Relation between pole and running heavy quark masses in QCD: $\mathcal{O}(a_s^4)$ level and beyond

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Relation between pole and running heavy quark masses in QCD

Consider the $\overline{\mathrm{MS}}$-on-shell heavy quark mass relation:

$$M_q = \overline{m}_q(\overline{m}_q^2) \sum_{n=0}^{\infty} t_n^M a_s^n(\overline{m}_q^2) \ ,$$

where M_q is the pole mass of q-th quark and $\overline{m}_q$ its running $\overline{\mathrm{MS}}$-scheme analogue, $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$ is defined in the $\overline{\mathrm{MS}}$-scheme. Coefficients t_n^M with $1 \leqslant n \leqslant 3$ are calculated in analytical form for gauge color SU(N_c)-group. For case of the SU(3_c)-group with n_l massless flavors ($n_l = n_f - 1$):

$$t_1^M = \frac{4}{3} \ , \textit{(Tarrach, 1981)}$$

$$t_2^M = 13.443 - 1.0414 n_l \ , \textit{(Gray, Broadhurst, 1990; Avdeev, Kalmykov, 1997)}$$

$$t_3^M = 190.60 - 26.655 n_l + 0.6527 n_l^2 \ , \textit{(Melnikov, Ritbergen, 2000; Chetyrkin, Steinhauser, 2000)} \ .$$

t_4^M-coefficient

Any-order term t_n^M may be expanded in powers of n_l, namely $t_n^M = \sum\limits_{k=0}^{n-1} t_{nk}^M n_l^k$. In particular, the four-loop coefficient t_4^M is presented

$$t_4^M = t_{40}^M + t_{41}^M n_l + t_{42}^M n_l^2 + t_{43}^M n_l^3 \ .$$

In this expression the last two terms are known analytically $t_{42}^M = 43.396$, $t_{43}^M = -0.6781$ *(Lee, Marquard, Smirnov A., Smirnov V., Steinhauser, 2013)*, and the first two are computed numerically by diagram calculations $t_{40}^M = 3567.60 \pm 1.64$, $t_{41}^M = -745.721 \pm 0.040$ *(Marquard, Smirnov A., Smirnov V., Steinhauser, Wellmann, 2016)* and are evaluated by means of the least squares method $t_{40}^M = 3567.60 \pm 1.34$, $t_{41}^M = -745.72 \pm 0.15$ *(Kataev, Molokoedov, 2016)* from data, obtained for t_4^M-coefficient at fixed number of n_l.

Estimates of the multiloop corrections by the ECH-motivated method, defined in the Euclidean region

The effective charges (ECH) motivated method *(Kataev, Starshenko, 1995)* gives possibility to estimate high-order corrections to the mass conversion formula *(Kataev, Kim, 2010)*.
We start from the following dispersion relation for the Euclidean quantity $F(Q^2)$, related to its image $T(s) = \overline{m}_q(s) \sum_{n=0}^{\infty} t_n^M a_s^n(s)$ in the Minkowski region through the Källen-Lehmann type spectral representation *(Chetyrkin, Kniehl, Sirlin, 1997)*:

$$F(Q^2) = Q^2 \int\limits_0^\infty ds \frac{T(s)}{(s+Q^2)^2} = \overline{m}_q(Q^2) \sum_{n=0}^{\infty} f_n^E a_s^n(Q^2) \ .$$

The RG functions

Then taking into account the scale dependence of the $\overline{\text{MS}}$-scheme coupling constant and the running heavy quark masses, defined by the RG equations

$$\beta(\alpha_s) = \mu^2 \frac{\partial}{\partial \mu^2}\left(\frac{\alpha_s(\mu^2)}{\pi}\right) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{\pi}\right)^{i+2} ,$$

$$\gamma_m(\alpha_s) = \mu^2 \frac{\partial}{\partial \mu^2} \log \overline{m}_q(\mu^2) = -\sum_{i=0}^{\infty} \gamma_i \left(\frac{\alpha_s}{\pi}\right)^{i+1} ,$$

we can fix the relations between t_n^M and f_n^E-coefficients.
Nowadays $\beta(\alpha_s)$ and $\gamma_m(\alpha_s)$ are calculated in analytical form at the 5-loop order in the $\overline{\text{MS}}$-scheme *(Baikov, Chetyrkin, Kühn, 2014, 2017)* and independently *(Herzog, Ruijl, Ueda, Vermaseren, Vogt, 2017)*.

