Effective Field Theories

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Plan

- Classical mechanics
- Heisenberg–Euler effective theory
  - QED: muons
  - QCD: heavy quarks
- Method of regions
- Bloch–Nordsieck effective theory
  - HQET

We don't know all physics up to infinitely high energies (or down to infinitely small distances)

All our theories are effective low-energy (or large-distance) theories

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There is a high energy scale M where an effective theory breaks down. Its Lagrangian describes light particles ($m_i \ll M$) and their interactions at $p_i \ll M$ (distances $\gg 1/M$); physics at distances $\lesssim 1/M$ produces local interactions of these light fields.

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The Lagrangian contains all possible operators (allowed by symmetries). Coefficients of operators of dimension $n + 4$ contain $1/M^n$. If M is much larger than energies we are interested in, we can retain only renormalizable terms (dimension 4), and, maybe, a power correction or two.

EFT in classical mechanics

- Slow motion – characteristic time $1/\omega$
- Fast motion – characteristic time $1/\Omega$

$$\Omega \gg \omega$$

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Average over fast oscillations
Effective Lagrangian describes slow motion
Poincaré, Krylov, Bogoliubov, Kapitza, ...

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$$m\ddot{X} = -\frac{dU}{dX} + \overline{\xi\frac{dF}{dX}} = -\frac{dU}{dX} - \frac{1}{m\Omega^2}\overline{F\frac{dF}{dX}} = -\frac{dU_{\mathrm{eff}}}{dX}$$

$$U_{\mathrm{eff}} = U + \frac{1}{2m\Omega^2}\overline{F^2}$$

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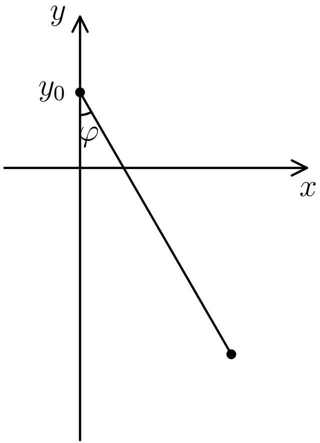
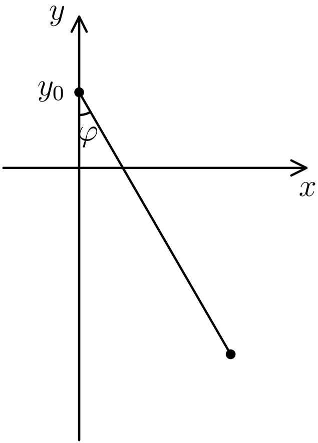
Kapitza pendulum



$$y_0 = a \cos \Omega t$$

Kapitza pendulum



$$y_0 = a\cos\Omega t$$

$$x = l\sin\varphi$$
$$y = a\cos\Omega t - l\cos\varphi$$

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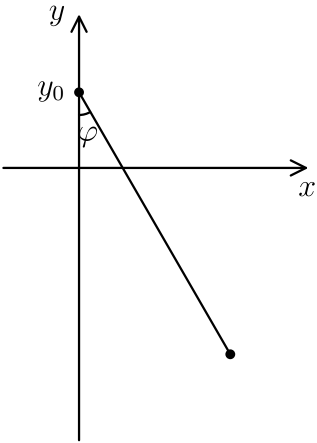

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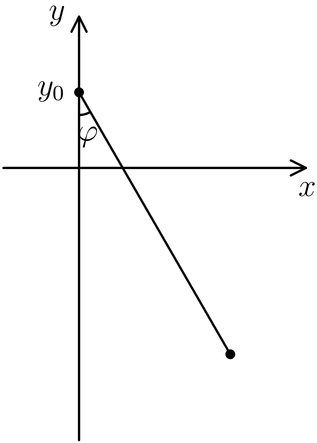

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$$L = \frac{ml^2}{2}\dot{\varphi}^2 - ml a\Omega\sin\Omega t\ \sin\varphi\,\dot{\varphi}$$
$$+ \frac{ma^2\Omega^2}{2}\sin^2\Omega t + mgl\cos\varphi$$

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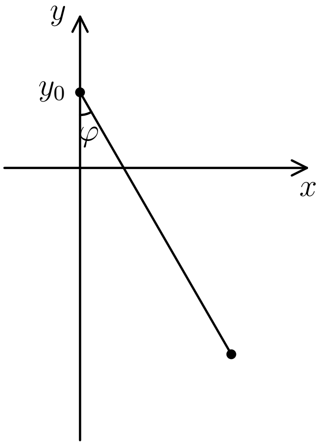

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$$-mla\Omega\sin\Omega t\sin\varphi\,\dot{\varphi}$$
$$= mla\Omega\frac{d}{dt}\left(\sin\Omega t\ \cos\varphi\right)$$
$$-mla\Omega^2\cos\Omega t\cos\varphi$$

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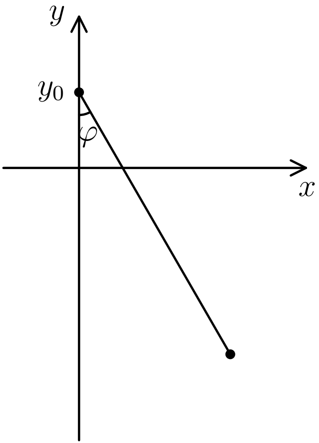

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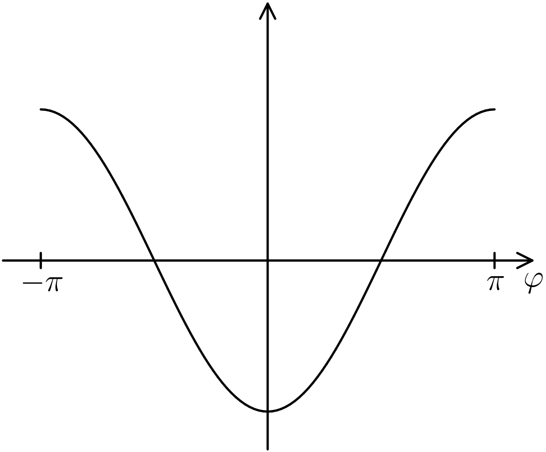
$$L = \frac{ml^2}{2}\dot{\varphi}^2 - mla\Omega^2\cos\Omega t\,\cos\varphi + mgl\cos\varphi$$

Kapitza pendulum

$$\Omega \gg \sqrt{\frac{g}{l}} \qquad \lambda = \frac{a^2\Omega^2}{2gl}$$

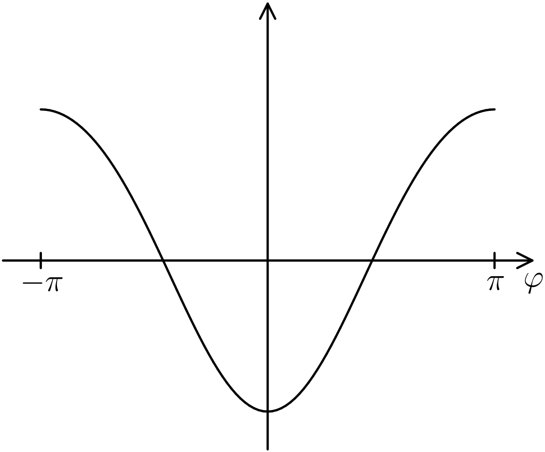
$$U_{\text{eff}} = mgl\left(-\cos\varphi + \frac{\lambda}{2}\sin^2\varphi\right)$$

Kapitza pendulum



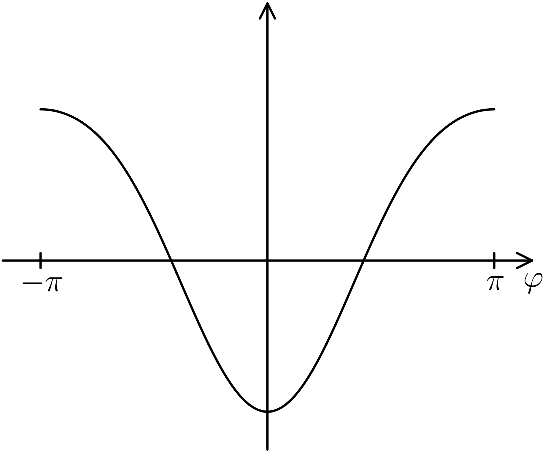
$\lambda = 0$

Kapitza pendulum



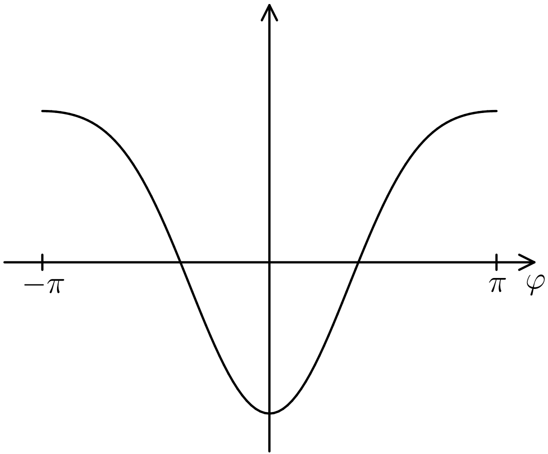
$\lambda = 0.25$

Kapitza pendulum



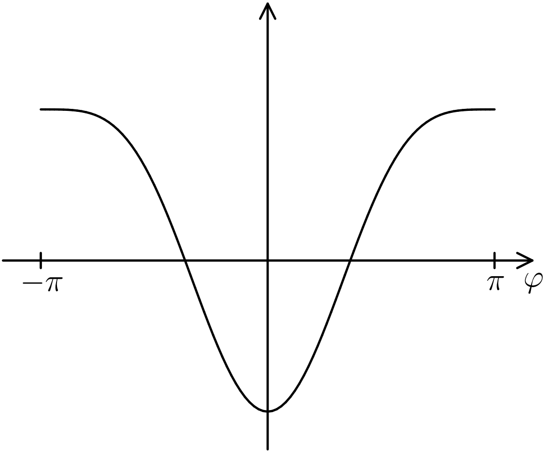
$\lambda = 0.5$

Kapitza pendulum



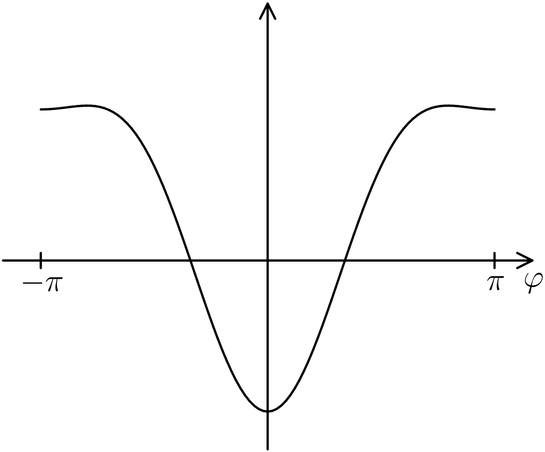
$\lambda = 0.75$

Kapitza pendulum



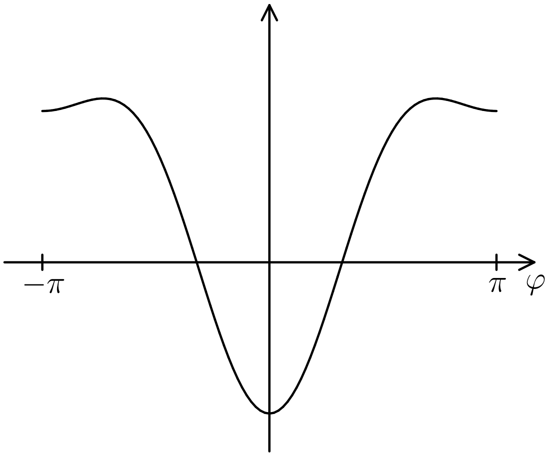
$\lambda = 1$

Kapitza pendulum



$\lambda = 1.25$

Kapitza pendulum



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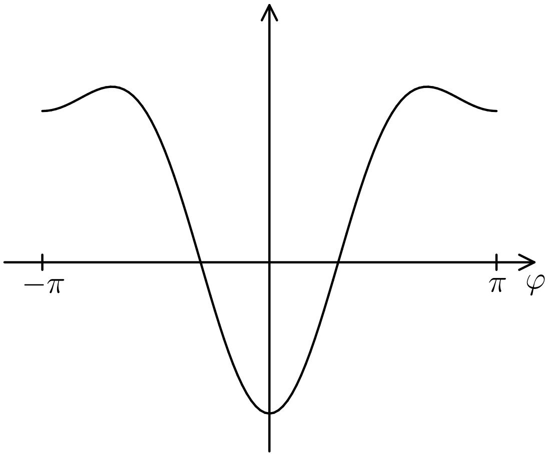

