Helmholtz International School "Modern Colliders - Theory and Experiment 2018" and Workshop "Calculations for Modern and Future Colliders (CALC2018)"

On the Renormalization in Maximally Supersymmetric Gauge Theories

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Dubna 2018

Introduction

Maximal SYM

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

Characteristics

- Partial or total cancelation of UV divergencies
- First UV divergent diagrams arise on L=6/(D-4)
- Common structure of integrands
- Conformal or dual conformal symmetry

D=4 N=8 Supergravity

- On-shell finite up to 8 loops
- Similar to higher dim SYM

The coupling g^2 has dimension $[g^2] = \dfrac{1}{M^{D-4}}$

The aim: to get all loop exact result at least for the leading divergencies

Z.Bern, L.Dixon 10
J.M.Drummond, J.Henn, G.P.Korchemsky, E.Sokatchev 10
N.Arkani-Hamed 12

Introduction

Spinor-helicity formalism

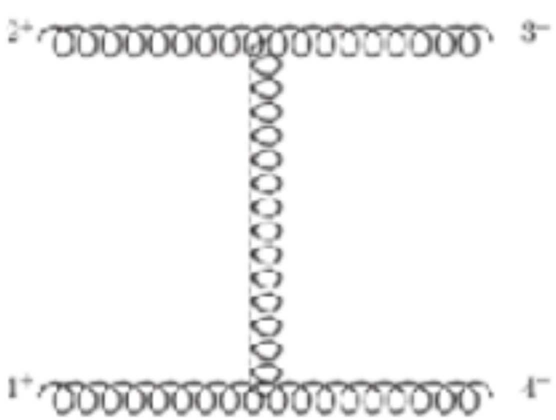
$k_i^\mu \to k_i^\mu (\sigma_\mu)_{\alpha\dot{\alpha}} = (\lambda_i)_\alpha (\tilde{\lambda}_i)_{\dot{\alpha}}$ $\qquad$ $\lambda_\alpha \in SL(2,C)$

$\langle ij\rangle \equiv \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_i)_\beta$ $\qquad$ $[ij] = \langle ij\rangle^*$

$[ij] \equiv \varepsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_i)_{\dot{\beta}}$ $\qquad$ $(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma_\mu)^{\beta\dot{\beta}} = 2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}}$

Lorentz invariant relation: $\langle ij\rangle[ij] = 2k_i k_j \equiv s_{ij}$

The object: 4-point helicity amplitudes on mass shell

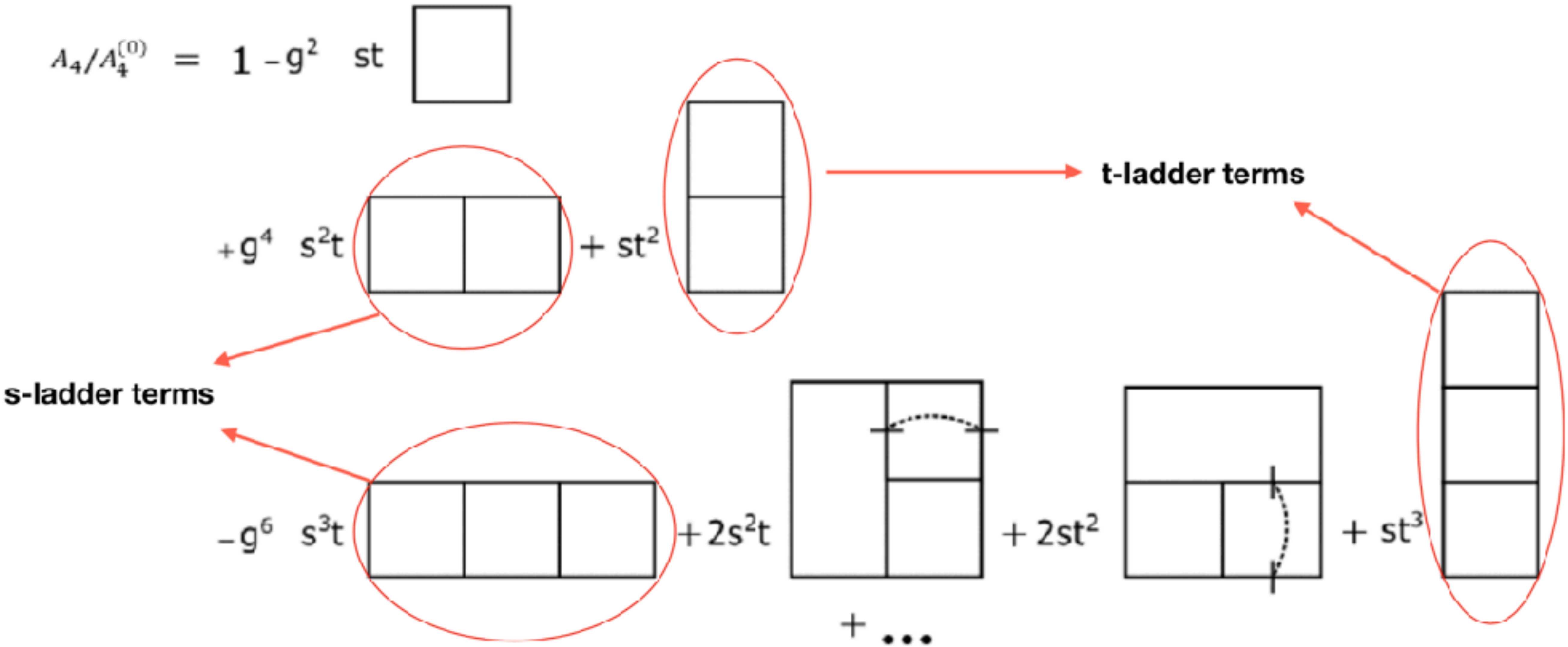


Parke-Taylor formula: $A_n[1^+ \dots i^- \dots j^- \dots n^+] = \dfrac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$

$$\boxed{\left|A_4^{(0)}\right|^2 = A_4^{(0)} A_4^{(0)*} = \frac{s_{34}^3}{s_{12}s_{23}s_{41}} = \frac{s^2}{t^2}}$$