Scale dependence of the coupling constant at six-loop level

$$\log \frac{\mu^2}{s} \approx \int\limits_{a_s(\mu^2)}^{a_s(s)} \frac{dx}{\beta_0 x^2 + \beta_1 x^3 + \beta_2 x^4 + \beta_3 x^5 + \beta_4 x^6 + \beta_5 x^7} \ ,$$

$$a_s(s) \approx a_s(\mu^2) + \sum_{n=1}^{6} \theta_n a_s^{n+1}(\mu^2) \ ,$$

$$\theta_1 = \beta_0 l \ , \qquad \theta_2 = \beta_0^2 l^2 + \beta_1 l \ , \qquad \theta_3 = \beta_0^3 l^3 + \frac{5}{2}\beta_0\beta_1 l^2 + \beta_2 l \ ,$$

$$\theta_4 = \beta_0^4 l^4 + \frac{13}{3}\beta_0^2\beta_1 l^3 + \left(3\beta_0\beta_2 + \frac{3}{2}\beta_1^2\right) l^2 + \beta_3 l \ ,$$

$$\theta_5 = \beta_0^5 l^5 + \frac{77}{12}\beta_0^3\beta_1 l^4 + \left(6\beta_0^2\beta_2 + \frac{35}{6}\beta_0\beta_1^2\right) l^3 + \frac{7}{2}\left(\beta_0\beta_3 + \beta_1\beta_2\right) l^2 + \beta_4 l \ ,$$

$$\theta_6 = \beta_0^6 l^6 + \frac{87}{10}\beta_0^4\beta_1 l^5 + \left(10\beta_0^3\beta_2 + \frac{85}{6}\beta_0^2\beta_1^2\right) l^4 + \left(8\beta_0^2\beta_3 + \frac{46}{3}\beta_0\beta_1\beta_2 + \frac{5}{2}\beta_1^3\right) l^3$$

$$+ (4\beta_0\beta_4 + 4\beta_1\beta_3 + 2\beta_2^2) l^2 + \beta_5 l \ , \quad \text{where } l = \log(\mu^2/s) \ ,$$

Scale dependence of the running heavy quark masses

$$\frac{\overline{m}_q(s)}{\overline{m}_q(\mu^2)} = \exp\left(\int_{a_s(\mu^2)}^{a_s(s)} dx\, \frac{\gamma_m(x)}{\beta(x)}\right) = 1 + \sum_{n=1}^{6} b_n a_s^n(\mu^2)\ ,$$

$$b_1 = \gamma_0 l\ , \qquad b_2 = \frac{\gamma_0}{2}(\beta_0+\gamma_0)l^2 + \gamma_1 l\ ,$$

$$b_3 = \frac{\gamma_0}{3}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)l^3 + \left(\beta_1\frac{\gamma_0}{2} + \gamma_1\beta_0 + \gamma_1\gamma_0\right)l^2 + \gamma_2 l\ ,$$

$$b_4 = \frac{\gamma_0}{4}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right)l^4 + \left(\frac{5}{6}\beta_1\beta_0\gamma_0 + \frac{\beta_1\gamma_0^2}{2}\right.$$
$$\left. + \gamma_1(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\right)l^3 + \left(\beta_2\frac{\gamma_0}{2} + \gamma_1\beta_1 + \frac{\gamma_1^2}{2} + \frac{3}{2}\gamma_2\beta_0 + \gamma_2\gamma_0\right)l^2 + \gamma_3 l\ ,$$

$$b_5 = \frac{\gamma_0}{5}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right)\left(\beta_0+\frac{\gamma_0}{4}\right)l^5 + \left(\gamma_1\beta_0^3 + \frac{13}{12}\gamma_0\beta_1\beta_0^2\right.$$
$$\left. + \frac{13}{12}\gamma_0^2\beta_1\beta_0 + \frac{11}{6}\gamma_0\gamma_1\beta_0^2 + \gamma_0^2\gamma_1\beta_0 + \frac{1}{4}\beta_1\gamma_0^3 + \frac{1}{6}\gamma_1\gamma_0^3\right)l^4 + \left(\gamma_0\beta_2\beta_0 + 2\gamma_0\beta_0\gamma_2\right.$$
$$\left. + \frac{7}{3}\gamma_1\beta_1\beta_0 + \frac{3}{2}\gamma_0\gamma_1\beta_1 + \frac{1}{2}\gamma_0\beta_1^2 + 2\beta_0^2\gamma_2 + \beta_0\gamma_1^2 + \frac{1}{2}\beta_2\gamma_0^2 + \frac{1}{2}\gamma_0\gamma_1^2 + \frac{1}{2}\gamma_2\gamma_0^2\right)l^3$$
$$+ \left(\frac{1}{2}\gamma_0\beta_3 + \gamma_1\beta_2 + \frac{3}{2}\beta_1\gamma_2 + 2\beta_0\gamma_3 + \gamma_1\gamma_2 + \gamma_0\gamma_3\right)l^2 + \gamma_4 l\ ,$$

