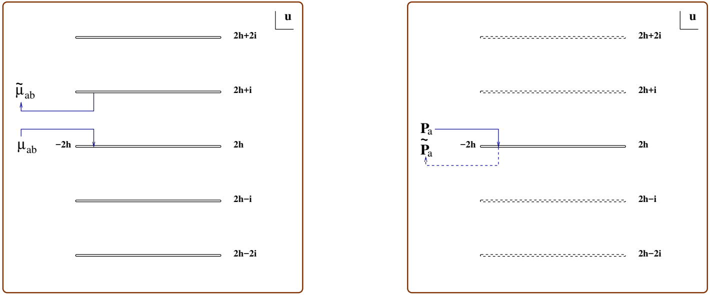
ABJM QSC and $\mathbf{P}\mu$ system

Vector form (CP^3 isometry group $SO(6) \simeq SU(4)$):

$$\mathbf{P}_A(u)\Big|_{A=1,\ldots,6}, \quad \mu_{AB}(u) = -\mu_{BA}(u)\Big|_{A,B=1,\ldots,6}$$

$$\widetilde{\mathbf{P}}_A - \mathbf{P}_A = \mu_{AB}\,\eta^{BC}\,\mathbf{P}_C, \qquad \widetilde{\mu}_{AB} - \mu_{AB} = \mathbf{P}_A\widetilde{\mathbf{P}}_B - \mathbf{P}_B\widetilde{\mathbf{P}}_A.$$

$$\mathbf{P}_5\mathbf{P}_6 - \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_1\mathbf{P}_4 = 1, \quad \mu_{AB}\,\eta^{BC}\,\mu_{CD} = 0, \quad \widetilde{\mu}_{AB}(u) = \mu_{AB}(u+i)$$



Gromov, Kazakov, Leurent, Volin, 2013
Cavaglia, Fioravanti, Gromov, Tateo, 2014

Spinor form (CP^3 isometry group $SO(6) \simeq SU(4)$): the matrix $\mu_{AB}(u)$ is decomposed in terms of $4+4$ functions ν_a, ν^a as

$$\mu_{AB} = \nu^a \, (\sigma_{AB})_a^{\;b} \, \nu_b \, , \quad \nu^a \, \nu_a = 0.$$

$$\widetilde{\nu}_a(u) = e^{i\mathcal{P}} \, \nu_a(u+i), \qquad \widetilde{\nu}^a(u) = e^{-i\mathcal{P}} \, \nu^a(u+i)$$

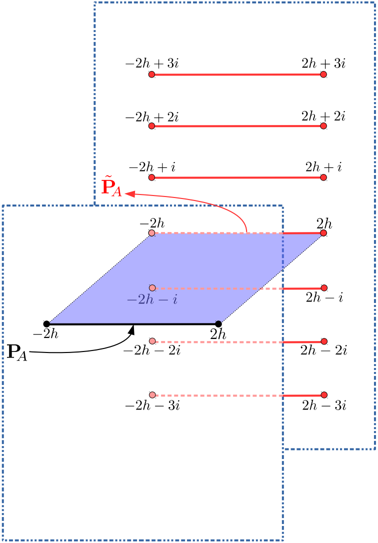
Riemann-Hilbert problem to solve:

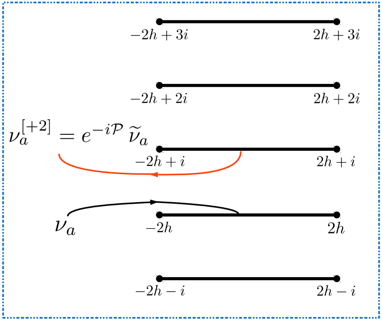
$$\widetilde{\mathbf{P}}_{ab} - \mathbf{P}_{ab} = \nu_a \tilde{\nu}_b - \nu_b \tilde{\nu}_a, \qquad \widetilde{\mathbf{P}}^{ab} - \mathbf{P}^{ab} = -\nu^a \tilde{\nu}^b + \nu^b \tilde{\nu}^a,$$
$$\tilde{\nu}_a = -\mathbf{P}_{ab} \, \nu^b, \qquad \tilde{\nu}^a = -\mathbf{P}^{ab} \, \nu_b.$$

$$\mathbf{P}_{ab} = \mathbf{P}_A \sigma^A_{ab} = \begin{pmatrix} 0 & -\mathbf{P}_1 & -\mathbf{P}_2 & -\mathbf{P}_5 \\ \mathbf{P}_1 & 0 & -\mathbf{P}_6 & -\mathbf{P}_3 \\ \mathbf{P}_2 & \mathbf{P}_6 & 0 & -\mathbf{P}_4 \\ \mathbf{P}_5 & \mathbf{P}_3 & \mathbf{P}_4 & 0 \end{pmatrix}, \quad \mathbf{P}^{ab} \text{ is inverse matrix}$$