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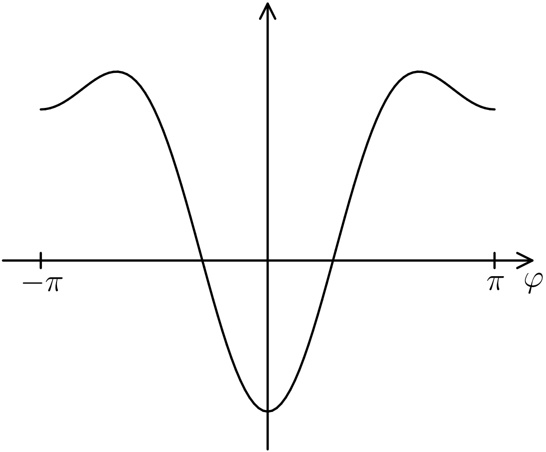
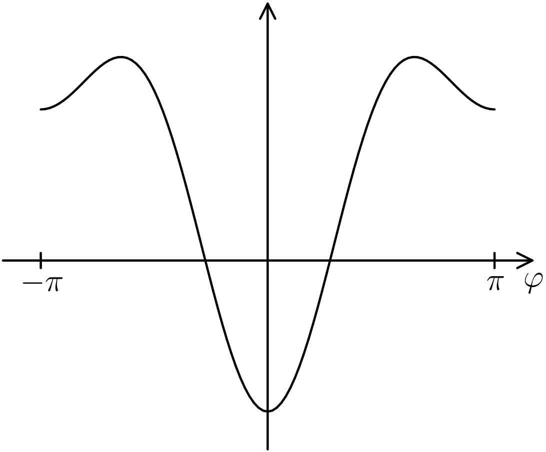


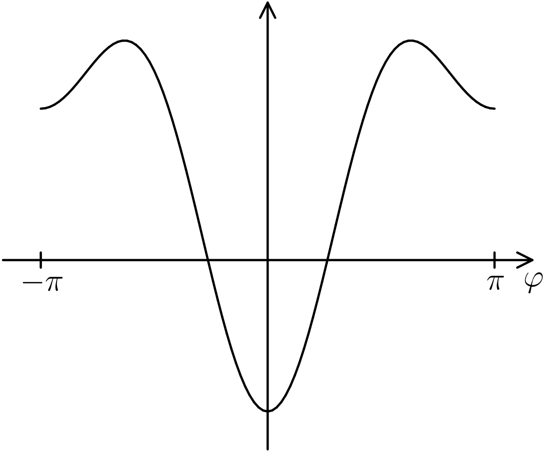
$\lambda = 2$

Kapitza pendulum



$\lambda = 2.25$

Kapitza pendulum



$\lambda = 2.5$

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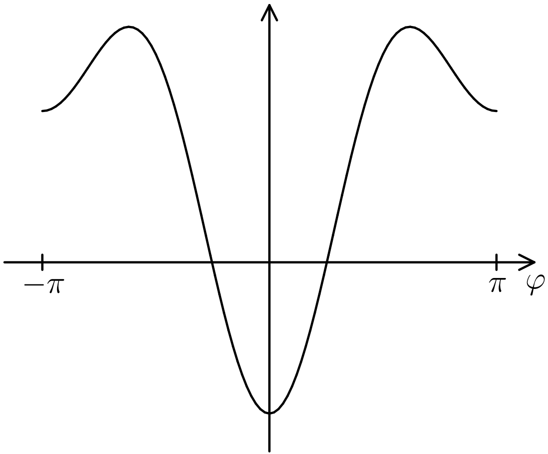

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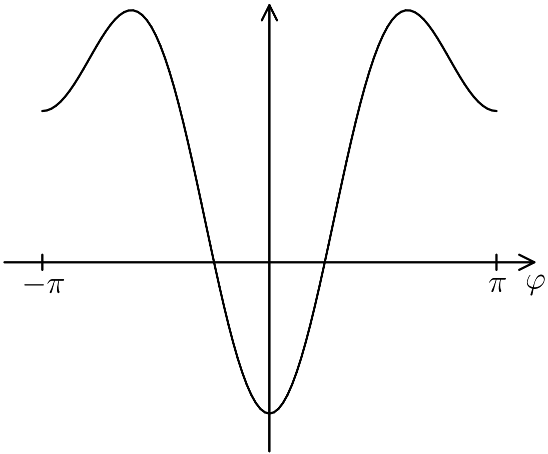

$\lambda = 3$

Photonia

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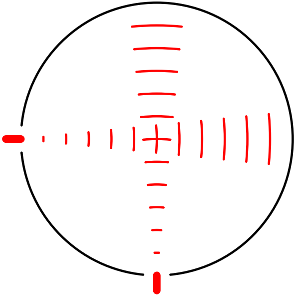
Here physicists have high-intensity sources and excellent detectors of low-energy photons, but they have no electrons and don't know that such a particle exists.

We indignantly refuse to discuss the question "What the experimantalists and their apparata are made of?" as irrelevant.

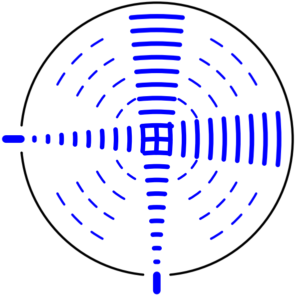
Photonia



Quantum PhotoDynamics (QPD)

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Photonia



Quantum PhotoDynamics (QPD)

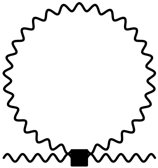
$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + c_1 O_1 + c_2 O_2$$

$$O_1 = (F_{\mu\nu} F^{\mu\nu})^2 \qquad O_2 = F_{\mu\nu} F^{\nu\alpha} F_{\alpha\beta} F^{\beta\mu} \qquad c_{1,2} \sim 1/M^4$$

Photonia

We work at the order $1/M^4$, there can be only 1 4-photon vertex

No corrections to the photon propagator



$= 0$

No renormalization of the photon field

No corrections to the 4-photon vertex
No renormalization of the operators $O_{1,2}$ and the couplings $c_{1,2}$

Qedland

Physicists in the neighboring Qedland are more advanced: in addition to photons, they know electrons and positrons, and investigate their interactions at energies $E \sim M$. They have constructed a nice theory, QED, which describes their experimental results.

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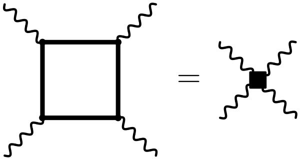
They understand that QPD (constructed in Photonia) is just a low-energy approximation to QED.

Matching

$c_{1,2}$ can be found by matching S-matrix elements



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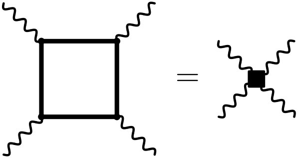

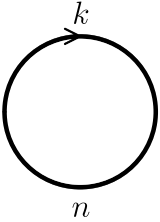

$$= \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} = M^{d-2n} V(n)$$

$$D = M^2 - k^2 - i0$$

$$V(n) = \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}$$

Matching

$$T^{\mu_1\mu_2\nu_1\nu_2} = \frac{e_0^4 M^{-4-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{(d-4)(d-6)}{2880} \times \left(-5T_1^{\mu_1\mu_2\nu_1\nu_2} + 14T_2^{\mu_1\mu_2\nu_1\nu_2}\right)$$

Matching

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Heisengerg–Euler Lagrangian

$$L_1 = \frac{\pi\alpha^2}{180M^4}(-5O_1 + 14O_2)$$

Wilson line

Physicists in Photonia have some classical (infinitely heavy) charged particles and can manipulate them.

$$S_{\text{int}} = e \int_l dx^\mu A_\mu(x)$$

Feynman path integral: $\exp(iS)$ contains

$$W_l = \exp\left(ie \int_l dx^\mu A_\mu(x) \right)$$

The vacuum-to-vacuum transition amplitude is the vacuum average of the Wilson lines

Potential

Charges e and $-e$ stay at some distance $\vec{r}$ during a large time T: the vacuum amplitude $e^{-iU(\vec{r})T}$

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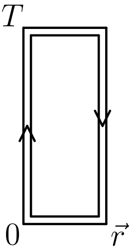
$T \gg r$



$$= e^{-iU(\vec{r})T}$$

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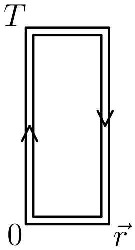


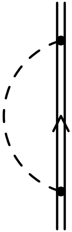
$= e^{-iU(\vec{r})T}$

Coulomb gauge

$$D^{00}(q) = -\frac{1}{\vec{q}^2}$$

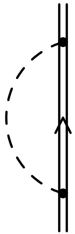
$$D^{ij}(q) = \frac{1}{q^2 + i0}\left(\delta^{ij} - \frac{q^i q^j}{\vec{q}^2}\right)$$

Wilson line



$$= 0$$

Wilson line



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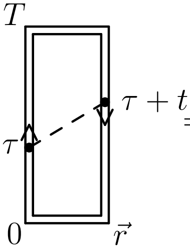

$$= -i\, e^2\, T \int D^{00}(t, \vec{r})\, dt$$

$$= -i\, e^2\, T \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}\, D^{00}(0, \vec{q})\, e^{i\,\vec{q}\cdot\vec{r}}$$

Coulomb potential

$$U(\vec{q}) = e^2 D^{00}(0, \vec{q}) = -\frac{e^2}{\vec{q}^2}$$

$$U(\vec{r}) = -\frac{\alpha}{r}$$

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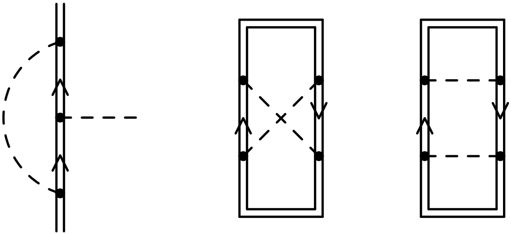

No corrections

Contact interaction

In the presence of sources

$$L_c = c\left(\partial^{\mu} F_{\lambda\mu}\right)\left(\partial_{\nu} F^{\lambda\nu}\right)$$

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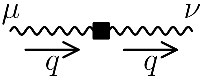

$$= 2icq^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

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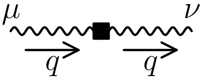

$$= 2icq^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

$$U_c(\vec{r}) = 2c\delta(\vec{r})$$

Qedland

$$D^{00}(\vec{q}) = -\frac{1}{\vec{q}^2}\frac{1}{1-\Pi(-\vec{q}^2)} \qquad U(\vec{q}) = e_0^2 D^{00}(\vec{q})$$

Qedland

$$D^{00}(\vec{q}) = -\frac{1}{\vec{q}^2}\frac{1}{1-\Pi(-\vec{q}^2)} \qquad U(\vec{q}) = e_0^2 D^{00}(\vec{q})$$

In macroscopic measurements $\vec{q} \to 0$

$$U(\vec{q}) \to -\frac{e_0^2}{\vec{q}^2}\frac{1}{1-\Pi(0)} = -\frac{e_{\rm os}^2}{\vec{q}^2}$$

On-shell renormalization scheme

$$e_0 = \left[Z_\alpha^{\rm os}\right]^{1/2} e_{\rm os} \qquad A_0 = \left[Z_A^{\rm os}\right]^{1/2} A_{\rm os}$$

$$D^{00}(\vec{q}) = Z_A^{\rm os} D_{\rm os}^{00}(\vec{q}) \qquad D_{\rm os}^{00}(\vec{q}) \to -\frac{1}{\vec{q}^2}$$

$$Z_\alpha^{\rm os} = \left[Z_A^{\rm os}\right]^{-1} = 1-\Pi(0)$$

$\overline{\text{MS}}$ renormalization scheme

$$e_0 = Z_\alpha^{1/2}(\alpha(\mu))e(\mu) \qquad A_0 = Z_A^{1/2}(\alpha(\mu))A(\mu)$$

$$D^{00}(\vec{q}) = Z_A D^{00}(\vec{q};\mu) \qquad D^{00}(\vec{q};\mu) = \text{finite}$$

$$U(\vec{q}) = e^2(\mu)D^{00}(\vec{q};\mu)Z_\alpha Z_A = \text{finite} \qquad Z_\alpha = Z_A^{-1}$$

$$\frac{\alpha(\mu)}{4\pi} = \frac{e^2(\mu)\,\mu^{-2\varepsilon}}{(4\pi)^{d/2}}e^{-\gamma\varepsilon}$$