$$
\begin{aligned}
b_6 = {} & \frac{\gamma_0}{6}(\beta_0+\gamma_0)\left(\beta_0+\frac{\gamma_0}{2}\right)\left(\beta_0+\frac{\gamma_0}{3}\right)\left(\beta_0+\frac{\gamma_0}{4}\right)\left(\beta_0+\frac{\gamma_0}{5}\right)l^6 \\
& + \left(\frac{1}{12}\beta_1\gamma_0^4+\gamma_1\beta_0^4+\frac{1}{24}\gamma_1\gamma_0^4+\frac{5}{3}\beta_0^2\beta_1\gamma_0^2+\frac{35}{24}\beta_0^2\gamma_0^2\gamma_1+\frac{2}{3}\beta_0\beta_1\gamma_0^3\right. \\
& \left. + \frac{77}{60}\beta_0^3\beta_1\gamma_0+\frac{5}{12}\beta_0\gamma_0^3\gamma_1+\frac{25}{12}\beta_0^3\gamma_0\gamma_1\right)l^5 + \left(\frac{1}{4}\beta_2\gamma_0^3+\frac{5}{2}\beta_0^3\gamma_2+\frac{1}{6}\gamma_0^3\gamma_2\right. \\
& + \frac{3}{2}\beta_0^2\gamma_1^2+\frac{5}{8}\beta_1^2\gamma_0^2+\frac{1}{4}\gamma_0^2\gamma_1^2+\frac{35}{24}\beta_0\beta_1^2\gamma_0+\frac{5}{4}\beta_0\beta_2\gamma_0^2+\frac{47}{12}\beta_0^2\beta_1\gamma_1 \\
& \left. + \frac{3}{2}\beta_0^2\beta_2\gamma_0+\frac{5}{4}\beta_0\gamma_0\gamma_1^2+\frac{5}{4}\beta_0\gamma_0^2\gamma_2+\beta_1\gamma_0^2\gamma_1+\frac{37}{12}\beta_0^2\gamma_0\gamma_2+\frac{25}{6}\beta_0\beta_1\gamma_0\gamma_1\right)l^4 \\
& + \left(\beta_1\gamma_1^2+\frac{4}{3}\beta_1^2\gamma_1+\frac{1}{2}\beta_3\gamma_0^2+\frac{10}{3}\beta_0^2\gamma_3+\frac{1}{2}\gamma_0^2\gamma_3+\frac{1}{6}\gamma_1^3+\frac{9}{2}\beta_0\beta_1\gamma_2+\frac{8}{3}\beta_0\beta_2\gamma_1\right. \\
& \left. + \frac{7}{6}\beta_0\beta_3\gamma_0+\frac{7}{6}\beta_1\beta_2\gamma_0+\frac{5}{2}\beta_0\gamma_0\gamma_3+\frac{5}{2}\beta_0\gamma_1\gamma_2+2\beta_1\gamma_0\gamma_2+\frac{3}{2}\beta_2\gamma_0\gamma_1+\gamma_0\gamma_1\gamma_2\right)l^3 \\
& + \left(\frac{1}{2}\gamma_2^2+\frac{3}{2}\beta_2\gamma_2+\frac{5}{2}\beta_0\gamma_4+2\beta_1\gamma_3+\beta_3\gamma_1+\frac{1}{2}\beta_4\gamma_0+\gamma_0\gamma_4+\gamma_1\gamma_3\right)l^2+\gamma_5 l\ .
\end{aligned}
$$

π^2-effects

The integration gives:

$$Q^2 \int_0^\infty ds \frac{\{1; l; l^2; l^3; l^4; l^5; l^6\}}{(s+Q^2)^2} = \Big\{1;\ \mathfrak{L};\ \mathfrak{L}^2 + \frac{\pi^2}{3};\ \mathfrak{L}^3 + \pi^2\mathfrak{L};\ \mathfrak{L}^4 + 2\pi^2\mathfrak{L}^2 + \frac{7\pi^4}{15};$$

$$\mathfrak{L}^5 + \frac{10}{3}\pi^2\mathfrak{L}^3 + \frac{7}{3}\pi^4\mathfrak{L};\ \mathfrak{L}^6 + 5\pi^2\mathfrak{L}^4 + 7\pi^4\mathfrak{L}^2 + \frac{31}{21}\pi^6\Big\},$$

with $l = \log(\mu^2/s)$ and $\mathfrak{L} = \log(\mu^2/Q^2)$. Fixing $\mu^2 = Q^2$ we obtain the relation between the above mentioned coefficients t_n^M and f_n^E with given from integration π^2-effects. This relation can be written as $f_n^E = t_n^M + \Delta_n$ and is presented as:

$$\Delta_0 = 0\,, \qquad \Delta_1 = 0\,, \qquad \Delta_2 = \frac{\pi^2}{6}\gamma_0(\beta_0+\gamma_0)t_0^M\,,$$

$$\Delta_3 = \frac{\pi^2}{3}\left[t_1^M(\beta_0+\gamma_0)\left(\beta_0+\frac{1}{2}\gamma_0\right) + t_0^M\left(\frac{1}{2}\beta_1\gamma_0 + \gamma_1\beta_0 + \gamma_1\gamma_0\right)\right],$$

$$\Delta_4 = \frac{\pi^2}{3}\left[t_2^M\left(3\beta_0^2 + \frac{5}{2}\beta_0\gamma_0 + \frac{1}{2}\gamma_0^2\right) + t_1^M\left(\frac{3}{2}\beta_1\gamma_0 + \frac{5}{2}\beta_1\beta_0 + 2\gamma_1\beta_0 + \gamma_1\gamma_0\right)\right.$$
$$\left. + t_0^M\left(\frac{1}{2}\beta_2\gamma_0 + \gamma_1\beta_1 + \frac{1}{2}\gamma_1^2 + \frac{3}{2}\gamma_2\beta_0 + \gamma_2\gamma_0\right)\right]$$
$$+ \frac{7\pi^4}{60}t_0^M\gamma_0(\beta_0+\gamma_0)\left(\beta_0+\frac{1}{2}\gamma_0\right)\left(\beta_0+\frac{1}{3}\gamma_0\right),$$