L.Dixon 13
H.Elvang, Y.Huang 13

Introduction



Universal expansion for any D in maximal SYM due to dual conformal invariance

T.Dennen, Y.Huang 10
S.Caron-Huot, D.O'Connell 10

R - operation and Recurrence Relations

- **In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of $1/\epsilon^n$ in n loops is**

$$\mathcal{R}'G = \sum_n \frac{a_n^{(n)}}{\epsilon^n} \qquad a_n^{(n)} = (a_1^{(1)})^n$$

- **In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

$$\mathcal{R}'G = 1 - \sum_\gamma K\mathcal{R}'_\gamma + \sum_{\gamma,\gamma'} K\mathcal{R}'_\gamma K\mathcal{R}'_{\gamma'} - ...,$$

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + ... + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$$+ \frac{B_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + ... + \frac{B_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-1}}$$

$$+\text{lower order terms}$$

Leading pole

SubLeading pole

$A_1^{(n)},\ B_1^{(n)}$ **1-loop graph**

$B_2^{(n)}$ **2-loop graph**

R - operation and Recurrence Relations

- In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation

All terms like $(log\mu^2)^m/\epsilon^k$ should cancel

$$A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n},$$

$$B_n^{(n)} = (-1)^n\left(\frac{2}{n}B_2^{(n)} + \frac{n-2}{n}B_1^{(n)}\right)$$

Leading pole from 1 loop diagrams

SubLeading pole from 2 loop diagrams

$$\mathcal{KR}'G_n = \sum_{k=1}^{n}\left(\frac{A_k^{(n)}}{\epsilon^n} + \frac{B_k^{(n)}}{\epsilon^{n-1}}\right) \equiv \frac{A_n^{(n)'}}{\epsilon^n} + \frac{B_n^{(n)'}}{\epsilon^{n-1}}.$$

$$A_n^{(n)'} = (-1)^{n+1}A_n^{(n)} = \frac{A_1^{(n)}}{n},$$

$$B_n^{(n)'} = \left(\frac{2}{n(n-1)}B_2^{(n)} + \frac{2}{n}B_1^{(n)}\right)$$

Just like in renormalizable theories one can deduce the leading, subleading, etc divergencies from 1, 2, etc loop diagrams

Ladder diagrams (leading order)

D=8 N=1 **Horizontal boxes** $A_n^{(n)} = s^{n-1} A_n$

$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

$A_1 = 1/6$ **1 loop box**

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Summation $\Sigma_m(z) = \sum_{n=m}^{\infty} A_n(-z)^n$

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$$-\frac{d}{dz}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1. \qquad \Sigma_3 = \Sigma_1 + A_1 z - A_2 z^2, \quad \Sigma_2 = \Sigma_1 + A_1 z, \quad A_1 = \frac{1}{3!}, \; A_2 = -\frac{1}{3!4!}$$

$\Sigma_A \equiv \Sigma_1$ **Diff eqn**

$$\boxed{\frac{d}{dz}\Sigma_A = -\frac{1}{3!} + \frac{2}{4!}\Sigma_A - \frac{2}{5!}\Sigma_A^2}$$

$z = g^2 s^2/\epsilon$

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$$\boxed{\Sigma_A(z) = -\sqrt{5/3}\frac{4\tan(z/(8\sqrt{15}))}{1 - \tan(z/(8\sqrt{15}))\sqrt{5/3}} = \sqrt{10}\frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15}) - z_0)}}$$

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$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots) \qquad z_0 = \arcsin(\sqrt{3/8})$$

All Loop Recurrence Relation

D=8 N=1

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$\boxed{\begin{aligned} nS_n(s,t) &= -2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu} \\ &+ s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times \\ &\times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p \\ S_1 = \frac{1}{12},\ T_1 = \frac{1}{12} & \end{aligned}}$$

All Loop Recurrence Relation

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s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

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$$+ \quad s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$

$$S_1 = \frac{1}{12},\ T_1 = \frac{1}{12} \qquad \times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p$$

summation $\Sigma_3(s,t,z) = \Sigma_1(s,t,z) - S_2(s,t)z^2 + S_1(s,t)z,\ \Sigma_2(s,t,z) = \Sigma_1(s,t,z) + S_1(s,t)z$