Bombardelli, Cavaglia, Fioravanti, Gromov, Tateo, 2017

ABJM QSC and $\mathbf{P}\mu$ system





Bombardelli, Cavaglia, Fioravanti, Gromov, Tateo, 2017

Boundary conditions in $sl(2)$ sector (large u):

$$\mathbf{P}_a \simeq (A_1 u^{-L},\ A_2 u^{-L-1},\ A_3 u^{+L+1},\ A_4 u^{+L},\ A_0 u^0),$$

$$-A_1 A_4 = \frac{(-\Delta + L - S)(-\Delta + L + S - 1)(\Delta + L - S + 1)(\Delta + L + S)}{L^2(2L+1)},$$

$$-A_2 A_3 = \frac{(-\Delta + L - S + 1)(-\Delta + L + S)(\Delta + L - S + 2)(\Delta + L + S + 1)}{(L+1)^2(2L+1)},$$

$$\nu_a \sim \left(u^{\Delta - L}, u^{\Delta+1}, u^{\Delta}, u^{\Delta+L+1}\right).$$

$L \in \mathbb{N}^+$ (twist), $S \in \mathbb{N}^+$ (spin) and Δ is the conformal dimension.
The anomalous dimension γ is given by $\gamma = \triangle - L - S$.

Cavaglia, Fioravanti, Gromov, Tateo, 2014

Solution for $sl(2)$ sector

We will look for solution at weak coupling in the form $(\mathbf{P}_0 = \mathbf{P}_5 = \mathbf{P}_6)$

$$\mathbf{P}_1 = (xh)^{-L}\mathbf{p}_1 = (xh)^{-L}\left(1 + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{1,k}^{(l)}\frac{h^{2l+k}}{x^k}\right), \quad \nu_i(u) = \sum_{l=1}^{\infty} h^{2l-L}\nu_i^{(l)}(u),$$

$$\mathbf{P}_2 = (xh)^{-L}\mathbf{p}_2 = (xh)^{-L}\left(\frac{h}{x} + \sum_{k=2}^{\infty}\sum_{l=0}^{\infty} c_{2,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_0 = (xh)^{-L}\mathbf{p}_0 = (xh)^{-L}\left(\sum_{l=0}^{\infty} A_0^{(l)} h^{2l} u^L + \sum_{j=0}^{L-1}\sum_{l=0}^{\infty} m_j^{(l)} h^{2l} u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{0,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_3 = (xh)^{-L}\mathbf{p}_3 = (xh)^{-L}\left(\sum_{l=0}^{\infty} A_3^{(l)} h^{2l} u^{2L+1} + \sum_{j=0}^{2L}\sum_{l=0}^{\infty} k_j^{(l)} h^{2l} u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{3,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

where

$$x \equiv x(u) = \frac{u + \sqrt{u^2 - 4h^2}}{2h}$$

$c_{i,k}^{(l)}$ are some functions of spin S only, otherwise they are just constants. The analytically continued though the cut functions are defined as

$$\tilde{\mathbf{P}}_i = \left(\frac{x}{h}\right)^L \tilde{\mathbf{p}}_i, \quad \tilde{\mathbf{p}}_i = \mathbf{p}_i\Big|_{x\to 1/x}.$$

Initial conditions for iterative solution:

$$\mathbf{p}_{1,0} = 1, \qquad \mathbf{p}_{2,0} = 0,$$
$$\widetilde{\mathbf{p}}_{1,0}(u) \sim 1 + O(u), \qquad \widetilde{\mathbf{p}}_{2,0}(u) \sim u + O(u^2).$$