π^2-effects

$$
\begin{aligned}
\Delta_5 = \frac{\pi^2}{3}\Bigg[& t_3^M\left(6\beta_0^2+\frac{7}{2}\beta_0\gamma_0+\frac{1}{2}\gamma_0^2\right)+t_2^M\left(7\beta_1\beta_0+3\gamma_1\beta_0+\frac{5}{2}\beta_1\gamma_0+\gamma_1\gamma_0\right)\\
&+t_1^M\left(\frac{3}{2}\beta_1^2+\frac{1}{2}\gamma_1^2+3\beta_2\beta_0+\frac{5}{2}\gamma_2\beta_0+2\beta_1\gamma_1+\frac{3}{2}\beta_2\gamma_0+\gamma_2\gamma_0\right)\\
&+t_0^M\left(\frac{1}{2}\beta_3\gamma_0+\beta_2\gamma_1+\frac{3}{2}\gamma_2\beta_1+2\gamma_3\beta_0+\gamma_1\gamma_2+\gamma_0\gamma_3\right)\Bigg]\\
&+\frac{7\pi^4}{15}\Bigg[t_1^M\left(\beta_0^4+\frac{25}{12}\beta_0^3\gamma_0+\frac{35}{24}\beta_0^2\gamma_0^2+\frac{5}{12}\beta_0\gamma_0^3+\frac{1}{24}\gamma_0^4\right)\\
&+t_0^M\left(\gamma_1\beta_0^3+\frac{13}{12}\gamma_0\beta_1\beta_0^2+\frac{13}{12}\gamma_0^2\beta_0\beta_1+\frac{11}{6}\gamma_0\gamma_1\beta_0^2+\gamma_0^2\beta_0\gamma_1+\frac{1}{4}\beta_1\gamma_0^3+\frac{1}{6}\gamma_1\gamma_0^3\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta_6 = \frac{\pi^2}{3}\Big[& t_4^M\left(10\beta_0^2 + \frac{9}{2}\beta_0\gamma_0 + \frac{1}{2}\gamma_0^2\right) + t_3^M\left(\frac{27}{2}\beta_0\beta_1 + 4\beta_0\gamma_1 + \frac{7}{2}\beta_1\gamma_0 + \gamma_0\gamma_1\right) \\
& + t_2^M\left(8\beta_0\beta_2 + \frac{7}{2}\beta_0\gamma_2 + 3\beta_1\gamma_1 + \frac{5}{2}\beta_2\gamma_0 + 4\beta_1^2 + \frac{1}{2}\gamma_1^2 + \gamma_0\gamma_2\right) \\
& + t_1^M\left(\frac{7}{2}\beta_0\beta_3 + \frac{7}{2}\beta_1\beta_2 + 3\beta_0\gamma_3 + \frac{5}{2}\beta_1\gamma_2 + 2\beta_2\gamma_1 + \frac{3}{2}\beta_3\gamma_0 + \gamma_0\gamma_3 + \gamma_1\gamma_2\right) \\
& + t_0^M\left(\frac{1}{2}\gamma_2^2 + \frac{3}{2}\beta_2\gamma_2 + \frac{5}{2}\beta_0\gamma_4 + 2\beta_1\gamma_3 + \beta_3\gamma_1 + \frac{1}{2}\beta_4\gamma_0 + \gamma_0\gamma_4 + \gamma_1\gamma_3\right)\Big] \\
& + \frac{7\pi^4}{15}\Big[t_2^M\left(5\beta_0^4 + \frac{77}{12}\beta_0^3\gamma_0 + \frac{71}{24}\beta_0^2\gamma_0^2 + \frac{7}{12}\beta_0\gamma_0^3 + \frac{1}{24}\gamma_0^4\right) + t_1^M\Big(\frac{77}{12}\beta_0^3\beta_1 \\
& + \frac{5}{12}\beta_1\gamma_0^3 + 4\beta_0^3\gamma_1 + \frac{1}{6}\gamma_0^3\gamma_1 + \frac{10}{3}\beta_0\beta_1\gamma_0^2 + \frac{25}{3}\beta_0^2\beta_1\gamma_0 + \frac{3}{2}\beta_0\gamma_0^2\gamma_1 + \frac{13}{3}\beta_0^2\gamma_0\gamma_1\Big) \\
& + t_0^M\Big(\frac{1}{4}\beta_2\gamma_0^3 + \frac{5}{2}\beta_0^3\gamma_2 + \frac{1}{6}\gamma_0^3\gamma_2 + \frac{3}{2}\beta_0^2\gamma_1^2 + \frac{5}{8}\beta_1^2\gamma_0^2 + \frac{1}{4}\gamma_0^2\gamma_1^2 + \frac{35}{24}\beta_0\beta_1^2\gamma_0 \\
& + \frac{5}{4}\beta_0\beta_2\gamma_0^2 + \frac{47}{12}\beta_0^2\beta_1\gamma_1 + \frac{3}{2}\beta_0^2\beta_2\gamma_0 + \frac{5}{4}\beta_0\gamma_0\gamma_1^2 + \frac{5}{4}\beta_0\gamma_0^2\gamma_2 + \beta_1\gamma_0^2\gamma_1 \\
& + \frac{37}{12}\beta_0^2\gamma_0\gamma_2 + \frac{25}{6}\beta_0\beta_1\gamma_0\gamma_1\Big)\Big] \\
& + \frac{31\pi^6}{126}t_0^M\gamma_0(\beta_0+\gamma_0)\left(\beta_0 + \frac{1}{2}\gamma_0\right)\left(\beta_0 + \frac{1}{3}\gamma_0\right)\left(\beta_0 + \frac{1}{4}\gamma_0\right)\left(\beta_0 + \frac{1}{5}\gamma_0\right) .
\end{aligned}
$$