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QPD

$$e_0' = e_{\text{os}}' = e'(\mu)$$

Charge decoupling

Macroscopically measured charge is the same in QED and QPD

$$e_{\text{os}} = e'_{\text{os}}$$

$$e_0 = \left[\zeta_\alpha^0\right]^{-1/2} e'_0 \qquad \zeta_\alpha^0 = \left[Z_\alpha^{\text{os}}\right]^{-1}$$

$$e(\mu) = \left[\zeta_\alpha(\mu)\right]^{-1/2} e'(\mu) \qquad \zeta_\alpha(\mu) = Z_\alpha \zeta_\alpha^0 = \frac{Z_\alpha}{Z_\alpha^{\text{os}}}$$

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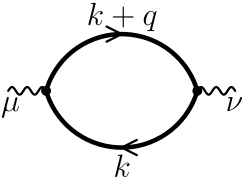
1 loop



$$= i\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) \Pi(q^2)$$

$$\Pi(q^2) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1 - \frac{d-4}{10}\frac{q^2}{M_0^2} + \cdots\right)$$

1 loop

$$[\zeta_\alpha(\mu)]^{-1} = \frac{Z_\alpha^{\text{os}}}{Z_\alpha} = Z_\alpha^{-1}\left[1 - \Pi(0)\right] = \text{finite}$$

$$Z_\alpha = 1 - \beta_0 \frac{\alpha}{4\pi\epsilon} + \cdot$$

1 loop

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$$\beta_0 = -\frac{4}{3}$$

$$[\zeta_\alpha(\mu)]^{-1} = 1 + \frac{4}{3}\left[\left(\frac{\mu}{M(\mu)}\right)^{2\varepsilon} e^{\gamma\epsilon}\Gamma(1+\varepsilon) - 1\right]\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots$$

1 loop

$$[\zeta_\alpha(\mu)]^{-1} = \frac{Z_\alpha^{\rm os}}{Z_\alpha} = Z_\alpha^{-1}\left[1 - \Pi(0)\right] = \text{finite}$$

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$$\to 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi}L \qquad L = 2\log\frac{\mu}{M(\mu)}$$

Electron charge

RG equation

$$\frac{d\,\log \alpha(\mu)}{d\,\log \mu} = -2\beta(\alpha(\mu)) \qquad \beta(\alpha) = \beta_0 \frac{\alpha}{4\pi} + \cdots$$

Initial condition

$$\alpha(M) = \alpha'(M)$$

Electron charge

RG equation

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Initial condition

$$\alpha(M) = \alpha'(M)$$

Contact interaction

$$c = -\frac{2}{15}\frac{\alpha}{4\pi}\frac{1}{M^2}$$

$$U_c(\vec{q}) = -\frac{4}{15}\frac{\alpha^2}{M^2} \qquad U_c(\vec{r}) = -\frac{4}{15}\frac{\alpha^2}{M^2}\delta(\vec{r})$$

Full theory and effective low-energy theory

QED

$$L = \bar{\Psi}_0 \left(i \not{D}_0 - M_0 \right) \Psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} \left(\partial_\mu A_0^\mu \right)^2$$

QPD

$$L' = -\frac{1}{4} F'_{0\mu\nu} F_0'^{\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0'^\mu \right)^2 + \frac{1}{M_0^4} \sum_i C_i^0 O_i'^0 + \cdots$$

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Bare decoupling

$$A_0 = \left[\zeta_A^0 \right]^{-1/2} A'_0 + \cdots$$

$$a_0 = \left[\zeta_A^0 \right]^{-1} a'_0 \qquad e_0 = \left[\zeta_\alpha^0 \right]^{-1/2} e'_0$$

$\overline{\text{MS}}$ renormalization scheme

QED

$$A_0 = Z_A^{1/2}(\alpha(\mu))\, A(\mu)$$

$$a_0 = Z_A(\alpha(\mu))\, a(\mu) \qquad e_0 = Z_\alpha^{1/2}(\alpha(\mu))\, e(\mu)$$

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$$a_0 = Z_A(\alpha(\mu))\, a(\mu) \qquad e_0 = Z_\alpha^{1/2}(\alpha(\mu))\, e(\mu)$$

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon}\frac{e^2(\mu)}{(4\pi)^{d/2}}e^{-\gamma\varepsilon}$$

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$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon}\frac{e^2(\mu)}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}$$

$$D_{\mu\nu}(p) = \frac{1}{p^2\left[1 - \Pi(p^2)\right]}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + a_0 \frac{p_\mu p_\nu}{(p^2)^2}$$

$$Z_A^{-1} D_{\mu\nu}(p) = D_{\mu\nu}(p;\mu)$$

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$$Z_A^{-1}D_{\mu\nu}(p) = D_{\mu\nu}(p;\mu)$$

QPD

$$Z'_A = 1 \qquad Z'_\alpha = 1$$

$\overline{\text{MS}}$ renormalization scheme

Renormalized decoupling

$$A(\mu) = \zeta_A^{-1/2}(\mu) A'(\mu)$$

$$a(\mu) = \zeta_A^{-1}(\mu) a'(\mu) \qquad e(\mu) = \zeta_\alpha^{-1/2}(\mu) e'(\mu)$$

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RG equations

$$\frac{d \log \zeta_A(\mu)}{d \log \mu} = \gamma_A(\alpha(\mu)) - \gamma'_A(\alpha'(\mu))$$

$$\frac{d \log \zeta_\alpha(\mu)}{d \log \mu} = 2 \left[\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))\right]$$

On-shell renormalization scheme

QED

$$A_0 = \left[Z_A^{\rm os}(e_0)\right]^{1/2} A_{\rm os}$$

$$a_0 = Z_A^{\rm os}(e_0)\, a_{\rm os} \qquad e_0 = \left[Z_\alpha^{\rm os}(e_0)\right]^{1/2} e_{\rm os}$$

On-shell renormalization scheme

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$$\text{At } p \to 0 \qquad D_\perp^{\text{os}}(p^2) \to D_\perp^0(p^2) = \frac{1}{p^2}$$

$$Z_A^{\text{os}}(e_0) = \frac{1}{1 - \Pi(0)}$$

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QPD

$$Z_A'^{\text{os}} = 1 \qquad Z_\alpha'^{\text{os}} = 1$$

On-shell renormalization scheme

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QPD

$$Z_A'^{\rm os} = 1 \qquad Z_\alpha'^{\rm os} = 1$$

Photon field decoupling

At $p^2 \to 0$, $D_\perp^{\rm os}(p) = D_\perp'^{\rm os}(p) = D_\perp^0(p)$

$$A^{\rm os} = A'^{\rm os}$$

$$\zeta_A^0(e_0) = \frac{Z_A'^{\rm os}(e_0')}{Z_A^{\rm os}(e_0)} = 1 - \Pi(0)$$

1 loop

$$\zeta_A^0 = \left[\zeta_\alpha^0\right]^{-1} = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

1 loop

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$$= 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \left(\frac{\mu}{M(\mu)}\right)^{2\varepsilon} \Gamma(1+\varepsilon) e^{\gamma\varepsilon}$$

where $M_0 = Z_m(\alpha(\mu))M(\mu)$

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$$\zeta_A(\mu) = Z_A \zeta_A^0 = \text{finite}$$

$$Z_A(\alpha) = Z_\alpha^{-1} = 1 - \frac{4}{3}\frac{\alpha}{4\pi\varepsilon}$$

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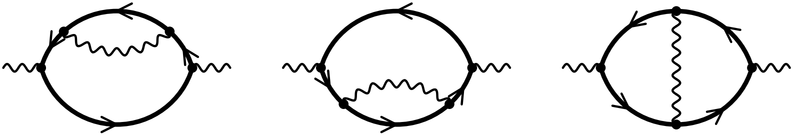
2 loops



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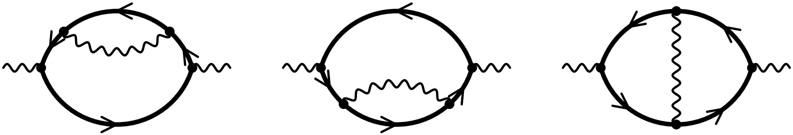

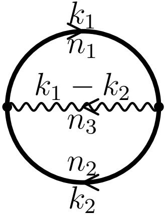

$$\frac{\Gamma\left(\frac{d}{2}-n_3\right)\Gamma\left(n_1+n_3-\frac{d}{2}\right)\Gamma\left(n_2+n_3-\frac{d}{2}\right)\Gamma(n_1+n_2+n_3-d)}{\Gamma\left(\frac{d}{2}\right)\Gamma(n_1)\Gamma(n_2)\Gamma(n_1+n_2+2n_3-d)}$$

A. Vladimirov (1980)

2 loops

$$\zeta_A^0 = \left[\zeta_\alpha^0\right]^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$
$$+ \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2$$

2 loops

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$$= 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon}e^{L\varepsilon}\left(1+\frac{\pi^2}{12}\varepsilon^2+\cdots\right)Z_\alpha(\alpha(\mu))Z_m^{-2\varepsilon}(\alpha(\mu))$$
$$- \varepsilon\left(6-\frac{13}{3}\varepsilon+\cdots\right)\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 e^{2L\varepsilon}$$

2 loops

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$$Z_\alpha = Z_A^{-1} = 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} \qquad Z_m = 1 - 3\frac{\alpha(\mu)}{4\pi\varepsilon}$$

2 loops

$$\zeta_A = Z_A \zeta_A^0 = \text{finite}$$

$$Z_A = Z_\alpha^{-1} = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} - 2\varepsilon\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2$$

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$$\zeta_A(\mu) = \zeta_\alpha^{-1}(\mu) = 1 + \frac{4}{3}\left[L + \left(\frac{L^2}{2} + \frac{\pi^2}{12}\right)\varepsilon\right]\frac{\alpha(\mu)}{4\pi}$$

$$+ \left(-4L + \frac{13}{3}\right)\left(\frac{\alpha(\mu)}{4\pi}\right)^2$$

2 loops

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$$+ \left(-4L + \frac{13}{3}\right)\left(\frac{\alpha(\mu)}{4\pi}\right)^2$$

Define $M(\bar{M}) = \bar{M}$, then $L = 0$

$$\zeta_A(\bar{M}) = \zeta_\alpha^{-1}(\bar{M}) = 1 + \frac{\pi^2}{9}\varepsilon\frac{\alpha(\bar{M})}{4\pi} + \frac{13}{3}\left(\frac{\alpha(\bar{M})}{4\pi}\right)^2$$

2 loops

Alternatively use M_{os}

$$\frac{M(\mu)}{M_{\mathrm{os}}} = 1 - 6\left(\log\frac{\mu}{M_{\mathrm{os}}} + \frac{2}{3}\right)\frac{\alpha}{4\pi} \qquad L = 8\frac{\alpha}{4\pi}$$

$$\zeta_A(M_{\mathrm{os}}) = \zeta_\alpha^{-1}(M_{\mathrm{os}}) = 1 + \frac{\pi^2}{9}\varepsilon\frac{\alpha(M_{\mathrm{os}})}{4\pi} + 15\left(\frac{\alpha(M_{\mathrm{os}})}{4\pi}\right)^2$$

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For any $\mu = M(1+\mathcal{O}(\alpha))$, $\zeta_\alpha = 1 + \mathcal{O}(\varepsilon)\alpha + \mathcal{O}(\alpha^2)$

Qedland

Physicists in Qedland suspect that QED is also a low-energy effective theory. They are right: muons exist (let's suppose that pions don't exist). Two ways to search for new physics:

- increase the energy of e^+e^- colliders to produce pairs of new particles
- performing high-precision experiments at low energies

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- increase the energy of e^+e^- colliders to produce pairs of new particles
- performing high-precision experiments at low energies

We were lucky: the scale of new physics in QED is $M \gg m_e$, loops of heavy particles also suppressed by α^n. μ_e agrees with QED without non-renormalizable corrections to a good precision. Physicists expected the same for proton. No luck here.