Baxter equations to solve (state quantum numbers are specified by LO Baxter polynomial $Q(u) \sim \nu_1^{[1]}(u)$) :

$$\frac{\nu_1^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_1^{[-1]}}{\mathbf{P}_1^{[-1]}} - \sigma \left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}} \right) \nu_1^{[1]} = -\sigma \left(\frac{\mathbf{P}_2^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_2^{[-1]}}{\mathbf{P}_1^{[-1]}} \right) \nu_2^{[1]}.$$

$$\frac{\nu_2^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_2^{[-1]}}{\mathbf{P}_1^{[-1]}} + \sigma \left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}} \right) \nu_2^{[1]} = \sigma \left(\frac{\mathbf{P}_3^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_3^{[-1]}}{\mathbf{P}_1^{[-1]}} \right) \nu_1^{[1]}.$$

where

$$\sigma \equiv e^{i\mathcal{P}} = Q^{[1]}(0)/Q^{[-1]}(0)\,, \quad Q \text{ is LO Baxter polynomial}$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

Coefficients are fixed from equations:

$$\left.\begin{aligned} \nu_a(u) + \widetilde{\nu}_a(u) &= \nu_a(u) + \sigma\, \nu_a^{[2]}(u) \\ \frac{\nu_a(u) - \widetilde{\nu}_a(u)}{\sqrt{u^2-4h^2}} &= \frac{\nu_a(u) - \sigma\, \nu_a^{[2]}(u)}{\sqrt{u^2-4h^2}} \end{aligned}\right\} \quad \text{free of cuts on real axis}$$

$$\begin{aligned} \left(\nu_1 + \sigma\, \nu_1^{[2]}\right)\left(\mathbf{p}_0 - (hx)^L\right) &= \mathbf{p}_2 \left(\nu_2 + \sigma\, \nu_2^{[2]}\right) - \mathbf{p}_1 \left(\nu_3 + \sigma\, \nu_3^{[2]}\right), \\ \left(\nu_2 + \sigma\, \nu_2^{[2]}\right)\left(\mathbf{p}_0 + (hx)^L\right) &= \mathbf{p}_3 \left(\nu_1 + \sigma\, \nu_1^{[2]}\right) + \mathbf{p}_1 \left(\nu_4 + \sigma\, \nu_4^{[2]}\right). \end{aligned}$$

$$\sigma\, \nu_1^{[2]} = \mathbf{P}_0\, \nu_1 - \mathbf{P}_2\, \nu_2 + \mathbf{P}_1\, \nu_3, \quad \widetilde{\mathbf{P}}_2 - \mathbf{P}_2 = \sigma \left(\nu_3 \nu_1^{[2]} - \nu_1 \nu_3^{[2]}\right),$$

$$\sigma\, \nu_2^{[2]} = -\mathbf{P}_0\, \nu_2 + \mathbf{P}_3\, \nu_1 + \mathbf{P}_1\, \nu_4, \quad \widetilde{\mathbf{P}}_1 - \mathbf{P}_1 = \sigma \left(\nu_2 \nu_1^{[2]} - \nu_1 \nu_2^{[2]}\right).$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

Solution of Baxter equations for integer spin values

Take as example first Baxter equation ($q_1 = \nu_1^{[1]}$):

$$(u+i/2)^L\, q_1^{[2]} - (u-i/2)^L\, q_1^{[-2]} - T_0\, q_1 = -U_1^{[-1]},$$

using an ansatz $q_1 = Q\, f_1^{[1]}$ we have ($\nabla_\pm g = g \mp g^{[2]}$):

$$\nabla_-\left(u^L Q^{[1]} Q^{[-1]} \nabla_+(f_1)\right) = U_1\, Q^{[1]},$$

introducing inverse operators $\nabla_\pm \Psi_\pm g = g$ we get

$$f_{1,\text{inhomo}} = \Psi_+\left(\frac{1}{u^L Q^{[1]} Q^{[-1]}}\Psi_-\left(U_1\, Q^{[1]}\right)\right).$$

full solution

$$q_1^{[-1]} = \Phi_{1,\text{per}} Q^{[-1]} + \Phi_{1,\text{anti}}\, \mathcal{Z}^{[-1]} + \Psi_+\left(\frac{1}{u^L Q^{[1]} Q^{[-1]}}\Psi_-\left(U_1\, Q^{[1]}\right)\right),$$

$$\mathcal{Z}^{[-1]} = Q^{[-1]}\, \Psi_-\left(\frac{1}{u^L Q^{[1]} Q^{[-1]}}\right),$$