Euclidean ECH-motivated approach

For $SU_c(3)$ case we have:

$$\begin{aligned}
\Delta_2 &= 5.89434 - 0.274156 n_l \ , \\
\Delta_3 &= 105.6221 - 10.04477 n_l + 0.198002 n_l^2 \ , \\
\Delta_4 &= 2272.002 - 403.9489 n_l + 20.67673 n_l^2 - 0.315898 n_l^3 \ , \\
\Delta_5 &= 56304.639 - 13767.2725 n_l + 1137.17794 n_l^2 - 37.745285 n_l^3 + 0.427523 n_l^4 \ , \\
\Delta_6 &= 1633115.62 \pm 347.65 + (-518511.694 \pm 56.723) n_l + (61128.1666 \pm 4.7791) n_l^2 \\
&+ (-3345.0818 \pm 0.1371) n_l^3 + 85.37937 n_l^4 - 0.818446 n_l^5 .
\end{aligned}$$

Note that the numerical values of these π^2-effects Δ_n are not negligible: they are comparable with t_n^M-coefficients.
The next stage is to determine the effective charge $a_s^{eff}(Q^2)$ for Euclidean quantity $F(Q^2)/\overline{m}_q(Q^2)$:

$$\frac{F(Q^2)}{\overline{m}_q(Q^2)} = f_0^E + f_1^E a_s^{eff}(Q^2) \ , \qquad a_s^{eff}(Q^2) = a_s(Q^2) + \sum_{k=2}^{\infty} \phi_k a_s^k(Q^2) \ ,$$

where terms ϕ_k are equal to $\phi_k = f_k^E / f_1^E$.

Euclidean ECH-motivated approach

After this we can define the ECH β-function for $a_s^{eff}(Q^2)$:

$$\begin{aligned}
\beta_0^{eff} &= \beta_0 \ , \qquad \beta_1^{eff} = \beta_1 \ , \qquad \beta_2^{eff} = \beta_2 - \phi_2\beta_1 + (\phi_3 - \phi_2^2)\beta_0 \ , \\
\beta_3^{eff} &= \beta_3 - 2\phi_2\beta_2 + \phi_2^2\beta_1 + (2\phi_4 - 6\phi_2\phi_3 + 4\phi_2^3)\beta_0 \ , \\
\beta_4^{eff} &= \beta_4 - 3\phi_2\beta_3 + (4\phi_2^2 - \phi_3)\beta_2 + (\phi_4 - 2\phi_2\phi_3)\beta_1 \\
&+ (3\phi_5 - 12\phi_2\phi_4 - 5\phi_3^2 + 28\phi_2^2\phi_3 - 14\phi_2^4)\beta_0 \ , \\
\beta_5^{eff} &= \beta_5 - 4\phi_2\beta_4 + (8\phi_2^2 - 2\phi_3)\beta_3 + (4\phi_2\phi_3 - 8\phi_2^3)\beta_2 \\
&+ (2\phi_5 - 8\phi_2\phi_4 + 16\phi_2^2\phi_3 - 3\phi_3^2 - 6\phi_2^4)\beta_1 \\
&+ (4\phi_6 - 20\phi_2\phi_5 - 16\phi_3\phi_4 + 48\phi_2\phi_3^2 - 120\phi_2^3\phi_3 \\
&+ 56\phi_2^2\phi_4 + 48\phi_2^5)\beta_0 \ .
\end{aligned}$$

The essence of evaluation

If we put $\beta_2^{eff} \approx \beta_2$ then we would obtain that $f_3^E \approx (f_2^E)^2/f_1^E + f_2^E\beta_1/\beta_0$ and using the relation $t_3^M = f_3^E - \Delta_3$ we would restore the value of t_3^M-term. Similarly, supposing that $\beta_3^{eff} \approx \beta_3$ we could estimate the value of the four-loop contribution f_4^E and then $t_4^M = f_4^E - \Delta_4$:

n_l	$t_3^{M,\, exact}$	$t_3^{M,\, ECH}$	$t_4^{M,\, exact}$	$t_4^{M,\, ECH}$
3	116.494	124	1702.70	1281
4	94.418	98	1235.66	986
5	73.637	74	839.14	719
6	54.161	52	509.07	483
7	35.991	32	241.37	279
8	19.126	15	31.99	111