Dimension 6

Massless electron. Dimension 5 operator

$$\bar{\psi} F_{\mu\nu} \sigma^{\mu\nu} \psi$$

violates the helicity conservation

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conserve helicity at odd n

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conserve helicity at odd n

$$(\partial_\mu F^{\lambda\mu})(\partial^\nu F_{\lambda\nu}) = \bar{\psi} \partial_\nu F^{\mu\nu} \gamma_\mu \psi = O_1$$

equations of motion

$$\bar{\psi} \partial_\lambda F_{\mu\nu} \gamma^{[\lambda} \gamma^\mu \gamma^{\nu]} \psi = 0$$

Contact interactions

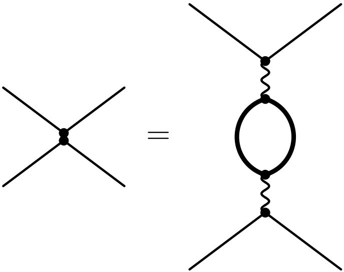
The q^2 term in the muon loop

$$\Delta L = cO \qquad c = -\frac{2}{15}\frac{\alpha}{4\pi}\frac{1}{M^2}+\mathcal{O}(\alpha^2) \qquad O = (\partial^\mu F_{\lambda\mu})(\partial_\nu F^{\lambda\nu})$$

EOM $O = e^2 O_1$

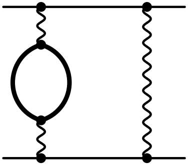
$$c_1(M) = -\frac{2}{15}\frac{\alpha^2(M)+\mathcal{O}(\alpha^3)}{M^2}$$

Matching



$$2ic_1^0 = i\frac{e_0^2}{q^2}\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-4}{10}\frac{q^2}{M_0^2} \Rightarrow -\frac{4}{15}i\frac{e_0^4}{(4\pi)^{d/2}}\frac{1}{M_0^{2+2\varepsilon}}$$

O_3



$$c_3(M) = \frac{\mathcal{O}(\alpha^3(M))}{M^2}$$

Decoupling

Full theory: QED with muons

$$L = \bar{\psi}_0 i \not{D}_0 \psi_0 + \bar{\Psi}_0 \left(i \not{D}_0 - M_0 \right) \Psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} \left(\partial_\mu A_0^\mu \right)^2$$

Effective theory: low-energy QED

$$L' = \bar{\psi}'_0 i \not{D}'_0 \psi'_0 - \frac{1}{4} F'_{0\mu\nu} F_0^{\prime\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0^{\prime\mu} \right)^2 + \frac{1}{M_0^2} \sum_i C_i^0 O_i^{\prime 0} + \cdots$$

Decoupling: fields

$$A_0 = \left[\zeta_A^0 \right]^{-1/2} A'_0 + \frac{1}{M_0^2} \sum_i C_{Ai}^0 O_{Ai}^{\prime 0} + \cdots$$

$$\psi_0 = \left[\zeta_\psi^0 \right]^{-1/2} \psi'_0 + \frac{1}{M_0^2} \sum_i C_{\psi i}^0 O_{\psi i}^{\prime 0} + \cdots$$

Decoupling: parameters

$$e_0 = \left[\zeta_\alpha^0 \right]^{-1/2} e'_0 \qquad a_0 = \left[\zeta_A^0 \right]^{-1} a'_0$$

Electron field

$$\psi_{\text{os}} = \psi'_{\text{os}} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

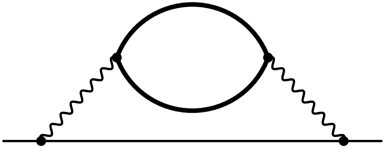
$$\zeta_\psi^0 = \frac{Z'^{\text{os}}_\psi(e'_0)}{Z^{\text{os}}_\psi(e_0)}$$

$$Z^{\text{os}}_\psi(e_0) = \frac{1}{1 - \Sigma_V(0)} \qquad Z'^{\text{os}}_\psi(e'_0) = \frac{1}{1 - \Sigma'_V(0)} = 1$$

$$\zeta_\psi(\mu) = \frac{Z_\psi(\alpha(\mu), a(\mu))}{Z'_\psi(\alpha'(\mu), a'(\mu))}\zeta_A^0 = \frac{Z_\psi(\alpha(\mu), a(\mu)) Z'^{\text{os}}_\psi(e'_0)}{Z^{\text{os}}_\psi(e_0) Z'_\psi(\alpha'(\mu), a'(\mu))}$$

- UV cancel in $Z_\psi / Z^{\text{os}}_\psi$, $Z'_\psi / Z'^{\text{os}}_\psi$
- IR cancel in $Z^{\text{os}}_\psi / Z'^{\text{os}}_\psi$
- $Z'^{\text{os}}_\psi = 1$: UV and IR cancel

Electron field



$$\Sigma_V(0) = \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} \left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

$$\zeta_\psi^0 = 1 - \varepsilon \left(1 - \frac{5}{6}\varepsilon + \cdots \right) \left(\frac{\alpha}{4\pi\varepsilon} \right)^2 + \cdots$$

$$\frac{Z_\psi(\alpha(\bar{M}), a(\bar{M}))}{Z'_\psi(\alpha'(\bar{M}), a'(\bar{M}))} = 1 + \varepsilon \left(\frac{\alpha}{4\pi\varepsilon} \right)^2$$

$$\zeta_\psi(\bar{M}) = 1 + \frac{5}{6} \left(\frac{\alpha(\bar{M})}{4\pi} \right)^2 + \cdots$$

Electron mass

$$\Sigma(p) = \not{p}\,\Sigma_V(p^2) + m_0 \Sigma_S(p^2)$$

$$S(p) = \frac{1}{1-\Sigma_V(p^2)} \; \frac{1}{\not{p} - \dfrac{1+\Sigma_S(p^2)}{1-\Sigma_V(p^2)} m_0}$$

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Near the mass shell

$$\frac{1}{1-\Sigma_V(p^2)}\,\frac{1}{\not{p} - \dfrac{1+\Sigma_S(p^2)}{1-\Sigma_V(p^2)}m_0} = \frac{\left[\zeta_\psi^0\right]^{-1}}{1-\Sigma'_V(p^2)}\,\frac{1}{\not{p} - \dfrac{1+\Sigma'_S(p^2)}{1-\Sigma'_V(p^2)}m'_0}$$

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$$\frac{1+\Sigma_S(0)}{1-\Sigma_V(0)} m_0 = \frac{1+\Sigma'_S(0)}{1-\Sigma'_V(0)} m'_0$$

Linear in m_0; $m_0 = 0$ in $\Sigma_{V,S}(0)$

Electron mass

$$m_0 = \left[\zeta_m^0\right]^{-1} m_0'$$

$$\zeta_m^0 = \left[\zeta_q^0\right]^{-1} \frac{1 + \Sigma_S(0)}{1 + \Sigma_S'(0)} = \frac{1 + \Sigma_S(0)}{1 - \Sigma_V(0)}$$

Electron mass

$$m_0 = \left[\zeta_m^0\right]^{-1} m_0'$$

$$\zeta_m^0 = \left[\zeta_q^0\right]^{-1} \frac{1+\Sigma_S(0)}{1+\Sigma_S'(0)} = \frac{1+\Sigma_S(0)}{1-\Sigma_V(0)}$$

$$m_{\text{os}} = m_{\text{os}}'$$

$$m_0 = Z_m^{\text{os}} m_{\text{os}} \qquad m_0' = Z_m'^{\text{os}} m_{\text{os}}'$$

$$\zeta_m^0 = \frac{Z_m'^{\text{os}}(e_0')}{Z_m^{\text{os}}(e_0)}$$

Neglect $m_{\text{os}}^2/M_{\text{os}}^2$ in $Z_m'^{\text{os}}$

Electron mass

$$m_0 = \left[\zeta_m^0\right]^{-1} m_0'$$

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$$\zeta_m^0 = \frac{Z_m'^{\rm os}(e_0')}{Z_m^{\rm os}(e_0)}$$

Neglect $m_{\rm os}^2/M_{\rm os}^2$ in $Z_m'^{\rm os}$

$$m(\mu) = \zeta_m^{-1}(\mu) m'(\mu) \qquad \zeta_m(\mu) = \frac{Z_m(\alpha(\mu))}{Z_m'(\alpha'(\mu))}\zeta_m^0$$

Electron mass

$$\Sigma_S(0) = -\frac{2(d-1)(d-6)}{(d-2)(d-5)(d-7)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 + \cdots$$

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$$\zeta_m^0 = 1 - \frac{8(d-1)(d-6)}{d(d-2)(d-5)(d-7)} \left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

$$= 1 + \left(2 - \frac{5}{3}\varepsilon + \frac{89}{18}\varepsilon^2 + \cdots \right) \left(\frac{\alpha}{4\pi\varepsilon} \right)^2 + \cdots$$

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$$\frac{Z_m(\alpha(\bar{M}))}{Z'_m(\alpha'(\bar{M}))} = 1 - \left(2 - \frac{5}{3}\varepsilon \right) \left(\frac{\alpha}{4\pi\varepsilon} \right)^2 + \cdots$$

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$$\frac{d\log\zeta_m(\mu)}{d\log\mu} + \gamma_m(\alpha(\mu)) - \gamma_m'(\alpha'(\mu)) = 0$$

Power counting

$$\lambda \sim \frac{p_i}{M}$$

$p \sim \lambda$, $x \sim 1/\lambda$, $\partial \sim \lambda$

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Soft photon

$$<0|T\left\{A_\mu(x)A_\nu(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\frac{1}{p^2}\left[g_{\mu\nu} - (1-a)\frac{p_\mu p_\nu}{p^2}\right]$$

$A \sim \lambda$, $D_\mu \sim \lambda$

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Soft electron

$$<0|T\left\{\psi(x)\bar{\psi}(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{1}{\not{p} - m},$$

$\psi \sim \lambda^{3/2}$

Power counting

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Lagrangian: $F_{\mu\nu}F^{\mu\nu} \sim \lambda^4$, $\bar{\psi} i \not{D} \psi \sim \lambda^4$

Action: ~ 1

Corrections to the Lagrangian $\sim \lambda^6$, to the action $\sim \lambda^2$

We can add higher-dimensional contributions to the Lagrangian, with further unknown coefficients. To any finite order in $1/M$, the number of such coefficients is finite, and the theory has predictive power.

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For example, if we want to work at the order $1/M^4$, then either a single $1/M^4$ (dimension 8) vertex or two $1/M^2$ ones (dimension 6) can occur in a diagram. UV divergences which appear in diagrams with two dimension 6 vertices are compensated by dimension 8 counterterms. So, the theory can be renormalized.

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The usual arguments about non-renormalizability are based on considering diagrams with arbitrarily many vertices of nonrenormalizable interactions (operators of dimensions > 4); this leads to infinitely many free parameters in the theory.

QCD

- QED: effects of decoupling of muon loops are tiny; pion pairs become important at about the same energies as muon pairs
- QCD: decoupling of heavy flavours is fundamental and omnipresent; everybody using QCD with $n_f < 6$ uses an effective field theory (even if he does not know that he speaks prose)

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Full theory QCD with n_l massless flavours
and 1 flavour of mass M

Effective theory QCD with n_l massless flavours

Dimension 6 operators

$$O_{g1}^{0} = g_0 f^{abc} G^{a}_{0\lambda}{}^{\mu} G^{b}_{0\mu}{}^{\nu} G^{c}_{0\nu}{}^{\lambda}$$
$$O_{g2}^{0} = (D^{\mu} G^{a}_{0\lambda\mu})(D_{\nu} G_{0}^{a\lambda\nu})$$

Dimension 6 operators

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$$O^0_{g2} = (D^\mu G^a_{0\lambda\mu})(D_\nu G^{a\lambda\nu}_0)$$

Quark operators

$$O^0_{qn} = \left(\sum_q \bar{q}_0 \gamma_{(n)} q_0\right)\left(\sum_q \bar{q}_0 \gamma_{(n)} q_0\right)$$
$$\tilde{O}^0_{qn} = \left(\sum_q \bar{q}_0 \gamma_{(n)} t^a q_0\right)\left(\sum_q \bar{q}_0 \gamma_{(n)} t^a q_0\right)$$

Only operators with odd n conserve the light-quark helicity
Equation of motion: $O^0_{g2} = g_0^2 \tilde{O}^0_{q1}$

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Only operators with odd n conserve the light-quark helicity
Equation of motion: $O^0_{g2} = g_0^2 \tilde{O}^0_{q1}$
If light quarks are not exactly massless, there is also chromomagnetic interaction

$$O^0_{cm} = m_0 \bar{q}_0 g_0 G^a_{0\mu\nu} t^a q_0$$

Contact interaction of quarks

$$c_{g2}(M) = -\frac{2}{15}\frac{T_F}{M^2}\left(\frac{\alpha_s(M)}{4\pi} + \mathcal{O}(\alpha_s^2(M))\right)$$

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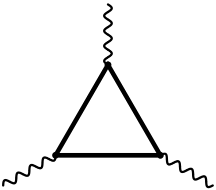
Eliminating this term in favour of $\tilde{c}^0_{q1}\tilde{O}^0_{q1}$

$$\tilde{c}_{q1}(M) = -\frac{2}{15}\frac{T_F}{M^2}\left(\alpha_s^2(M) + \mathcal{O}(\alpha_s^3(M))\right)$$