All Loop Recurrence Relation

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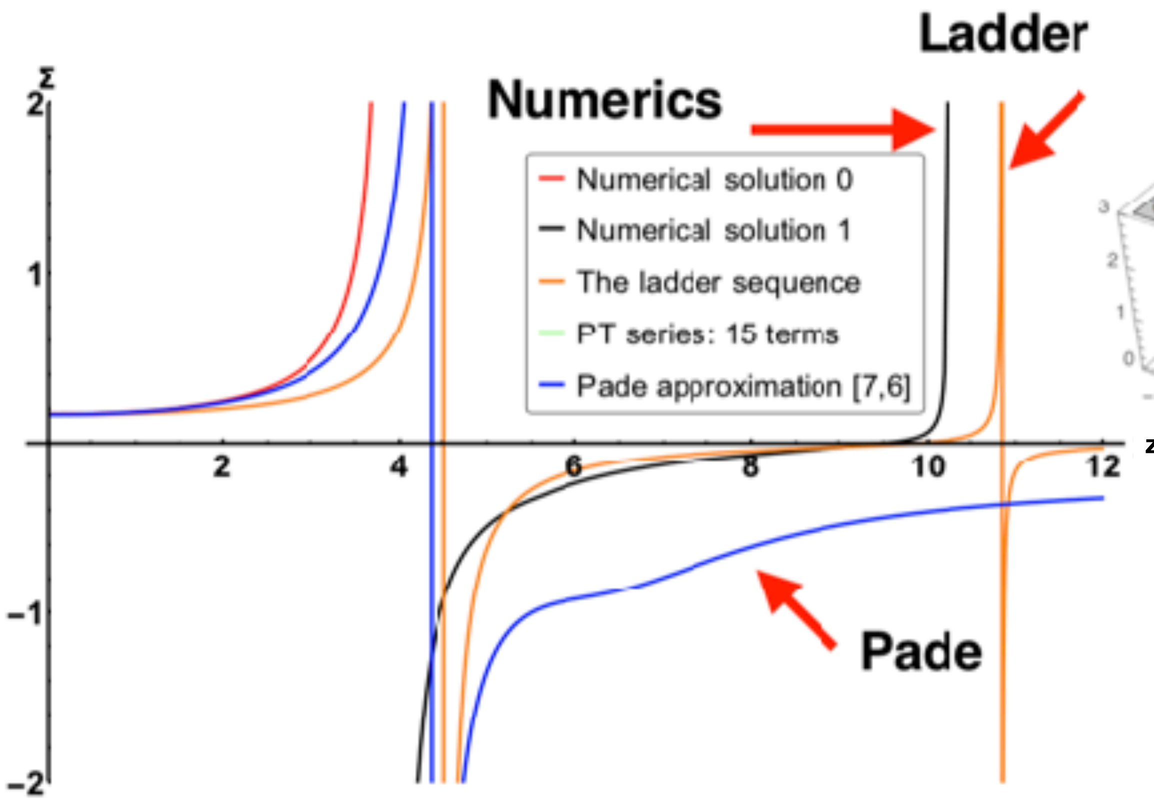
Diff eqn

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=tx+yu}$$

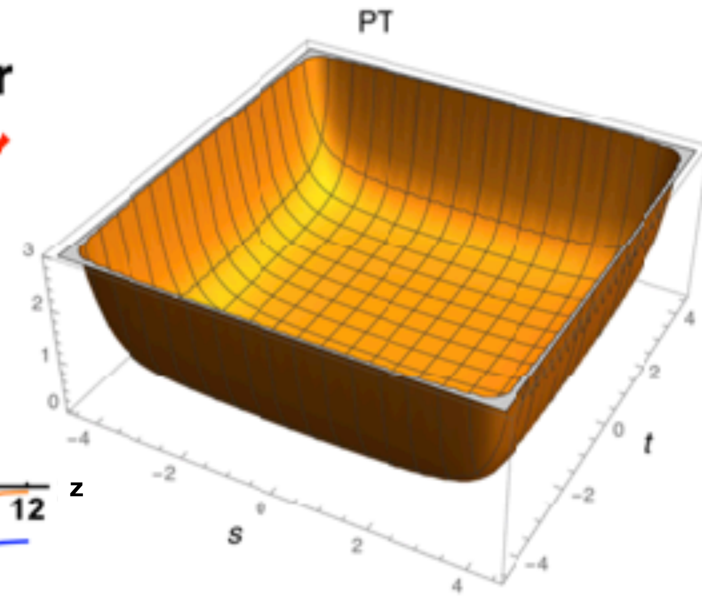
$$-s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!} (\frac{d^p}{dt'^p}(\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=-sx})^2\ (tsx(1-x))^p.$$

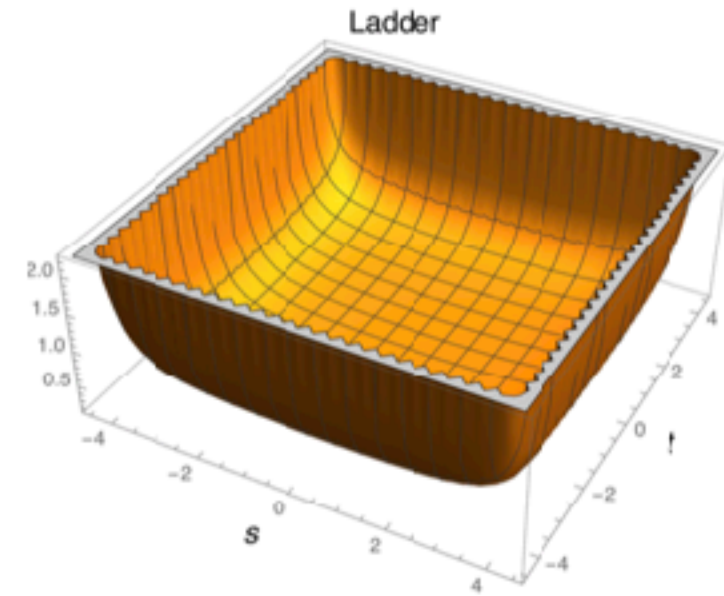
Solutions (leading order)