The essence of evaluation

Therefore, we have reason to believe that conditions $\beta_4^{eff} \approx \beta_4$, $\beta_5^{eff} \approx \beta_5$ and $t_5^M = f_5^E - \Delta_5$, $t_6^M = f_6^E - \Delta_6$ allow us to estimate values of t_5^M and t_6^M-terms with satisfactory accuracy.

$$f_5^E \approx \frac{1}{3\beta_0}\left[3f_2^E\beta_3 + \left(f_3^E - 4\frac{(f_2^E)^2}{f_1^E}\right)\beta_2 + \left(2\frac{f_2^E f_3^E}{f_1^E} - f_4^E\right)\beta_1\right] + 4\frac{f_2^E f_4^E}{f_1^E} + \frac{5}{3}\frac{(f_3^E)^2}{f_1^E} - \frac{28}{3}f_3^E\left(\frac{f_2^E}{f_1^E}\right)^2 + \frac{14}{3}\frac{(f_2^E)^4}{(f_1^E)^3}\ ,$$

$$f_6^E \approx \frac{1}{4\beta_0}\left[4f_2^E\beta_4 + \left(2f_3^E - 8\frac{(f_2^E)^2}{f_1^E}\right)\beta_3 + \left(8\frac{(f_2^E)^3}{(f_1^E)^2} - 4\frac{f_2^E f_3^E}{f_1^E}\right)\beta_2 + \left(6\frac{(f_2^E)^4}{(f_1^E)^3} + 3\frac{(f_3^E)^2}{f_1^E} + 8\frac{f_2^E f_4^E}{f_1^E} - 16f_3^E\left(\frac{f_2^E}{f_1^E}\right)^2 - 2f_5^E\right)\beta_1\right] + 5\frac{f_2^E f_5^E}{f_1^E} + 4\frac{f_3^E f_4^E}{f_1^E} + 30f_3^E\left(\frac{f_2^E}{f_1^E}\right)^3 - 12f_2^E\left(\frac{f_3^E}{f_1^E}\right)^2 - 12\frac{(f_2^E)^5}{(f_1^E)^4} - 14f_4^E\left(\frac{f_2^E}{f_1^E}\right)^2 .$$

ECH-motivated method in the Minkowskian region

Repeating in part the foregoing reasoning for quantity $T(s)/\overline{m}_q(s)$, defined in the Minkowskian region, we obtain:

$$t_5^{M,\,ECH\ direct} \approx \frac{1}{3\beta_0(t_1^M)^3}\Big[3t_2^M(t_1^M)^3\beta_3 + t_3^M(t_1^M)^3\beta_2 - 4(t_2^M t_1^M)^2\beta_2 + 2t_3^M t_2^M(t_1^M)^2\beta_1 - t_4^M(t_1^M)^3\beta_1 + 12t_4^M t_2^M(t_1^M)^2\beta_0 + 5(t_3^M t_1^M)^2\beta_0 + 14(t_2^M)^4\beta_0 - 28t_3^M(t_2^M)^2 t_1^M\beta_0\Big],$$

$$t_6^{M,\,ECH\ direct} \approx \frac{1}{12\beta_0^2(t_1^M)^4}\Big[48t_4^M t_3^M(t_1^M)^3\beta_0^2 + 72t_4^M(t_1^M t_2^M)^2\beta_0^2 + 136(t_2^M)^5\beta_0^2 - 200t_3^M t_1^M(t_2^M)^3\beta_0^2 - 20t_4^M t_2^M(t_1^M)^3\beta_0\beta_1 + 48t_3^M(t_1^M t_2^M)^2\beta_0\beta_1 - 10t_1^M(t_2^M)^4\beta_0\beta_1 - 44t_2^M(t_1^M t_3^M)^2\beta_0^2 + 36(t_1^M)^3(t_2^M)^2\beta_0\beta_3 - 56(t_1^M)^2(t_2^M)^3\beta_0\beta_2 + 2t_4^M(t_1^M)^4\beta_1^2 + 8t_3^M t_2^M(t_1^M)^3\beta_0\beta_2 - 6t_2^M(t_1^M)^4\beta_1\beta_3 - 2t_3^M(t_1^M)^4\beta_1\beta_2 + 6t_3^M(t_1^M)^4\beta_0\beta_3 - (t_1^M)^3(t_3^M)^2\beta_0\beta_1 - 4t_3^M t_2^M(t_1^M)^3\beta_1^2 + 8(t_1^M)^3(t_2^M)^2\beta_1\beta_2 + 12t_2^M(t_1^M)^4\beta_0\beta_4\Big].$$