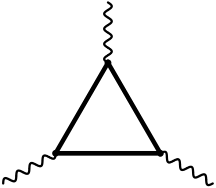
3-gluon interaction



$$T_F f^{a_1 a_2 a_3} \frac{g_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left[-\frac{4}{3} g_0 V^{\mu_1 \mu_2 \mu_3} \right.$$

$$\left. + i \frac{d-4}{180} (T_1^{\mu_1 \mu_2 \mu_3} + 12 T_2^{\mu_1 \mu_2 \mu_3}) + \mathcal{O}(p_i^5) \right]$$

3-gluon interaction



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$$c_{g1}(M) = -\frac{T_F}{90 M^2} \left(\frac{\alpha_s(M)}{4\pi} + \mathcal{O}(\alpha_s^2(M)) \right)$$

QCD decoupling

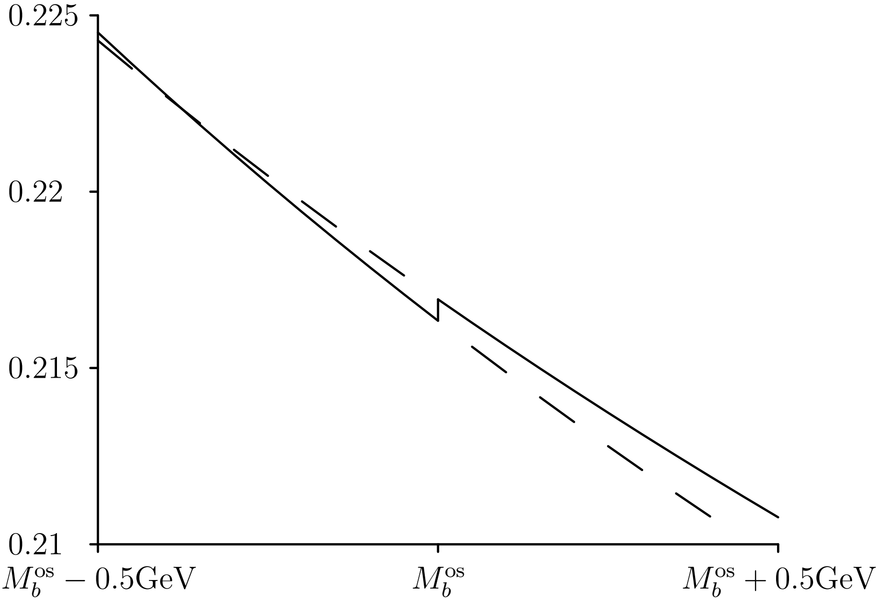
$$\alpha_s^{(n_l+1)}(\mu) = \zeta_\alpha^{-1}(\mu)\alpha_s^{(n_l)}(\mu)$$

$$\zeta_\alpha(\bar{M}) = 1 - \left(\frac{13}{3}C_F - \frac{32}{9}C_A\right) T_F \left(\frac{\alpha_s(\bar{M})}{4\pi}\right)^2 + \cdots$$

RG equation

$$\frac{d\log\zeta_\alpha(\mu)}{d\log\mu} - 2\beta^{(n_l+1)}(\alpha_s^{(n_l+1)}(\mu)) + 2\beta^{(n_l)}(\alpha_s^{(n_l)}(\mu)) = 0$$

QCD

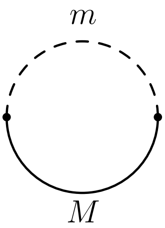


Light-quark masses

$$m^{(n_l+1)}(\mu) = \zeta_m^{-1}(\mu) m^{(n_l)}(\mu)$$

$$\zeta_m(\bar{M}) = 1 - \frac{89}{18} C_F T_F \left(\frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

Method of regions



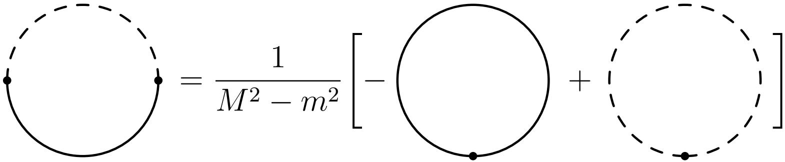
$$d = 2 - 2\varepsilon$$

$$M \gg m$$

$$I = \int \frac{d^d k}{\pi^{d/2}} \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

Exact solution

$$I = \frac{1}{M^2 - m^2} \int \frac{d^d k}{\pi^{d/2}} \left[-\frac{1}{k^2 + M^2} + \frac{1}{k^2 + m^2} \right]$$



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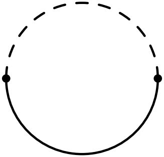
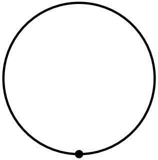
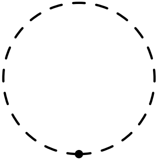


$$= \frac{1}{M^2 - m^2} \Bigg[-$$



$$+$$



$$\Bigg]$$

$$I = -\Gamma(\varepsilon) \frac{M^{-2\varepsilon} - m^{-2\varepsilon}}{M^2 - m^2} \to \frac{\log \dfrac{M^2}{m^2}}{M^2 - m^2}$$

$$= \frac{1}{M^2} \log \frac{M^2}{m^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right]$$

Method of regions

$$I = I_h + I_s$$

hard $k \sim M$

soft $k \sim m$

Hard region

$$k \sim M$$

$$I_h = \int \frac{d^d k}{\pi^{d/2}} T_h \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

$$T_h \frac{1}{(k^2 + M^2)(k^2 + m^2)} = \frac{1}{k^2 + M^2} \frac{1}{k^2} \left[1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} - \cdots \right]$$

$$I_h = -\frac{M^{-2\varepsilon}}{M^2} \Gamma(\varepsilon) \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right]$$

- IR divergence
- Taylor series in m
- Loop integrals with a single scale $M \Rightarrow M^{-2\varepsilon}$

Soft region

$$k \sim m$$

$$I_s = \int \frac{d^d k}{\pi^{d/2}} T_s \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

$$T_s \frac{1}{(k^2 + M^2)(k^2 + m^2)} = \frac{1}{M^2} \frac{1}{k^2 + m^2} \left[1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} - \cdots \right]$$

$$I_s = \frac{m^{-2\varepsilon}}{M^2} \Gamma(\varepsilon) \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right]$$

- UV divergence
- Taylor series in $1/M$
- Loop integrals with a single scale $m \Rightarrow m^{-2\varepsilon}$

Result

$$I = I_h + I_s = -\Gamma(\varepsilon)\frac{M^{-2\varepsilon} - m^{-2\varepsilon}}{M^2}\left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots\right]$$

$$\to \frac{1}{M^2}\log\frac{M^2}{m^2}\left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots\right]$$

Proof

$m \ll \Lambda \ll M$

$$I = \int_{k>\Lambda} \frac{d^d k}{\pi^{d/2}} \quad \frac{1}{(k^2+M^2)(k^2+m^2)}$$
$$+ \int_{k<\Lambda} \frac{d^d k}{\pi^{d/2}} \quad \frac{1}{(k^2+M^2)(k^2+m^2)}$$

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&\quad + \int_{k>\Lambda} \frac{d^d k}{\pi^{d/2}} T_h T_s \frac{1}{(k^2+M^2)(k^2+m^2)}
\end{aligned}$$

Proof

$$T_s T_h \frac{1}{(k^2+M^2)(k^2+m^2)} = T_h T_s \frac{1}{(k^2+M^2)(k^2+m^2)}$$
$$= \frac{1}{M^2k^2}\left[1-\frac{m^2}{k^2}+\frac{m^4}{k^4}-\cdots\right]\left[1-\frac{k^2}{M^2}+\frac{k^4}{M^4}-\cdots\right]$$
$$\Delta I = \int \frac{d^dk}{\pi^{d/2}} T_h T_s \frac{1}{(k^2+M^2)(k^2+m^2)} = 0$$

No scale

Photonia

Photonia has imported a single electron from Qedland, and physicists are studying its interaction with soft photons (both real and virtual)

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Velocity

$$\vec{v} = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{M} \to 0$$

Lagrangian

$$L = h^{+} i\partial_0 h$$

equation of motion

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Not Lorentz-invariant

Lagrangian

+ Lagrangian of the photon field

$$\partial_\mu F^{\mu\nu} = j^\nu$$
$$j^0 = -eh^+h$$

The electron produces the Coulomb field

Spin symmetry

At the leading order in $1/M$, the electron spin does not interact with electromagnetic field
We can rotate it without affecting physics
In addition to the $U(1)$ symmetry $h \to e^{i\alpha} h$,
also the $SU(2)$ spin symmetry

$$h \to Uh$$

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The electron magnetic moment $\vec{\mu} = \mu\vec{\sigma}$
interacts with magnetic field: $-\vec{\mu} \cdot \vec{B}$
By dimensionality $\mu \sim e/M$
(Bohr magneton $e/(2M)$ up to radiative corrections)

$$L_m = -\frac{e}{2M} h^+ \vec{B} \cdot \vec{\sigma} h$$

Violates the $SU(2)$ spin symmetry at the $1/M$ level

Spin-flavour symmetry

n_f flavours of heavy fermions

$$L = \sum_{i=1}^{n_f} h_i^+ i D_0 h_i$$

$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction

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$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction
At the leading order in $1/M$, not only the spin direction but also its magnitude is irrelevant
We can, for example, switch the electron spin off:

$$L = \varphi^* i D_0 \varphi$$

Superflavour symmetry

The scalar and the spinor fields together

$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry

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$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry
The superflavour $SU(3)$ symmetry:

- $\varphi \to e^{2i\alpha}\varphi$, $h \to e^{-i\alpha}h$
- $SU(2)$ spin rorations
- $$\delta \begin{pmatrix} \varphi \\ h \end{pmatrix} = i \begin{pmatrix} 0 & \varepsilon^+ \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ h \end{pmatrix}$$

  ε — an infinitesimal spinor

Broken at $1/M$

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Broken at $1/M$
We can consider, e. g., spins $\frac{1}{2}$ and 1
$SU(5)$ superflavour symmetry

Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

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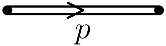
The usual photon propagator

The momentum-space free electron propagator



$$= i S_0(p) \qquad S_0(p) = \frac{1}{p_0 + i0}$$

depends only on p_0, not on $\vec{p}$

(spin-$\frac{1}{2}$ field h_0 — the unit 2×2 spin matrix)

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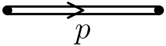

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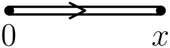
The coordinate-space propagator



$$= iS_0(x) \qquad S_0(x) = S_0(x_0)\delta(\vec{x}\,) \qquad S_0(t) = -i\theta(t)$$

Static electron does not move

Feynman rules

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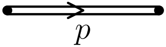

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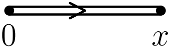

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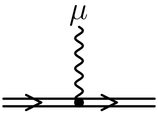
Solving the equation

$$i\partial_0 S_0(x) = \delta(x)$$

Feynman rules

Vertex



$$= ie_0 v^\mu$$

$$v^\mu = (1, \vec{0}\,)$$

Feynman rules

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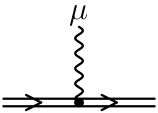

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The static field φ_0 (or h_0) describes only particles, there are no antiparticles.
No loops formed by static-electron propagators.
The electron propagates only forward in time;
the product of θ functions for a loop vanishes.
In the momentum space: all poles of the propagators are in the lower p_0 half-plane;
closing the integration contour upwards, we get 0.

Wilson line

In an external field $A^\mu(x)$

$$iD_0 S(x,x') = (i\partial_0 + e_0 A^0(x)) S(x,x') = \delta(x-x')$$

Solution

$$S(x,x') = S(x_0,x_0')\delta(\vec{x}-\vec{x}\,') \qquad S(x_0,x_0') = S_0(x_0-x_0')W(x_0,x_0')$$

Wilson line from x' to x (along v)

$$W(x_0,x_0') = \exp ie_0 \int_{x_0'}^{x_0} A^\mu(t,\vec{x}\,) v_\mu dt$$

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The HQET Lagrangian has been introduced as a device to investigate of Wilson lines

Gauge $A^0 = 0$

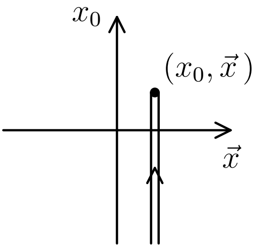
The field $\varphi_0(x)$ does not interact with the electromagnetic field (and thus becomes free).
However, this gauge is rather pathological.
The static electron creates the Coulomb electric field $\vec{E}$.
In the $A^0 = 0$ gauge, $\vec{A}$ has to depend on t linearly.