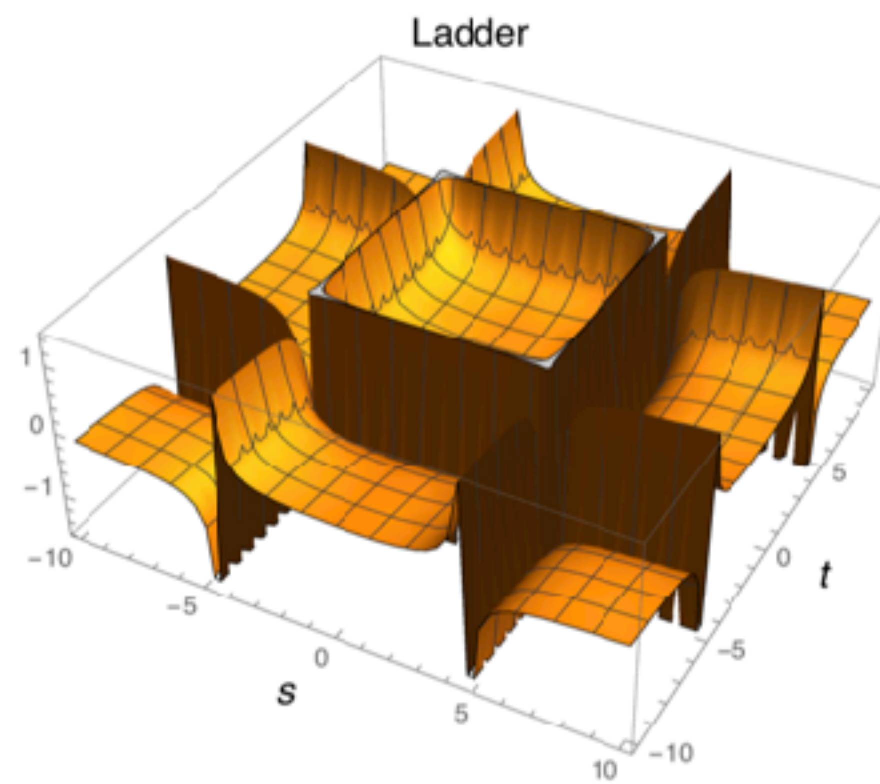
D=8 N=1

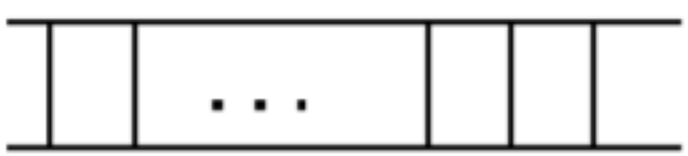


PT and Pade versus ladder for t=s









$$z = \frac{g^2}{\epsilon}$$

$$\boxed{\Sigma_L(s,z) = -\sqrt{5/3}\,\frac{4\tan(zs^2/(8\sqrt{15}))}{1-\tan(zs^2/(8\sqrt{15}))\sqrt{5/3}}}$$

Ladder diagrams (subleading order)

$$\Sigma'_{sB} = \sum_{n=2}^{\infty} z^n B'_{sn}$$

$$\boxed{\frac{d^2\Sigma'_{sB}(z)}{dz^2} + f_1(z)\frac{d\Sigma'_{sB}(z)}{dz} + f_2(z)\Sigma'_{sB}(z) = f_3(z)}$$

Diff eqn

$$f_1(z) = -\frac{1}{6} + \frac{\Sigma_A}{15},$$

$$f_2(z) = \frac{1}{80} - \frac{\Sigma_A}{360} + \frac{\Sigma_A^2}{600} + \frac{1}{15}\frac{d\Sigma_A}{dz},$$

$$f_3(z) = \frac{2321}{5!5!2}\Sigma_A + \frac{11}{1800}\Sigma'_{tB} - \frac{47}{5!45}\Sigma_A^2 - \frac{1}{5!72}\Sigma_A\Sigma'_{tB} + \frac{23}{6750}\Sigma_A^3 + \frac{1}{1200}\Sigma_A^2\Sigma'_{tB}$$
$$-\frac{19}{36}\frac{d\Sigma_A}{dz} - \frac{1}{15}\frac{d\Sigma'_{tB}}{dz} + \frac{23}{225}\frac{d\Sigma_A^2}{dz} + \frac{1}{30}\frac{d(\Sigma_A\Sigma'_{tB})}{dz} - \frac{3}{32}$$

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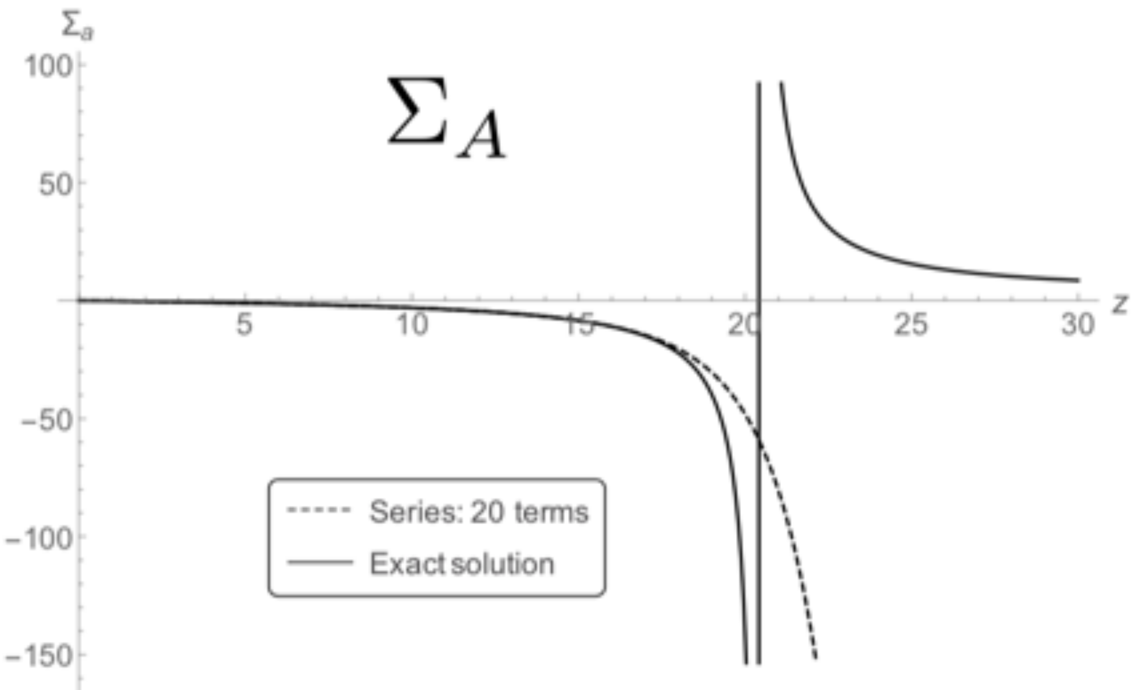
Solution to Diff eqn