The renormalon-based analysis

The renormalon dominance hypothesis leads to the following factorial growth of the t_n^M-terms at $\mu^2 = \overline{m}_q^2$ renormalization point *(Beneke, Braun, 94-95)* ,

$$t_n^{M,\, r-n} \xrightarrow{n\to\infty} \pi N_m (2\beta_0)^{n-1} \frac{\Gamma(n+b)}{\Gamma(1+b)} \left(1 + \sum_{k=1}^{3} \frac{s_k}{(n+b-1)\dots(n+b-k)} + \mathcal{O}(n^{-4})\right) ,$$

with $b = \beta_1/(2\beta_0^2)$. The normalization factor N_m depends on n_l and n.

$$s_1 = \frac{1}{4\beta_0^4}(\beta_1^2 - \beta_0\beta_2) \ ,$$

$$s_2 = \frac{1}{32\beta_0^8}(\beta_1^4 - 2\beta_1^3\beta_0^2 - 2\beta_1^2\beta_2\beta_0 + 4\beta_1\beta_2\beta_0^3 + \beta_2^2\beta_0^2 - 2\beta_3\beta_0^4) \ ,$$

$$s_3 = \frac{1}{384\beta_0^{12}}(\beta_1^6 - 6\beta_1^5\beta_0^2 + 8\beta_1^4\beta_0^4 - 3\beta_1^4\beta_2\beta_0 + 18\beta_1^3\beta_2\beta_0^3 - 24\beta_1^2\beta_2\beta_0^5 + 6\beta_2\beta_3\beta_0^5 + 3\beta_1^2\beta_2^2\beta_0^2 - 6\beta_1^2\beta_3\beta_0^4 - 12\beta_1\beta_2^2\beta_0^4 + 16\beta_1\beta_3\beta_0^6 - \beta_2^3\beta_0^3 + 8\beta_2^2\beta_0^6 - 8\beta_4\beta_0^7) \ .$$

We use the following four-loop numerical results of N_m for c, b and t-quarks *(Beneke, Marquard, Nason, Steinhauser, 2017)*

n_l	3	4	5
N_m	0.54	0.51	0.46

Numerical results

n_l	$t_5^{M,\ ECH}$	$t_5^{M,\ ECHdirect}$	$t_5^{M,\ r-n}$	$t_6^{M,\ ECH}$	$t_6^{M,\ ECHdirect}$	$t_6^{M,\ r-n}$
3	28435	26871	34048	476522	437146	829993
4	17255	17499	22781	238025	255692	511245
5	9122	10427	13882	90739	133960	283902
6	3490	5320	—	8412	57920	—
7	-127	1871	—	-29701	15798	—
8	-2153	-196	—	-39432	-2184	—

ECH – Euclidean ECH-motivated method; ECH direct – Minkowskian ECH-motivated method; r-n – renormalon-based approach.
Note that renormalon approach with fourth-order values of N_m for interval $3 \leq n_l \leq 8$ does not reproduce the sign-alternating structure of t_n^M-terms (which may indicate the impossibility of applying the fourth-order results N_m for higher-order estimations of t_n^M with big number n_l and the need to take into account additional ambiguities). Therefore we do not consider $6 \leq n_l \leq 8$.
Our five-loop estimates for b-quark are in rather good agreement with results, obtained in the process of global fits to $Q\bar{Q}$ bound states (Mateu, Ortega, 2017).

Numerical results

Using data in Table and expansion $t_n^M = \sum_{k=0}^{n-1} t_{nk}^M n_l^k$ we find:

for five-loop terms:

$$t_5^{M,\,ECH} = 2.5n_l^4 - 136n_l^3 + 2912n_l^2 - 26976n_l + 86620\ ,$$
$$t_5^{M,\,ECHdirect} = 1.2n_l^4 - 77n_l^3 + 1959n_l^2 - 20445n_l + 72557\ .$$

and for six-loop terms:

$$t_6^{M,\,ECH} = -4.9n_l^5 + 352n_l^4 - 9708n_l^3 + 131176n_l^2 - 855342n_l + 2096737\ ,$$
$$t_6^{M,\,ECHdirect} = -2.2n_l^5 + 148n_l^4 - 4561n_l^3 + 71653n_l^2 - 538498n_l + 1519440\ .$$

Asymptotic structure: QCD and QED series

For c, b and t-quarks the $\overline{\text{MS}}$-on-shell mass relations contain significantly growing and strictly sign-constant coefficients ($\overline{a}_s = \alpha_s(\overline{m}_q^2)/\pi$):

$$M_c^{ECH} \approx \overline{m}_c(\overline{m}_c^2)(1 + 1.3333\,\overline{a}_s + 10.318\,\overline{a}_s^2 + 116.49\,\overline{a}_s^3 + (1702.70 \pm 1.41)\,\overline{a}_s^4 + \underline{28435\,\overline{a}_s^5 + 476522\,\overline{a}_s^6})\ ,$$

$$M_b^{ECH} \approx \overline{m}_b(\overline{m}_b^2)(1 + 1.3333\,\overline{a}_s + 9.277\,\overline{a}_s^2 + 94.41\,\overline{a}_s^3 + (1235.66 \pm 1.47)\,\overline{a}_s^4 + \underline{17255\,\overline{a}_s^5 + 238025\,\overline{a}_s^6})\ ,$$

$$M_t^{ECH} \approx \overline{m}_t(\overline{m}_t^2)(1 + 1.3333\,\overline{a}_s + 8.236\,\overline{a}_s^2 + 73.63\,\overline{a}_s^3 + (839.14 \pm 1.54)\,\overline{a}_s^4 + \underline{9122\,\overline{a}_s^5 + 90739\,\overline{a}_s^6})\ .$$