Gauge $A^0 = 0$

We can formally express the field $\varphi_0(x)$ in any gauge via a free field $\varphi^{(0)}(x)$:

$$\varphi_0(x) = W(x)\varphi^{(0)}(x)$$

$$W(x_0, \vec{x}\,) = P \exp i \int_{-\infty}^{x_0} A_0^{\mu}(t, \vec{x}\,) v_\mu dt$$



Then $W^{-1}(x) D_0 W(x) = \partial_0$, and

$$L = \varphi^{(0)+} i\partial_0 \varphi^{(0)}$$

Residual momentum

The full-theory energy M is the HEET zero level

$$E = M + \varepsilon$$

ε — the residual energy

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$$E = M + \varepsilon$$

ε — the residual energy

$$P^\mu = M v^\mu + p^\mu$$

- P^μ — 4-momentum of some state (containing a single electron) in the full theory
- p^μ — its momentum in HEET (the residual momentum)

v^μ — 4-velocity of a reference frame in which the electron always stays approximately at rest

Reparametrization invariance

HEET is applicable if there exists such v that

$$p^\mu \ll M \qquad p^\mu_{\gamma i} \ll M$$

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This condition does not fix v uniquely: $v \to v + \delta v$, $\delta v \sim p/M$.
Effective theories corresponding to different choices of v must produce identical physical predictions: reparametrization invariance.
Relations between quantities at different orders in $1/M$.

Relativistic notation

Lagrangian

$$L = \varphi_0^* i v \cdot D \varphi_0 + \text{(light fields)}$$

Free propagator

$$S_0(p) = \frac{1}{p \cdot v + i0}$$

Mass shell

$$p \cdot v = 0$$

Spin $\frac{1}{2}$

4-component spinor field

$$\not{v} h_v = h_v$$

Lagrangian

$$L = \bar{h}_{v0} i v \cdot D h_{v0} + (\text{light fields})$$

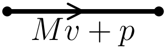
Propagator

$$S_0(p) = \frac{1+\not{v}}{2} \frac{1}{p \cdot v + i0}$$

Vertex $ie_0 v^\mu$

Qedland

$$S_0(Mv+p) = \frac{M + M\not{v} + \not{p}}{(Mv+p)^2 - M^2 + i0} = \frac{1+\not{v}}{2}\,\frac{1}{p\cdot v + i0} + \mathcal{O}\left(\frac{p}{M}\right)$$



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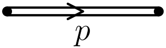
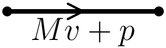


$+\,\mathcal{O}\left(\frac{p}{M}\right)$

Qedland

$$S_0(Mv+p) = \frac{M + M\not{v} + \not{p}}{(Mv+p)^2 - M^2 + i0} = \frac{1+\not{v}}{2}\,\frac{1}{p\cdot v + i0} + \mathcal{O}\left(\frac{p}{M}\right)$$



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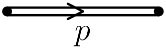


$+\,\mathcal{O}\left(\frac{p}{M}\right)$

$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$$

We may insert the projectors $(1+\not{v})/2$ before $u(P_i)$ and after $\bar{u}(P_i)$, too, because

$$\not{v}u(Mv+p) = u(Mv+p) + \mathcal{O}\left(\frac{p}{M}\right)$$

Qedland

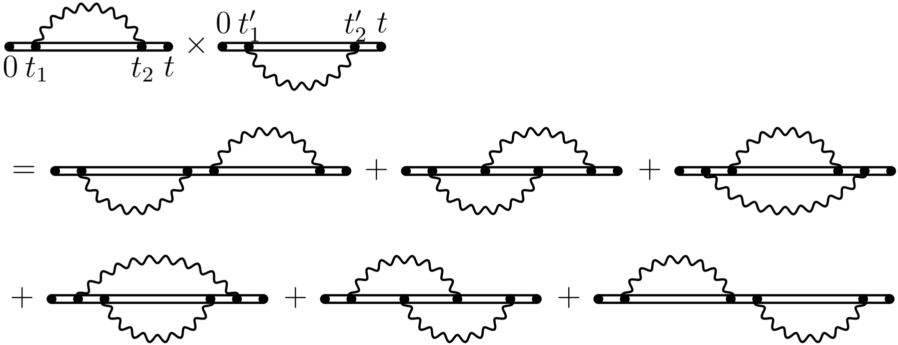
We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules.

Qedland

We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules. We have thus proved that at the tree level any QED diagram is equal to the corresponding HEET diagram up to $\mathcal{O}(p/m)$ corrections. This is not true at loops, because loop momenta can be arbitrarily large. Renormalization properties of HEET (anomalous dimensions, etc.) differ from those in QED.

Exponentiation

1-loop correction to x-space propagator, multiply by itself
Integral in t_1, t_2, t'_1, t'_2 with $0 < t_1 < t_2 < t$, $0 < t'_1 < t'_2 < t$
Ordering of primed and non-primed t's can be arbitrary
6 regions corresponding to 6 diagrams



Exponentiation

This is 2× the 2-loop correction
1-loop correction cubed is 3!× the 3-loop correction, ...

$$S(t) = S_0(t)\,\exp w_1$$
$$w_1 = -\frac{e_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon)\left(1+\frac{2}{d-3}-a_0\right)$$

In the d-dimensional Yennie gauge the exact propagator is free

Exponentiation

No corrections to the photon propagator $Z_A = 1$: $a = a_0$, $e = e_0$

$$Z_h = \exp\left[-(a-3)\frac{\alpha}{4\pi\varepsilon}\right]$$
$$\gamma_h = 2(a-3)\frac{\alpha}{4\pi}$$

exactly!

Current

$$J_0 = \varphi_0^* \varphi_0 \qquad Q_0 = \int d^{d-1}\vec{x}\, J_0(x_0, \vec{x}) = 1$$

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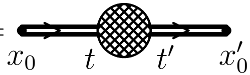
Ward identity

Green function

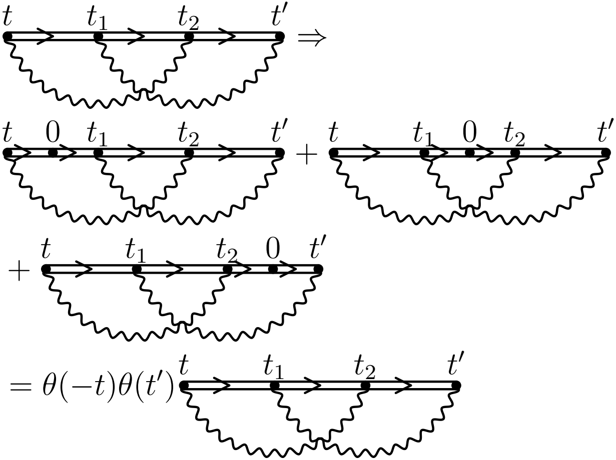
$$<0|\varphi_0^*(x) J_0(0) \varphi_0(x')|0> = \delta(\vec{x})\delta(\vec{x}\,')G(x_0, x_0')$$

$=$ 

Vertex $\Gamma(t, t') = \delta(t' - t) + \Lambda(t, t')$ and 2 full propagators

Coordinate space

Each diagram for $\Sigma \Rightarrow$ a set of diagrams for Λ



Coordinate space

Regions $t \leq 0 \leq t_1 \leq t_2 \leq t'$, $t \leq t_1 \leq 0 \leq t_2 \leq t'$,
$t \leq t_1 \leq t_2 \leq 0 \leq t'$
union — the region for Σ $(t \leq t_1 \leq t_2 \leq t')$

$$\Lambda(t, t') = -i\theta(-t)\theta(t')\Sigma(t' - t)$$

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Or we can start from diagrams for $S(t, t')$

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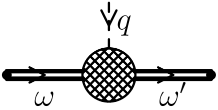
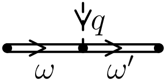
LHS $= Z_h Z_J G_r$, RHS $= Z_h S_r \Rightarrow Z_J = 1$

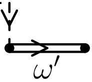
Momentum space

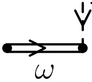
$G(\omega, \omega') =$  $= iS(\omega)\,\Gamma(\omega,\omega')\,iS(\omega')$

$$\Gamma(\omega,\omega') = 1 + \Lambda(\omega,\omega') \qquad q_0 = \omega' - \omega$$

Each diagram for $\Sigma \Rightarrow$ a set of diagrams for Λ

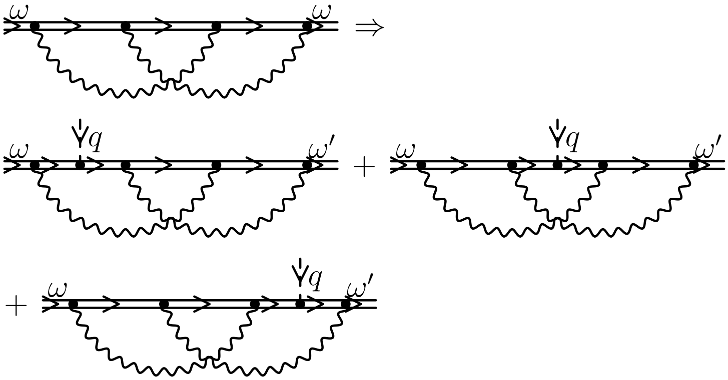


$= -\dfrac{i}{\omega' - \omega}\Bigg[$ 

$-$ 

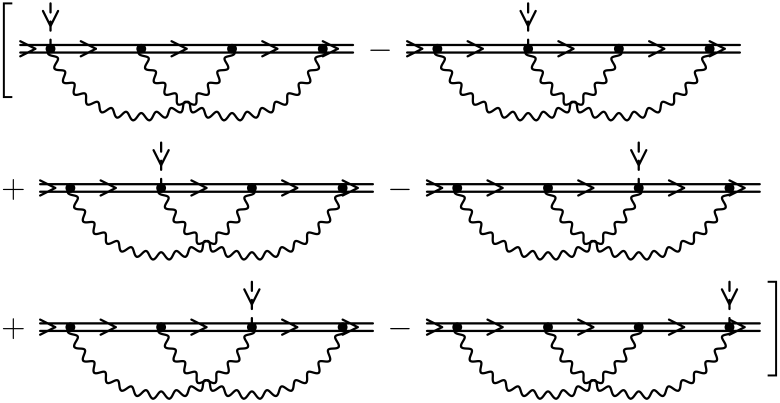
$\Bigg]$

Momentum space



Momentum space

$$= -\frac{i}{\omega' - \omega} \times$$



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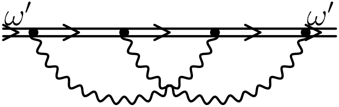

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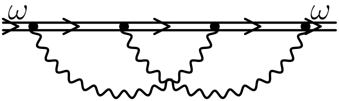

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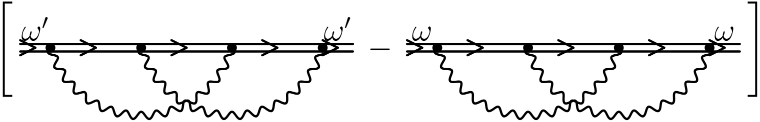

$$\Lambda(\omega, \omega') = -\frac{\Sigma(\omega') - \Sigma(\omega)}{\omega' - \omega}$$

$$\Gamma(\omega, \omega') = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}$$

(also Fourier from coordinate space)

Momentum space

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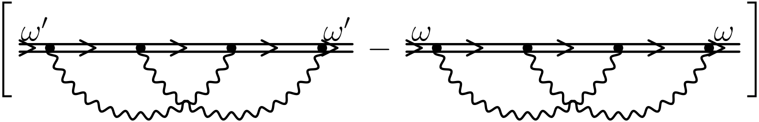

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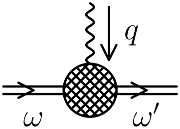
$$G(\omega, \omega') = \frac{S(\omega') - S(\omega)}{\omega' - \omega}$$

also from all diagrams for G, or Fourier

Vertex



$$= ie_0 v^\mu \Gamma(\omega, \omega')$$

$$Z_\Gamma Z_h = 1$$

$$Z_\alpha = (Z_\Gamma Z_h)^{-2} Z_A^{-1} = Z_A^{-1} = 1$$

Operators

Full QED operators — series in $1/M$
via HEET operators

$$O(\mu) = C(\mu)\tilde{O}(\mu) + \frac{1}{2M}\sum_i B_i(\mu)\tilde{O}_i(\mu) + \cdots$$

Matching on-shell matrix elements

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

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On-shell matrix elements

$$<0|\psi_0|e(p)> = \left(Z_\psi^{\rm os}\right)^{1/2} u(p)$$
$$<0|h_{v0}|e(p)> = (Z_h^{\rm os})^{1/2} u_v(k)$$