smooth monotonic function

$$\Sigma'_{sB}(z) = \frac{d\Sigma_A}{dz}u(z) \qquad u(z) = \int_0^z dy \int_0^y dx \frac{f_3(x)}{d\Sigma_A(x)/dx}$$

Solutions (subleading order)

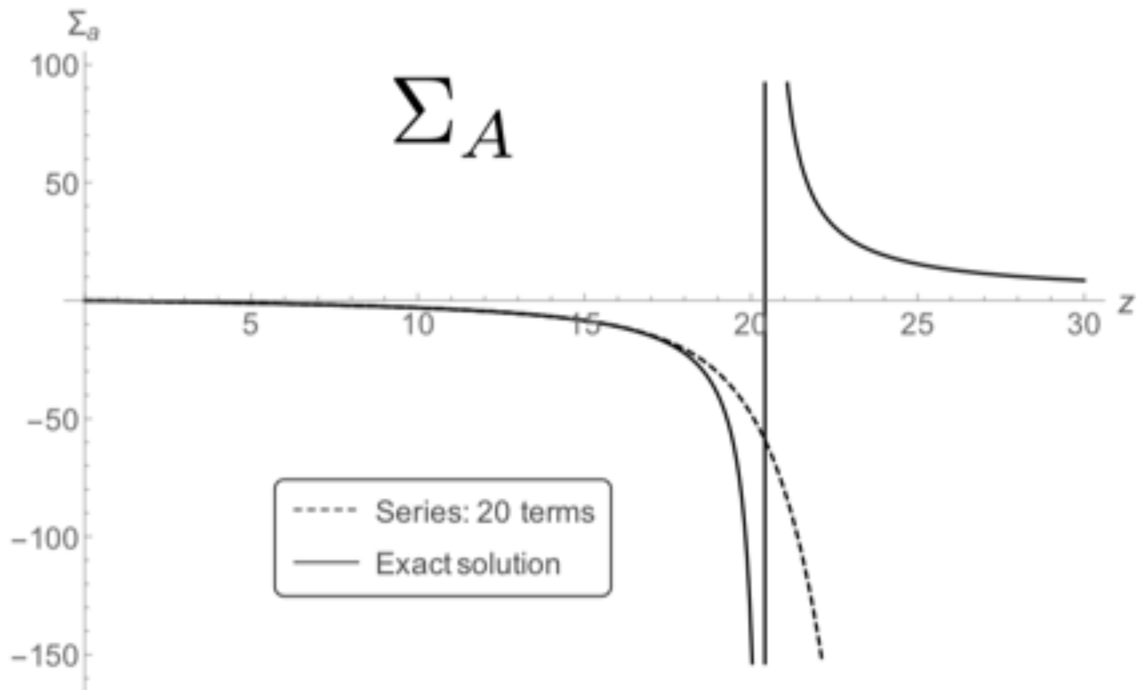
Leading divs



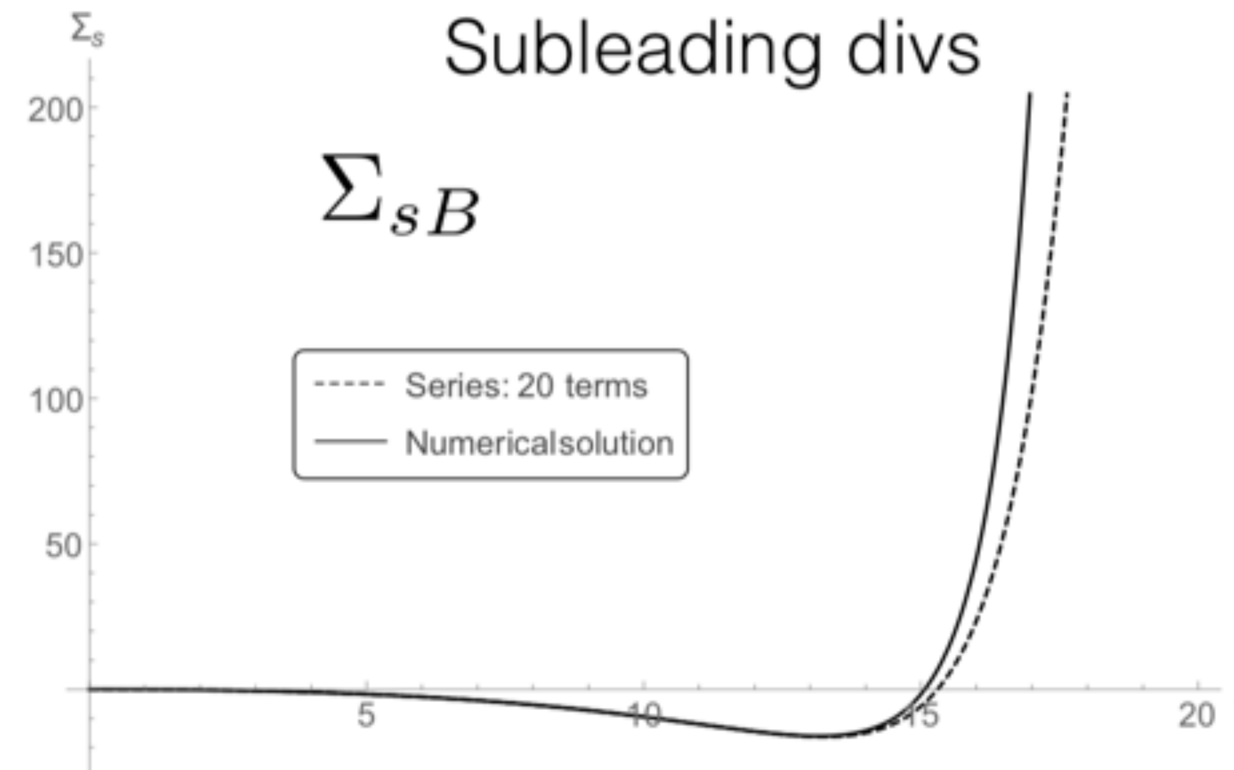
Infinite number of poles