Marboe, Volin, 2014; Anselmetti, Bombardelli, Cavaglia, Tateo, 2015

Next, introducing polynomials A, B :

$$A\,Q^{[1]} + B\,Q^{[-1]} = 1.$$

from Baxter equation we get (R is polynomial of degree $L-2$):

$$-A^{[1]}(u^{[-1]})^L + B^{[-1]}(u^{[1]})^L = QR,$$

$$\frac{R}{(u^{[1]}u^{[-1]})^L} = \sum_{k=1}^{L}\left(\frac{r_{k,+}}{(u^{[1]})^k} + \frac{r_{k,-}}{(u^{[-1]})^k}\right), \quad C = \frac{A}{u^L} - Q^{[-1]}\sum_{k=1}^{L}\frac{r_{k,+}}{u^k},$$

Then the solution reads

$$\mathcal{Z}^{[-1]} = \left(C + Q^{[-1]}\sum_{k=1}^{L}(-r_{k,+} + r_{k,-})\,\eta_{-k}(u)\right),$$

$$f_{1,\mathrm{inhomo}} = \Psi_-\left(U_1 Q^{[1]}\right)C + Q^{[-1]}\Psi_+\left(\Psi_-\left(U_1 Q^{[1]}\right)\sum_{k=1}^{L}\frac{-r_{k,+}+r_{k,-}}{u^k} + C^{[2]}U_1\right)$$

Solution is expressed in terms of polynomials, rational functions and generalized Hurwitz functions

$$\eta_{a_1,a_2,\ldots,a_k}(u) = \sum_{n_k > n_{k-1} > \cdots > n_1 \geq 0}\ \prod_{i=1}^{k}\frac{(\mathrm{sgn}(a_i))^{n_i - n_{i-1} - 1}}{(u + i n_i)^{|a_i|}},$$

Solution of Baxter equations in Mellin space

Studies with Mellin space techniques:

Faddeev, Korchemsky, 1995

Kotikov, Rej, Zieme, 2008

Beccaria, Belitsky, Kotikov, Zieme, 2010

Lee, Onishchenko, 2017

Baxter equations for twist $L = 1$:

$$(u + i/2)q_1^{(k)}(u + i) - i(2S + 1)q_1^{(k)}(u) - (u - i/2)q_1^{(k)}(u - i) = V_1^{(k)} ,$$
$$(u + i/2)q_2^{(k)}(u + i) + i(2S + 1)q_2^{(k)}(u) - (u - i/2)q_2^{(k)}(u - i) = V_2^{(k)} .$$

$$q_1^{(k)}(u) = \nu_1^{(k)[1]}(u) , \quad q_2^{(k)}(u) = \nu_2^{(k)[1]}(u)$$

As coefficients are linear functions, then Mellin transform will result in a first order differential equation!

Solution of Baxter equations in Mellin space

At four loop order we obtained

$$\gamma(S) = \gamma^{(0)}(S)h^2 + \gamma^{(1)}(S)h^4 + \dots$$

where

$$\gamma^{(0)}(S) = 4\left(\bar{H}_1 + \bar{H}_{-1} - 2\bar{H}_i\right)$$

$$\begin{aligned}\gamma^{(1)}(S) = 16\Big\{ & 3\bar{H}_{-2,-1} - 2\bar{H}_{-2,i} - \bar{H}_{-2,1} - \bar{H}_{-1,-2} + 2\bar{H}_{-1,2i} - \bar{H}_{-1,2} - 6\bar{H}_{i,-2} \\ & + 12\bar{H}_{i,2i} - 6\bar{H}_{i,2} - 6\bar{H}_{2i,-1} + 4\bar{H}_{2i,i} + 2\bar{H}_{2i,1} - \bar{H}_{1,-2} + 2\bar{H}_{1,2i} - \bar{H}_{1,2} + 3\bar{H}_{2,-1} \\ & - 2\bar{H}_{2,i} - \bar{H}_{2,1} + 2\bar{H}_{-1,i,-1} - 2\bar{H}_{-1,i,1} + 8\bar{H}_{i,-1,-1} - 12\bar{H}_{i,-1,i} + 4\bar{H}_{i,-1,1} - 16\bar{H}_{i,i,-1} \\ & + 16\bar{H}_{i,i,i} + 4\bar{H}_{i,1,-1} - 4\bar{H}_{i,1,i} + 2\bar{H}_{1,i,-1} - 2\bar{H}_{1,i,1}\Big\} + 8\left(H_{-1} - H_1\right)\zeta_2\end{aligned}$$