Bare decoupling $Z_h^{\rm os} = 1$

$$z_0 = \frac{Z_\psi^{\rm os}(e_0^{(1)})}{Z_h^{\rm os}(e_0^{(0)})}$$

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Renormalized decoupling

$$z(\mu) = \frac{Z_h(\alpha^{(0)}(\mu), a^{(0)}(\mu))}{Z_\psi(\alpha_s^{(1)}(\mu), a^{(1)}(\mu))} z_0$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$S(x) = S_L(x)$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\, e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

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Landau, Khalatnikov (1955)
Fradkin (1955)
Bogoliubov, Shirkov (1957)
Zumino (1960)

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x)e^{\sigma(x)}$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon)$$

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$$= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$\begin{aligned}
S(x) &= S_0(x) e^{\sigma(x)} \\
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\end{aligned}$$

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\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}
\end{aligned}
$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

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$$= \log Z_\psi + \sigma_r$$

$$\log Z_\psi(\alpha, a) = \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}$$

$$\gamma_\psi(\alpha, a) = 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)$$

$d \log(a(\mu)\alpha(\mu))/d \log \mu = -2\varepsilon$ exactly
$\gamma_L(\alpha)$ starts from α^2

Gauge dependence of Z_ψ, γ_ψ

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$d \log(a(\mu)\alpha(\mu)) / d \log \mu = -2\varepsilon$ exactly
$\gamma_L(\alpha)$ starts from α^2
known to 5 loops

Gauge independence of $z(\mu)$ in QED

- $z_0 = Z_\psi^{\text{os}}$ gauge invariant
- $\log Z_h = (3 - a^{(0)})\dfrac{\alpha^{(0)}}{4\pi\varepsilon}$
  $\alpha^{(0)} = \alpha_{\text{os}} \approx 1/137$
- $\log Z_\psi = -a^{(1)}(\mu)\dfrac{\alpha^{(1)}(\mu)}{4\pi\varepsilon} + (\text{gauge invariant})$
- Decoupling $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$
  Gauge dependence cancels in $\log(\tilde{Z}_\psi/Z_\psi)$

Result

$$z(M_{\text{os}}) = 1 - \frac{\alpha}{\pi} + \left(\pi^2 \log 2 - \frac{3}{2}\zeta_3 - \frac{55}{48}\pi^2 + \frac{5957}{1152}\right)\left(\frac{\alpha}{\pi}\right)^2 + \cdots$$

Electron propagator near the mass shell

On-shell mass $M = M_0 + \delta M$, $\omega \ll M$

$$P = (M + \omega)v \qquad \Sigma(P) = \Sigma_0(\omega) + \Sigma_1(\omega)(\not{v} - 1)$$

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$$\begin{aligned} S(P) &= \frac{1}{\not{p} - M_0 - \Sigma(p)} \\ &= \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\not{v} - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)} \end{aligned}$$

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$$= \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\not{v} - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)}$$

The denomunator

$$[M + \omega - \Sigma_1(\omega)]^2 - [M - \delta M + \Sigma_0(\omega) - \Sigma_1(\omega)]^2$$

should vanish at $\omega = 0$:

$$\delta M = \Sigma_0(0)$$

Electron propagator near the mass shell

$$S(P) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)]^2}$$

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The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

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The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

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The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

$$S(P) \approx \frac{\not{v}+1}{2}\,\frac{1}{\omega - \Sigma_0(\omega) + \Sigma_0(0)}$$

Regions

$$\Sigma_0(\omega) = \Sigma_h(\omega) + \Sigma_s(\omega)$$

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Hard

$$\Sigma_h(\omega) = \frac{e_0^2 M^{1-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} \left(1 - \frac{\omega}{M} + \cdots\right)$$

$$\delta M = M \left[\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} + \cdots\right]$$

$$Z_\psi^{\mathrm{os}} = \frac{1}{1 - \Sigma_0'(0)} = 1 - \frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} + \cdots$$

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Soft

$$\Sigma_s(\omega) = \Sigma(\omega) \left(1 + \mathcal{O}\left(\frac{\omega}{M}\right)\right)$$

$$\Sigma(\omega) = \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(1 + \frac{2}{d-3} - a_0\right)$$

Electron propagator in QED and HEET

$$S(p) = \frac{1+\not{v}}{2} \frac{1}{\omega - \Sigma'_h(0)\omega - \Sigma_s(\omega)} = z_0 S(\omega)$$

$$z_0 = Z_\psi^{\text{os}} = \frac{1}{1-\Sigma'_h(0)}$$

$$S(\omega) = \frac{1+\not{v}}{2} \frac{1}{\omega - \Sigma(\omega)}$$

$S(\omega)$ — HEET propagator

Electron propagator in QED and HEET

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$S(\omega)$ — HEET propagator

- Higher terms in $\Sigma_h \Rightarrow$ corrections to ψ_0 via h_{v0}
- Higher terms in $\Sigma_s \Rightarrow$ corrections to $S(\omega)$ due to $1/M$ terms in the HEET Lagrangian

Power counting

Small parameter (p — residual momentum)

$$\lambda \sim \frac{p}{M}$$

Soft fields: $\partial \sim \lambda$, $A \sim \lambda$, $D \sim \lambda$

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$$<T\{\varphi(x)\varphi^+(0)\}> \sim \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{1}{k\cdot v + i0}$$

$$\varphi \sim \lambda^{3/2}$$

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$$\varphi \sim \lambda^{3/2}$$

$$\varphi^+ i D_0 \varphi \sim \lambda^4$$

$$\varphi^+ \vec{D}^{\,2} \varphi \sim \lambda^5 \qquad \varphi^+ \vec{B} \cdot \vec{\sigma} \varphi \sim \lambda^5$$

Action: main ~ 1, corrections $\sim \lambda$

Heavy–heavy current

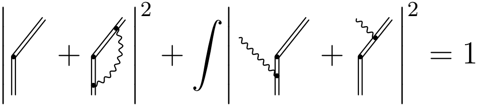
$$J_0 = \varphi^*_{v'0}\varphi_{v0} = Z_J(\alpha(\mu))J(\mu) \qquad \cosh\varphi = v\cdot v'$$

$$\Gamma(\vartheta) = \frac{d\log Z_J}{d\log\mu}$$

Exponentiation: 1-loop formula is exact

Cusp anomalous dimension

Unitarity



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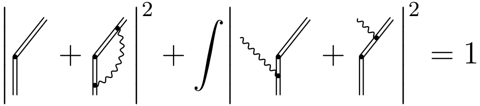


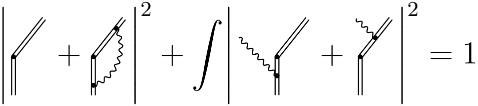
Classical electrodynamics

$$dE = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\, d\omega$$

$$dw = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\frac{d\omega}{\omega}$$

Cusp anomalous dimension

Unitarity



Classical electrodynamics

$$dE = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\, d\omega$$

$$dw = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}}$$

Cusp anomalous dimension

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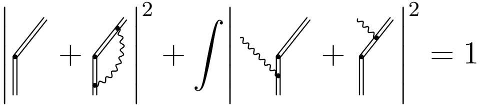

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$$F = 1 - \frac{1}{2}\int_\lambda^\infty \frac{e^2}{2\pi^2}(\vartheta \coth\vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}} = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta \coth\vartheta - 1)$$

$$\Gamma = 4\frac{\alpha}{4\pi}(\vartheta \coth\vartheta - 1)$$

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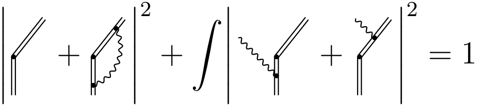

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The Guiness Book of Records: the anomalous dimension known for the longest time ($>$ 100 years)

Limiting cases

$\vartheta \ll 1$ Series in ϑ^2

$$\Gamma(\vartheta) = \frac{\alpha}{3\pi}\vartheta^2 + \mathcal{O}(\vartheta^4)$$

$\vartheta \gg 1$ $\Gamma(\vartheta) = \Gamma_l \vartheta + \mathcal{O}(\vartheta^0)$

$$\Gamma_l = \frac{\alpha}{\pi}$$

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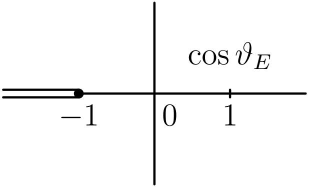
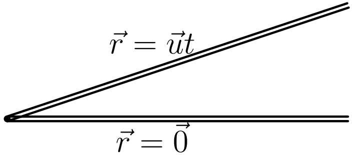
Euclidean space $\cos\vartheta_E = v \cdot v'$

$$\Gamma(\vartheta_E) = 4\frac{\alpha}{4\pi}(\vartheta_E \cot\vartheta_E - 1)$$



$\vartheta_E \to \pi$

Heavy-particle pair production



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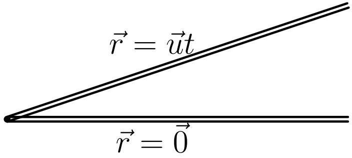
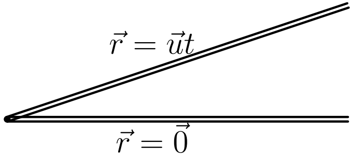


$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r}$$

$\vartheta_E \to \pi$

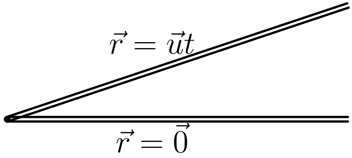
Heavy-particle pair production



$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r^{1-2\varepsilon}}$$

$\vartheta_E \to \pi$

Heavy-particle pair production



$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r^{1-2\varepsilon}}$$

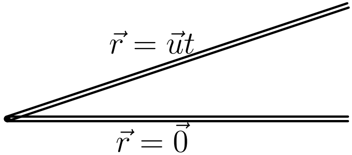
$$W = \exp\left[-i\int_0^T dt\,U(ut)\right] = \exp\left[i\frac{e^2}{4\pi}\frac{T^{2\varepsilon}}{2\varepsilon u^{1-2\varepsilon}}\right]$$

$$Z_J = \exp\left[i\frac{\alpha}{2\varepsilon u}\right]$$

$$\Gamma = -i\frac{\alpha}{u}$$

$\vartheta_E \to \pi$

Heavy-particle pair production



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$$Z_J = \exp\left[i\frac{\alpha}{2\varepsilon u}\right]$$

$$\Gamma = -i\frac{\alpha}{u} \qquad u \Rightarrow i\delta \qquad \Gamma(\pi - \delta) = -\frac{\alpha}{\delta}$$

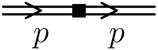
Kinetic energy

$$L = L_0 + \frac{C_k^0}{2M} O_k^0 = L_0 + \frac{C_k(\mu)}{2M} O_k(\mu)$$
$$L_0 = \varphi_0^* i D_0 \varphi_0$$
$$O_k^0 = \varphi_0^* \vec{D}^2 \varphi_0 = -\varphi_0^* D_\perp^2 \varphi_0 = Z_k(\alpha(\mu)) Z_k(\mu)$$

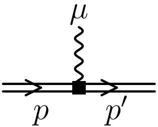
Mass shell

$$\varepsilon(\vec{p}) = \frac{C_k^0 \vec{p}^2}{2M} \quad \Rightarrow \quad C_k^0 = 1 \text{ at tree level}$$

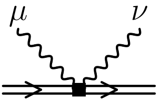
Feynman rules



$$= i\frac{C_k^0}{2M}p_\perp^2$$



$$= i\frac{C_k^0}{2M}e_0'(p+p')_\perp^\mu$$



$$= i\frac{C_k^0}{2M}e_0'^2 g_\perp^{\mu\nu}$$

Ward identity

Sum of 1PI diagrams at $1/M$

$$-i\frac{C_k^0}{2M}\Sigma_k(\omega,\vec{p}_\perp^{\,2})$$

Ward identity

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$$-i\frac{C_k^0}{2M}\Sigma_k(\omega,\vec{p}_\perp^{\,2})$$

$$\Sigma_k(\omega,p_\perp^2)=\frac{d\Sigma(\omega)}{d\omega}p_\perp^2+\Sigma_{k0}(\omega)$$

Ward identity

- $\delta\Sigma$ for $v \to v + \delta v$ $(v \cdot \delta v = 0)$
  - propagators $1/(p \cdot v + i0) \Rightarrow i p_i \cdot \delta v$
  - vertices $i e_0' \delta v^\mu$

Ward identity

- $\delta\Sigma$ for $v \to v + \delta v$ $(v \cdot \delta v = 0)$
  - propagators $1/(p \cdot v + i0) \Rightarrow ip_i \cdot \delta v$
  - vertices $ie_0' \delta v^\mu$
- $\delta\Sigma_k$ for $p_\perp \to p_\perp + \delta p_\perp$
  - 0-photon vertices $i(C_k^0/M)p_i \cdot \delta p_\perp$
  - 1-photon vertices $i(C_k^0/M)e_0' \delta p_\perp^\mu$