What about QED? Using the $U(1)$-limit of the QCD numerical results with $SU(N_c)$ gauge group (Marquard, Smirnov A., Smirnov V., Steinhauser, Wellmann, 2016) one can obtain following expansions for e, μ and τ-leptons ($\overline{a}_s = \alpha_s(\overline{m}_l^2)/\pi$):

$$M_e \approx \overline{m}_e(\overline{m}_e^2)(1 + \overline{a} + 0.1659\,\overline{a}^2 - 2.1314\,\overline{a}^3 + (7.487 \pm 1.030)\,\overline{a}^4)\ ,$$

$$M_\mu \approx \overline{m}_\mu(\overline{m}_\mu^2)(1 + \overline{a} - 1.3961\,\overline{a}^2 - 0.6460\,\overline{a}^3 + (3.169 \pm 1.045)\,\overline{a}^4)\ ,$$

$$M_\tau \approx \overline{m}_\tau(\overline{m}_\tau^2)(1 + \overline{a} - 2.9582\,\overline{a}^2 + 4.7556\,\overline{a}^3 + (-21.238 \pm 1.090)\,\overline{a}^4)\ .$$

These expressions demonstrate the absence of any sign-constant or sign-alternating structure of QED series for mass relation (unlike sign-alternating series for the anomalous magnetic moment of electron (Aoyama, Kinoshita, Nio, Volkov, 2018))

Numerical results

For numerical studies we use following values of the running masses of c, b and t-quarks, namely $\overline{m}_c(\overline{m}_c^2) = 1.275$ GeV, $\overline{m}_b(\overline{m}_b^2) = 4.180$ GeV, $\overline{m}_t(\overline{m}_t^2) = 164.3$ GeV, and the corresponding values of the $\overline{\text{MS}}$-scheme strong coupling constant, normalized at these running masses, viz $\alpha_s(\overline{m}_c^2) = 0.3947$, $\alpha_s(\overline{m}_b^2) = 0.2256$, $\alpha_s(\overline{m}_t^2) = 0.1085$:

$$\frac{M_c}{1\ \text{GeV}} \approx 1.275 + 0.214 + 0.208 + 0.295 + 0.541$$

$$+ \Big\{ \underbrace{1.135 + 2.389}_{\text{ECH}};\ \underbrace{1.072 + 2.192}_{\text{ECH direct}};\ \underbrace{1.359 + 4.162}_{\text{IRR},\ N_m = 0.54} \Big\},$$

$$\frac{M_b}{1\ \text{GeV}} \approx 4.180 + 0.400 + 0.200 + 0.146 + 0.137$$

$$+ \Big\{ \underbrace{0.137 + 0.137}_{\text{ECH}};\ \underbrace{0.140 + 0.147}_{\text{ECH direct}};\ \underbrace{0.182 + 0.293}_{\text{IRR},\ N_m = 0.51} \Big\},$$

$$\frac{M_t}{1\ \text{GeV}} \approx 164.300 + 7.566 + 1.614 + 0.498 + 0.196$$

$$+ \Big\{ \underbrace{0.074 + 0.025}_{\text{ECH}};\ \underbrace{0.084 + 0.037}_{\text{ECH direct}};\ \underbrace{0.112 + 0.079}_{\text{IRR},\ N_m = 0.46} \Big\}.$$

Manifestation of the asymptotic nature in the PT series for pole mass of t-quark

Neglecting the dependence of the N_m-factor on the order of PT and putting $N_m = 0.46$ we estimate roughly values of multiloop corrections within the IRR-based approach:

$$\frac{M_t}{1\ \mathrm{GeV}} \approx 164.300 + 7.566 + 1.614 + 0.498 + 0.196 + \boxed{0.112 + 0.079}$$
$$+ \boxed{0.066 + 0.064 + 0.071 + 0.088 + \ldots}$$

This estimate procedure permit us to understand approximately from what level of PT the asymptotic behavior of the QCD series for pole mass of t-quark starts to manifest itself. The first traces of this effect can already be observed in the seven order of PT. The eighth and ninth contributions are either comparable or exceed the value of the seventh correction.

Conclusion

- The numerical studies of all considered by us estimate procedures indicate the growth of the five and six-loop corrections to the pole mass of charm-quark.
- The ECH-motivated method with arising π^2-effects of the analytic continuation from the Euclidean to Minkowskian region for b-quark pole mass leads to effect of plateau and the rest two methods outline the increase of these corrections.
- For t-quark the asymptotic nature of the corresponding PT series is not observed even at six-loop level. Therefore the concept of the pole mass of top-quark is applicable up to 6 order of PT for sure.
- It may be interesting to calculate explicitly the leading in n_l terms to the relation between pole and running masses at 5-loops and beyond.

Thank you for your attention!