and we introduced new sums

$$H_{a,b,\dots}(S) = \sum_{k=1}^{S} \frac{\Re[(a/|a|)^k]}{k^{|a|}} H_{b,\dots}(k) \quad H_{a,\dots} = H_{a,\dots}(S) \quad \bar{H}_{a,\dots} = H_{a,\dots}(2S)$$

Lee, Onishchenko, 2017

Solution of Baxter equations in u-space

Taking as example first Baxter equation and using ansatz $q_1^{(k)}(u) = Q_S(u) F_S^{(k)}(u + i/2)$ we have

$$-\nabla_- \left(u Q_S^{[1]} Q_S^{[-1]} \nabla_+ F_S^{(k)} \right) = Q_S^{[1]} V_1^{(k)[1]}$$

The solution as before could be written as

$$F^{(k)} = -\Psi_+ \left(\frac{1}{u Q_S^{[1]} Q_S^{[-1]}} \Psi_- \left(Q_S^{[1]} V_1^{(k)[1]} \right) \right) .$$

Now using the nontrivial relation

$$\frac{1}{u Q_S^{[1]} Q_S^{[-1]}} = \frac{(-1)^S}{u} + i(-1)^S \sum_{k=0}^{\left[\frac{S-1}{2}\right]} \frac{1}{S-k} \left(\frac{Q_{S-1-2k}^{[-1]}}{Q_S^{[-1]}} + \frac{Q_{S-1-2k}^{[1]}}{Q_S^{[1]}} \right)$$

and summation by parts we get ($P_S(u) = i \sum_{k=0}^{\left[\frac{S-1}{2}\right]} \frac{1}{S-k} Q_{S-1-2k}(u)$):

$$F_S^{(k)}(u) = -(-1)^S \Psi_+ \left(\frac{1}{u} \Psi_- \left(Q_S V_1^{(k)} \right)^{[1]} \right) + (-1)^S \frac{P_S^{[-1]}}{Q_S^{[-1]}} \Psi_- \left(Q_S V_1^{(k)} \right)^{[-1]} - (-1)^S \Psi_+ \left(P_S V_1^{(k)} \right)^{[-1]}$$

Solution of Baxter equations could be written as:

$$q_1^{(k)} = \mathcal{F}_1^S\left[V_1^{(k)}\right] + Q_S\Phi_1^{per,(k)} + \mathcal{Z}_S\Phi_1^{anti,(k)}$$

$$q_2^{(k)} = \mathcal{F}_2^S\left[V_1^{(k)}\right] + Q_S\Phi_2^{anti,(k)} + \mathcal{Z}_S\Phi_2^{per,(k)}$$

where

$$\mathcal{F}_1^S[f] = -Q_S\Psi_+\left(\frac{1}{u+i/2}\Psi_-\left(Q_S(-1)^S f\right)^{[2]}\right) - Q_S\Psi_+\left(P_S(-1)^S f\right) + P_S\Psi_-\left(Q_S(-1)^S f\right)$$

$$\mathcal{F}_2^S[f] = -Q_S\Psi_-\left(\frac{1}{u+i/2}\Psi_+\left(Q_S(-1)^S f\right)^{[2]}\right) + Q_S\Psi_-\left(P_S(-1)^S f\right) - P_S\Psi_+\left(Q_S(-1)^S f\right)$$

and

$$P_S(u) = i\sum_{k=0}^{\left[\frac{S-1}{2}\right]}\frac{1}{S-k}Q_{S-1-2k}(u)$$

$$Q_S\left(u\right) = \frac{(-1)^S\Gamma\left(\frac{1}{2}+iu\right)}{S!\Gamma\left(\frac{1}{2}+iu-S\right)}\,{}_2F_1\left(-S,\frac{1}{2}+iu;\frac{1}{2}+iu-S;-1\right),$$

$$\mathcal{Z}_S(u) = i\sigma\sum_{k=0}^{\left\lfloor\frac{S-1}{2}\right\rfloor}\frac{1}{S-k}Q_{S-1-2k}\left(u\right) + \sigma\eta_{-1}(u+i/2)Q_S\left(u\right)$$