Solutions (subleading order)

Leading divs



Subleading divs



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Solutions (subleading order)

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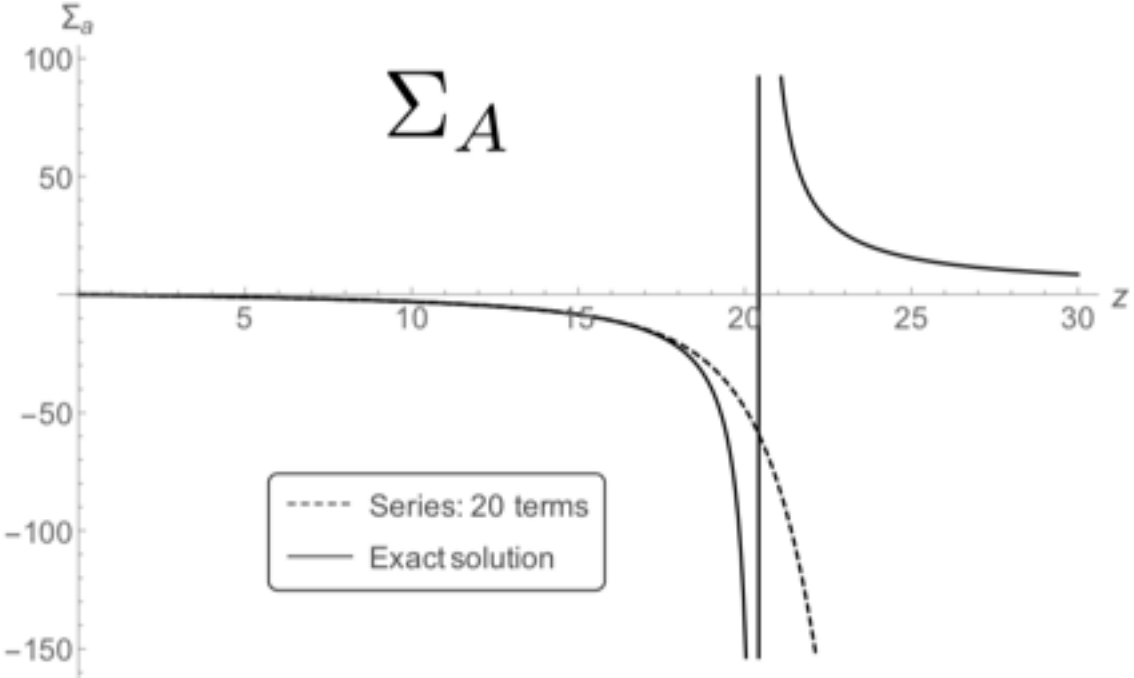
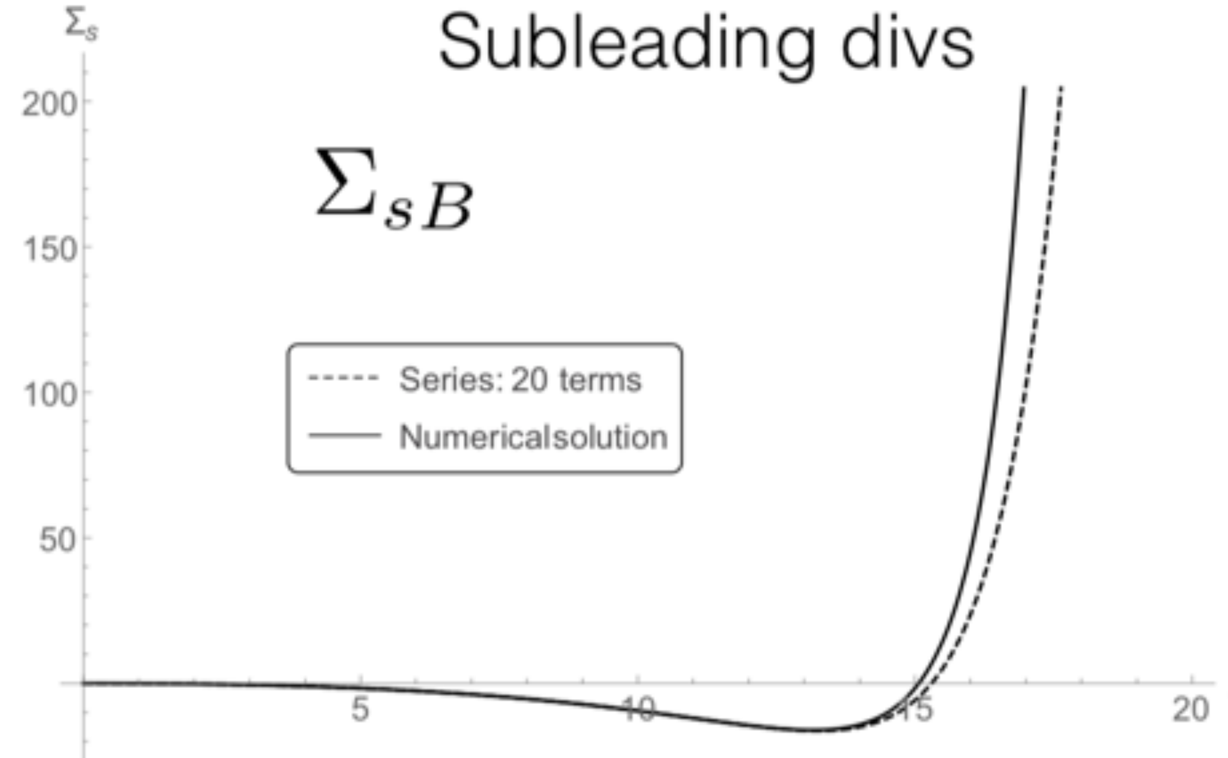