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$$\frac{\partial \Sigma_k}{\partial p_\perp^\mu} = 2\frac{\partial \Sigma}{\partial v^\mu}$$

Ward identity

- $\delta\Sigma$ for $v \to v + \delta v$ $(v \cdot \delta v = 0)$
  - propagators $1/(p \cdot v + i0) \Rightarrow ip_i \cdot \delta v$
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$$\frac{\partial \Sigma_k}{\partial p_\perp^\mu} = 2\frac{\partial \Sigma}{\partial v^\mu}$$

$$\frac{\partial \Sigma_k}{\partial p_\perp^\mu} = 2\frac{\partial \Sigma_k}{\partial p_\perp^2} p_\perp^\mu \qquad \frac{\partial \Sigma}{\partial v^\mu} = \frac{d\Sigma}{d\omega} p_\perp^\mu$$

$$\frac{\partial \Sigma_k}{\partial p_\perp^2} = \frac{d\Sigma}{d\omega}$$

Mass shell

$$\omega - \Sigma(\omega) - \frac{C_k^0}{2M}\left[\vec{p}^{\,2} - \frac{d\Sigma(\omega)}{d\omega}\vec{p}^{\,2} + \Sigma_{k0}(\omega)\right] = 0$$

Expand in ω up to ω^1

$$\omega = \frac{C_k^0}{2M}\vec{p}^{\,2} \quad \Rightarrow \quad C_k^0 = 1$$

On-shell scattering

Full theory

$$e_{\text{os}}\,\varphi^*(P')F(q^2)(P+P')^\mu\varphi(P) = e_{\text{os}}\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M}\right]$$

$$F(q^2) = 1 + F'(0)\frac{q^2}{M^2} + \cdots$$

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Full theory

$$e_{\rm os}\,\varphi^*(P')F(q^2)(P+P')^\mu\varphi(P) = e_{\rm os}\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M}\right]$$

$$F(q^2) = 1 + F'(0)\frac{q^2}{M^2} + \cdots$$

Effective theory loops vanish

$$e_0'\left[v^\mu + \frac{C_k^0}{2M}(p+p')^\mu_\perp\right]$$

Reparametrization invariance

$$v' = v + \delta v$$

$$L_{v'} = \varphi^*_{v'} i v' \cdot D \varphi_{v'} - \frac{C_k}{2M} \varphi^*_{v'} D'^2_{\perp} \varphi_{v'}$$

$$\varphi_{v'} = e^{iM\,\delta v \cdot x} \left(1 + \frac{i\,\delta v \cdot D}{2M} \right) \varphi_v$$

$$L_{v'} = L_v - (C_k - 1) \varphi^*_v i\,\delta v \cdot D \varphi_v$$

Magnetic moment

$$L = L_0 + \frac{C_k^0}{2M} O_k^0 + \frac{C_m^0}{2M} O_m^0 = L_0 + \frac{C_k(\mu)}{2M} O_k(\mu) + \frac{C_m(\mu)}{2M} O_m(\mu)$$

$$O_k^0 = h_0^+ \vec{D}^{\,2} h_0 = -\bar{h}_{v0} D_\perp^2 h_{v0} = Z_k(\mu) O_k(\mu)$$

$$O_m^0 = -e_0 h_0^+ \vec{B}_0 \cdot \vec{\sigma} h_0 = \frac{1}{2} e_0 \bar{h}_{v0} F_{\mu\nu}^0 \sigma^{\mu\nu} h_{v0} = Z_m(\mu) O_m(\mu)$$

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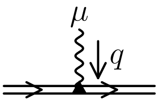
Breaks spin symmetry



$$= \frac{i e_0' C_m^0}{2M} [\not{q}, \gamma^\mu]$$

On-shell scattering

Full theory

$$e_{\text{os}}\,\bar{u}'(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

$$= e_{\text{os}}\,\bar{u}'(P')\left[\left(F_1(q^2)+F_2(q^2)\right)\gamma^\mu - F_2(q^2)\frac{(P+P')^\mu}{2M}\right]u(P)$$

$$= e_{\text{os}}\,\bar{u}'(P')\left[F_1(q^2)\frac{(P+P')^\mu}{2M} + \left(F_1(q^2)+F_2(q^2)\right)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

$$F_1(q^2) = 1 + F_1'(0)\frac{q^2}{M^2} + \cdots \qquad F_2(q^2) = F_2(0) + \cdots$$

On-shell scattering

Full theory

$$e_{\rm os}\, \bar{u}'(P') \left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{[\not{q}, \gamma^\mu]}{4M} \right] u(P)$$

$$= e_{\rm os}\, \bar{u}'(P') \left[\left(F_1(q^2) + F_2(q^2)\right) \gamma^\mu - F_2(q^2)\frac{(P+P')^\mu}{2M} \right] u(P)$$

$$= e_{\rm os}\, \bar{u}'(P') \left[F_1(q^2)\frac{(P+P')^\mu}{2M} + \left(F_1(q^2) + F_2(q^2)\right) \frac{[\not{q}, \gamma^\mu]}{4M} \right] u(P)$$

$$F_1(q^2) = 1 + F_1'(0)\frac{q^2}{M^2} + \cdots \qquad F_2(q^2) = F_2(0) + \cdots$$

Foldy–Wouthuysen $P = Mv + p$

$$u(P) = \left(1 + \frac{\not{p}}{2M}\right) u_v(p)$$

Matching

Full theory

$$e_{\text{os}}\,\bar{u}'_v(p')\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} + (1+F_2(0))\,\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u_v(p)\,.$$

Matching

Full theory

$$e_{\text{os}}\,\bar{u}'_v(p')\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} + (1+F_2(0))\,\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u_v(p)\,.$$

Effective theory

$$e'_0\,\bar{u}'_v(p')\left[v^\mu + \frac{C_k^0}{2M}(p+p')^\mu_\perp + \frac{C_m^0}{2M}i\sigma^{\mu\nu}q_\nu\right]u_v(p)$$

Matching

Full theory

$$e_{\rm os}\,\bar{u}'_v(p')\left[v^\mu+\frac{(p+p')^\mu_\perp}{2M}+(1+F_2(0))\,\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u_v(p)\,.$$

Effective theory

$$e'_0\,\bar{u}'_v(p')\left[v^\mu+\frac{C^0_k}{2M}(p+p')^\mu_\perp+\frac{C^0_m}{2M}i\sigma^{\mu\nu}q_\nu\right]u_v(p)$$

$$C^0_k=1 \qquad C^0_m=1+F_2(0)$$

C^0_m is finite $\Rightarrow Z_m=1$

Reparametrization invariance

$$h_{v'} = e^{iM\,\delta v\cdot x}\left(1 - \frac{\delta \not{v}}{2} + \frac{i\,\delta v\cdot D}{2M}\right) h_v$$

$$L_{v'} = L_v - (C_k - 1)\bar{h}_v i\,\delta v \cdot D h_v$$

HQET

$$L = L_0 + \frac{C_k^0}{2M} O_k^0 + \frac{C_m^0}{2M} O_m^0 + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$L_0 = h_0^+ i D_0 h_0$$

$$O_k^0 = h_0^+ \vec{D}^2 h_0 = Z_k(\alpha_s(\mu)) O_k(\mu)$$

$$O_m^0 = g_0 h_0^+ \vec{B}^a \cdot \vec{\sigma} t_a h_0 = Z_m(\alpha_s(\mu)) O_m(\mu)$$

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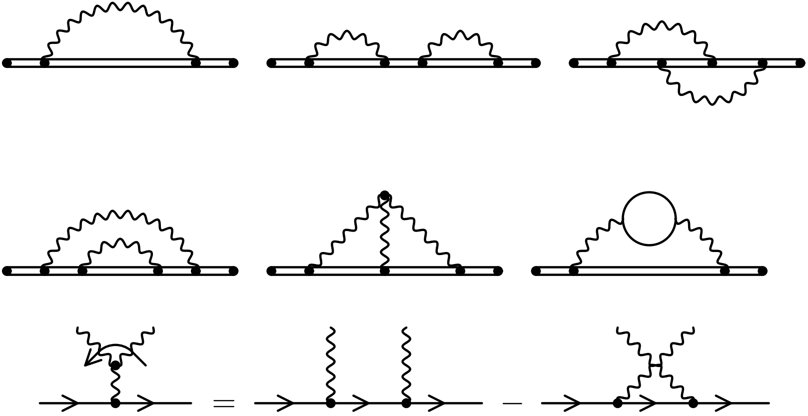
Reparametrization invariance

$$Z_k = 1 \qquad O_k = O_k^0$$

$$C_k^0 = 1 \qquad C_k(\mu) = Z_k^{-1} C_k^0 = 1$$

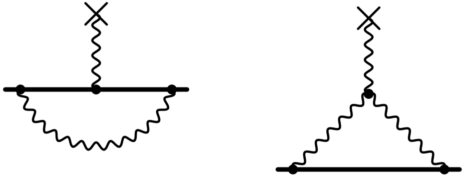
Propagator



$$S(t) = S_0(t) \exp\left[C_F \frac{g_0^2}{(4\pi)^{d/2}} \left(\frac{it}{2}\right)^{2\varepsilon} S \right.$$

$$\left. + C_F \frac{g_0^4}{(4\pi)^{d}} \left(\frac{it}{2}\right)^{4\varepsilon} (C_A S_A + T_F n_l S_l) \right]$$

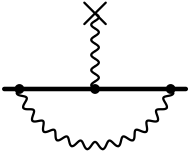
Chromomagnetic interaction

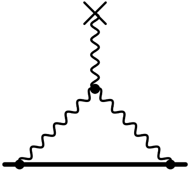


$$F_2(0) = \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\varepsilon)}{2(d-3)} \times \left[2(d-4)(d-5)C_F - (d^2 - 8d + 14)C_A\right]$$

IR divergent (unlike QED)

Chromomagnetic interaction





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$$\times \left[2(d-4)(d-5)C_F - (d^2 - 8d + 14)C_A\right]$$

IR divergent (unlike QED)

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A (17C_A - 13T_F n_l) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$
$$C_m(\mu) = 1 + 2\left(-C_A \log\frac{M}{\mu} + C_F + C_A\right) \frac{\alpha_s(M)}{4\pi} + \cdots$$

Mass splitting

$$M_{B^*}^2 - M_B^2 = \frac{4}{3} C_m^{(4)}(\mu)\mu_{G(4)}^2(\mu) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{M_b}\right)$$

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} = \left(\frac{\alpha_s^{(4)}(M_c)}{\alpha_s^{(4)}(M_b)}\right)^{-9/25}\left[1 + \mathcal{O}\left(\alpha_s, \frac{\Lambda_{\rm QCD}}{M_{b,c}}\right)\right]$$

In the past

Only renormalizable theories were considered well-defined: they contain a finite number of parameters, which can be extracted from a finite number of experimental results and used to predict an infinite number of other potential measurements. Non-renormalizable theories were rejected because their renormalization at all orders in non-renormalizable interactions involve infinitely many parameters, so that such a theory has no predictive power. This principle is absolutely correct, if we are impudent enough to pretend that our theory describes the Nature up to arbitrarily high energies (or arbitrarily small distances).

At present

We accept the fact that our theories only describe the Nature at sufficiently low energies (or sufficiently large distances). They are effective low-energy theories. Such theories contain all operators (allowed by the relevant symmetries) in their Lagrangians. They are necessarily non-renormalizable. This does not prevent us from obtaining definite predictions at any fixed order in the expansion in E/M, where E is the characteristic energy and M is the scale of new physics. Only if we are lucky and M is many orders of magnitude larger than the energies we are interested in, we can neglect higher-dimensional operators in the Lagrangian and work with a renormalizable theory.

Conclusion

Practically all physicists believe that the Standard Model is also a low-energy effective theory. But we don't know what is a more fundamental theory whose low-energy approximation is the Standard Model. Maybe, it is some supersymmetric theory (with broken supersymmetry); maybe, it is not a field theory, but a theory of extended objects (superstrings, branes); maybe, this more fundamental theory lives in a higher-dimensional space, with some dimensions compactified; or maybe it is something we cannot imagine at present.

Conclusion

The only model-independent method to search for physics beyond the Standard Model (without inventing arbitrary scenarios) is to use SMEFT: add operators having higher dimensions (5, 6) to the Standard Model Lagrangian with unknown coefficients, and to try to measure these coefficients experimantally. As soon as some coefficient(s) is proved to be non-zero, we know that the Standard Model is not exact. After measuring sufficiently many such coefficients we can start inventing a more fundamental theory which explains them.