Lee, Onishchenko, 2018

Then at four loop order Baxter equations are solved with

$$\mathcal{F}_1^{S_1}[Q_{S_2}] = \frac{i}{2}\frac{Q_{S_2}}{S_1 - S_2}\,, \quad S_1 \neq S_2$$

$$\mathcal{F}_1^{S}[Q_S] = -\frac{1}{2}Q_S\,\eta_1(u+i/2) - \frac{i}{2}\sum_{k=1}^{S}\frac{1+(-1)^k}{k}Q_{S-k}\,,$$

$$\mathcal{F}_2^{S_1}[Q_{S_2}] = -\frac{i}{2}\frac{Q_{S_2}}{S_1+S_2+1}\,,$$

$$\mathcal{F}_2^{S_1}\left[\eta_1^{[1]}Q_{S_2}\right] = \frac{1}{2i(S_1+S_2+1)}\left\{\eta_1^{[1]}Q_{S_2} + \mathcal{F}_2^{S_1}\left[Q_{S_2}^{[2]}\right] + \mathcal{F}_2^{S_1}\left[Q_{S_2}^{[-2]}\right]\right\}.$$

and

$$\frac{Q_S}{u \pm i/2} = \frac{(\mp 1)^S}{u \pm i/2} - 2i\sum_{k=1}^{S}(\pm 1)^{k+1}Q_{S-k}\sum_{l=0}^{k-1}\frac{(-1)^l}{S-l}\,,$$

$$Q_S^{[\pm 2]} = Q_S + 2\sum_{k=1}^{S}(\pm 1)^k Q_{S-k}\,,$$

$$u\,Q_S = \frac{i}{2}(S+1)Q_{S+1} - \frac{i}{2}SQ_{S-1}\,.$$

Lee, Onishchenko, 2018

To automate solutions of Baxter equations it is convenient to introduce W and WQ-sums

$$W_{\tilde{S}}\left(\{S_1,a_1\},\{S_2,a_2\},\ldots,\{S_n,a_n\}\right)=\sum_{\Omega_{\tilde{S}}}\prod_{i=1}^{n} w_{S_i,a_i}(k_i)\,,$$

where

$$\Omega_{\tilde{S}}=\tilde{S}\geq k_1>k_2>\ldots k_n>0$$

and

$$w_{S,\pm n}(k)=\frac{(\pm 1)^k}{(S-k)^n}\,,\;\; w_{S_1,a_1}(k+S_2)=(\text{sign}(a_1))^{S_2}w_{S_1-S_2,a_1}(k)\,,\;\; \widetilde{w}_{S,\pm n}(k)=(\pm 1)^k(S-k)^n\,.$$

WQ sums are defined as

$$WQ_{\tilde{S}}\left(\{S_1,a_1,S_Q\},\{S_2,a_2\},\ldots,\{S_n,a_n\}\right)=\sum_{\Omega_{\tilde{S}}}Q_{S_Q-k_1}\prod_{i=1}^{n} w_{S_i,a_i}(k_i)\,.$$

In terms of WQ-sums the reduction operations take the form ($\tilde{S} < S$)

$$\frac{1}{u \pm i/2} WQ_{\tilde{S}}\left(\{S_1, a_1, S\}, \{S_2, a_2\}, \ldots\right) = \frac{(\mp 1)^S}{u \pm i/2} W_{\tilde{S}}\left(\{S_1, \mp a_1\}, \{S_2, a_2\}, \ldots\right)$$
$$\pm 2i WQ_{\tilde{S}}\left(\{S+1, \mp 1, S\}, \{S_1, \mp a_1\}, \{S_2, a_2\}, \ldots\right)$$
$$\pm 2i WQ_{\tilde{S}}\left(\{0, \pm\infty, S\}, \{S+1, -1\}, \{S_1, \mp a_1\}, \{S_2, a_2\}, \ldots\right)$$
$$+ 2i \sum_{k_1=\tilde{S}+1}^{S} (\pm 1)^{k_1+1} Q_{S-k_1} W_{\tilde{S}}\left(\{S+1, -1\}, \{S_1, \mp a_1\}, \{S_2, a_2\}, \ldots\right)$$
$$+ 2i \sum_{k_1=\tilde{S}+1}^{S} (\pm 1)^{k_1+1} Q_{S-k_1} \sum_{k_2=\tilde{S}+1}^{k_1} \frac{(-1)^{k_2}}{S+1-k_2} W_{\tilde{S}}\left(\{S_1, \mp a_1\}, \{S_2, a_2\}, \ldots\right).$$