Subleading divs



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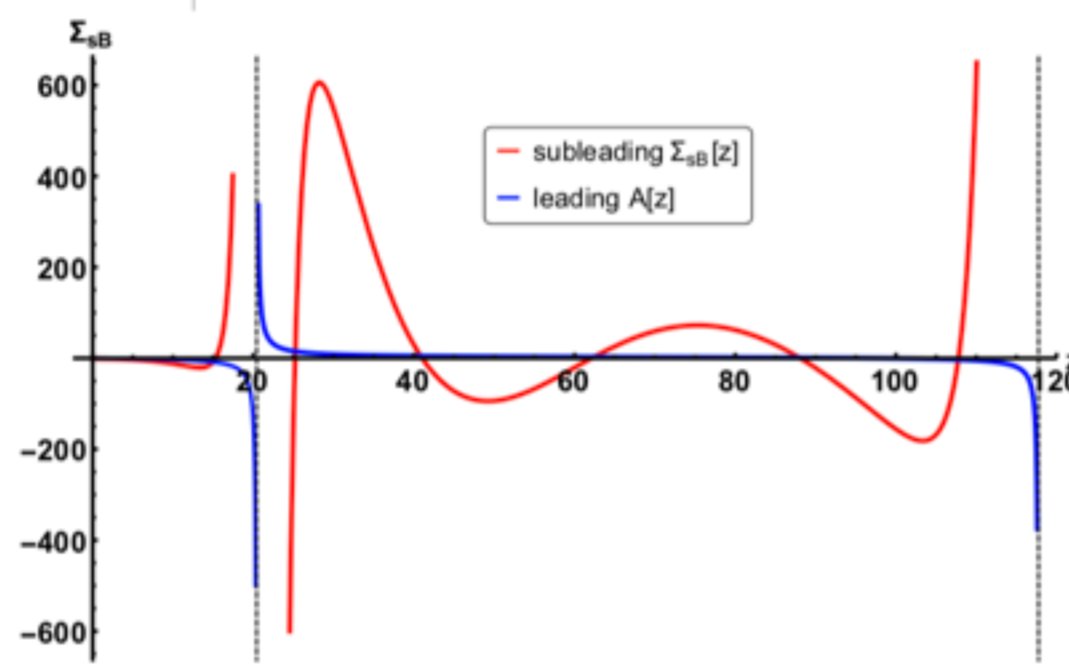


Infinite number of poles at the same position

Scheme dependence

subleading case

$$A'_1 + B'_{s1} = \frac{1}{6\epsilon}(1 + c_1\epsilon) \qquad \Delta\Sigma'_{sB} = c_1 z \frac{d\Sigma'_A}{dz}. \quad \longrightarrow \quad z \to z(1 + c_1\epsilon).$$

sub-subleading case

$$A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2}\left(1 - \frac{5}{12}\epsilon + 2c_1\epsilon + c_2\epsilon^2\right) \qquad \Delta\Sigma'_{sC} = c_2 z^2 \frac{d\Sigma'_A}{dz}.$$

$$\longrightarrow \quad z \to z(1 + c_1\epsilon) + z^2 c_2 \epsilon^2.$$

$$\Delta\Sigma'_{sC} = -c_1^2 \frac{z}{4!}\left(\frac{d\Sigma'_A}{dz} - 12\frac{d^2\Sigma'_A}{dz^2}\right) \quad \longrightarrow \quad z \to z(1 + c_1\epsilon) + z^2(c_2 + c_1^2/4!)\epsilon^2$$

Scheme dependence

subleading case

$$A'_1 + B'_{s1} = \frac{1}{6\epsilon}(1 + c_1\epsilon) \qquad \Delta\Sigma'_{sB} = c_1 z \frac{d\Sigma'_A}{dz}. \qquad \longrightarrow \qquad z \to z(1 + c_1\epsilon).$$

sub-subleading case

$$A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2}\left(1 - \frac{5}{12}\epsilon + 2c_1\epsilon + c_2\epsilon^2\right) \qquad \Delta\Sigma'_{sC} = c_2 z^2 \frac{d\Sigma'_A}{dz}.$$

$$\longrightarrow \qquad z \to z(1 + c_1\epsilon) + z^2 c_2 \epsilon^2.$$

$$\Delta\Sigma'_{sC} = -c_1^2 \frac{z}{4!}\left(\frac{d\Sigma'_A}{dz} - 12\frac{d^2\Sigma'_A}{dz^2}\right) \qquad \longrightarrow \qquad z \to z(1 + c_1\epsilon) + z^2(c_2 + c_1^2/4!)\epsilon^2$$

Finally

$$\boxed{z \to z(1 + c_1\epsilon) + z^2\left(c_2 + c_1^2/4! + \frac{569}{3!4!5!}c_1\right)\epsilon^2}$$

Kinematically dependent renormalization

<u>operator</u> kinematically dependent renormalization

at 2 loops

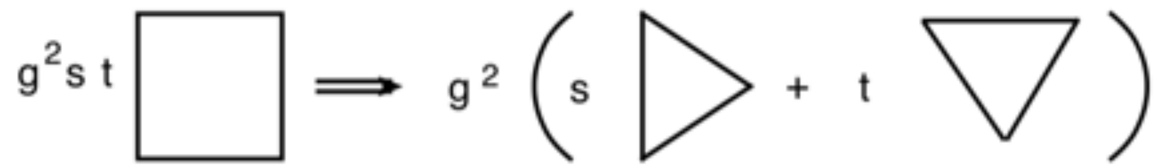
$$\bar{A}_4 = 1 - \frac{g_B^2 st}{3!\epsilon} - \frac{g_B^4 st}{3!4!}\left(\frac{s^2+t^2}{\epsilon^2} + \frac{27/4s^2 + 1/3st + 27/4t^2}{\epsilon}\right) + ...$$

$$\bar{A}_4 = Z_4(g^2)\bar{A}_4^{bare}|_{g^2_{bare} -> g^2 Z_4}$$

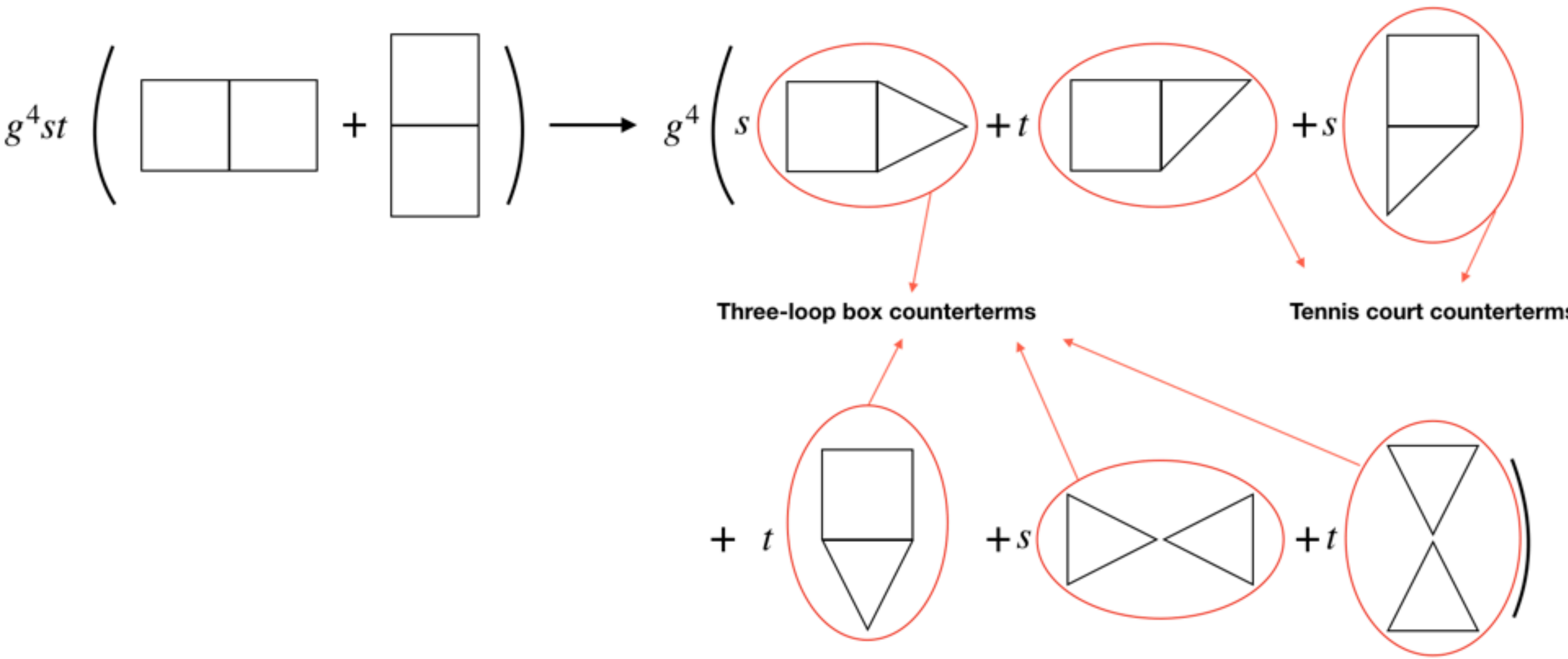
$$Z_4 = 1 + \frac{g^2 st}{3!\epsilon} + \frac{g^4 st}{3!4!}\left(-\frac{s^2+t^2}{\epsilon^2} + \frac{5/12s^2 + 1/3st + 5/12t^2}{\epsilon}\right)$$

$$g_B^2 = g^2(1 + \frac{g^2}{3!\epsilon})$$

Kinematically dependent renormalization



this is operator action!



Conclusions

- The recurrence relations allow to calculate the (sub)leading UV divergences algebraically starting from 1 and 2 loops
- The structure of UV in non-renormalizable theories essentially copies that of renormalizable one
- The main difference from renormalizable theories is that the coupling constant depends on kinematics and acts like operator

JHEP 1612 (2016) 154, arXiv:1610.05549v2 [hep-th] Phys.Rev. D95 (2017) no.4, 045006 arXiv: 1603.05501 [hep-th] arXiv:1712.04348 [hep-th], arXiv:1804.08387 [hep-th]

Thanks for the attention!