$$WQ_{\tilde{S}}^{[\pm 2]}\left(\{S_1, a_1, S\}, \ldots \{S_n, a_n\}\right) = WQ_{\tilde{S}}\left(\{S_1, a_1, S\}, \ldots \{S_n, a_n\}\right)$$
$$+ 2 WQ_{\tilde{S}}\left(\{0, \pm\infty, S\}, \{S_1, \pm a_1\}, \{S_2, a_2\}, \ldots \{S_n, a_n\}\right)$$
$$+ 2 \sum_{n=\tilde{S}+1}^{S} (\pm 1)^n Q_{S-n} W_{\tilde{S}}\left(\{S_1, \pm a_1\}, \{S_2, a_2\}, \ldots \{S_n, a_n\}\right).$$

Dictionary for $\mathcal{F}_1$ and $\mathcal{F}_2 = \mathcal{F}_{-1}$ images

$$\mathcal{B}_{\tilde\sigma}(\zeta_{a,A} Q_{S-k}) = \sigma_a \zeta_{a,A} \mathcal{B}_{\sigma_a \tilde\sigma} Q_{S-k} - \frac{\sigma_a}{(u+i/2)^{|a|-1}} \zeta_A^{[2]} Q_{S-k}^{[2]} - \frac{1}{(u-i/2)^{|a|-1}} \zeta_A Q_{S-k}^{[-2]}\,,$$

where $\mathcal{B}_{\tilde\sigma}(f) = (u+i/2) f^{[2]} - i\tilde\sigma(2S+1) f - (u-i/2) f^{[-2]}\,, \quad \zeta_A \equiv \eta_A^{[1]} = \eta(u+\frac{i}{2})\,.$

Acting with $\mathcal{F}_{\tilde\sigma}$ and using $\mathcal{B}_\sigma(Q_{S-k}) = 2i\left\{\frac{1-\sigma}{2}(2S+1) - k\right\} Q_{S-k}$ we get

$$\sigma_a \zeta_{a,A} Q_S = i(1-\sigma_s\tilde\sigma)(2S+1)\mathcal{F}_{\tilde\sigma}\left[\zeta_{a,A} Q_S\right] - \mathcal{F}_{\tilde\sigma}\left[\frac{1}{(u+i/2)^{|a|-1}} \zeta_A^{[2]} Q_S^{[2]}\right] - \sigma_a \mathcal{F}_{\tilde\sigma}\left[\frac{1}{(u-i/2)^{|a|-1}} \zeta_A Q_S^{[-2]}\right]$$

and ($\tilde S \le S$)

$$\begin{aligned}
&\mathcal{F}_{\tilde\sigma}\left[\zeta_{a,A} W Q_{\tilde S}\left(\{\{S_1,a_1\},S\},\dots,\{S_n,a_n\}\}\right)\right] = \\
&\quad \frac{\sigma_a \zeta_{a,A}}{2i} W Q_{\tilde S}\left(\{\{\frac{1-\sigma_a\tilde\sigma}{2}(2S+1),1\} \otimes \{S_1,a_1\},S\},\dots,\{S_n,a_n\}\right) \\
&+ \frac{1}{2i}\mathcal{F}_{\tilde\sigma}\left[\frac{\zeta_A^{[2]}}{(u+i/2)^{|a|-1}} W Q_{\tilde S}^{[2]}\left(\{\{\frac{1-\sigma_a\tilde\sigma}{2}(2S+1),1\} \otimes \{S_1,a_1\},S\},\dots,\{S_n,a_n\}\right)\right] \\
&+ \frac{\sigma_a}{2i}\mathcal{F}_{\tilde\sigma}\left[\frac{\zeta_A}{(u-i/2)^{|a|-1}} W Q_{\tilde S}^{[-2]}\left(\{\{\frac{1-\sigma_a\tilde\sigma}{2}(2S+1),1\} \otimes \{S_1,a_1\},S\},\dots,\{S_n,a_n\}\right)\right]\,.
\end{aligned}$$

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Some specific $\mathcal{F}_{\pm 1}$ images

$$\mathcal{F}_1\left[\zeta_{-1}Q_S\right] = \frac{i}{2(2S+1)}\zeta_{-1}Q_S - \frac{i}{(2S+1)}\mathcal{F}_1\left[WQ_S(\{\{0,\infty\},S\})\right] + \frac{i}{(2S+1)}\mathcal{F}_1\left[WQ_S(\{\{0,-\infty\},S\})\right]$$

$$\mathcal{F}_1\left[\zeta_1 Q_S\right] = -\frac{1}{2}(\zeta_{1,1}+\zeta_{1,-1})Q_S - \mathcal{F}_1\left[\zeta_1 WQ_S(\{\{0,\infty\},S\})\right] - \mathcal{F}_1\left[\zeta_1 WQ_S(\{\{0,-\infty\},S\})\right] + \mathcal{F}_1\left[\zeta_{-1}WQ_S(\{\{0,\infty\},S\})\right] - \mathcal{F}_1\left[\zeta_{-1}WQ_S(\{\{0,-\infty\},S\})\right] ,$$

$$\begin{aligned}\mathcal{F}_1\left[\zeta_1 WQ_{\tilde{S}}(\{\{S_1,a_1\},S\},\ldots,\{S_n,a_n\})\right] = & -\frac{i}{2}\zeta_1 WQ_{\tilde{S}}(\{\{0,1\}\otimes\{S_1,a_1\},S\},\ldots,\{S_n,a_n\}) \\ & - i\mathcal{F}_1\left[WQ_{\tilde{S}}(\{\{0,\infty\},S\},\{0,1\}\otimes\{S_1,a_1\},\ldots,\{S_n,a_n\})\right] \\ & - i\mathcal{F}_1\left[WQ_{\tilde{S}}(\{\{0,-\infty\},S\},\{0,1\}\otimes\{S_1,-a_1\},\ldots,\{S_n,a_n\})\right] \\ & - iW_{\tilde{S}}(\{0,1\}\otimes\{S_1,a_1\},\ldots,\{S_n,a_n\})\sum_{l=\tilde{S}+1}^{S}\mathcal{F}_1\left[Q_{S-l}\right] \\ & - iW_{\tilde{S}}(\{0,1\}\otimes\{S_1,-a_1\},\ldots,\{S_n,a_n\})\sum_{l=\tilde{S}+1}^{S}(-1)^l\mathcal{F}_1\left[Q_{S-l}\right] ,\end{aligned}$$

At six loops for fixed spin values we get

$$
\begin{aligned}
\gamma^{(2)}(5) &= \frac{16928\zeta(3)}{25} - \frac{749207584}{1771875} + \frac{92912\pi^2}{1575} + \frac{322\pi^4}{75} - \frac{33856}{225}\pi^2\log(2) \\
\gamma^{(2)}(10) &= \frac{10143008\zeta(3)}{11025} - \frac{303562045526159958 4}{143248910889459375} + \frac{4641541857896\pi^2}{173241313245} \\
&\quad + \frac{1126\pi^4}{225} - \frac{20286016\pi^2\log(2)}{99225} \\
\gamma^{(2)}(15) &= \frac{265411493888\zeta(3)}{225450225} - \frac{36242750794665141402652795479 04}{66590160573335764008440671875} \\
&\quad + \frac{3593709256322943648256\pi^2}{984550811209521 80625} + \frac{182144\pi^4}{32175} - \frac{530822987776\pi^2\log(2)}{2029052025} \\
\gamma^{(2)}(20) &= \frac{30827191890924032\zeta(3)}{23520996524025} + \frac{1948857047511423184964102975203491228085647584}{7482144284371332845393775248854242377015625} \\
&\quad + \frac{89737601240282891679060093596 8\pi^2}{1060349671937987064011897268 75} + \frac{62075752\pi^4}{10392525} - \frac{61654383781848064\pi^2\log(2)}{211688968716225}
\end{aligned}
$$

Conclusion and future directions

- Simplification of the obtained expressions in terms of WQ-sums
- An extension to eight loops and above
- An extension to twist 2 operators
- An extension of computational techniques to twisted ABJM and $\mathcal{N} = 4$ theories
- Study of untwisting limits

Thank you for your attention!