The AVV triangle diagram in $SU(N_c)$ QCD and the generalized Crewther relation : scheme (in)dependent results. The generalized Crewther relation: is it valid in the proposed recently C-scheme ?

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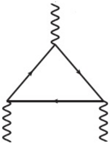
AVV Green function

Axial-Vector-Vecctor triangle diagram is related to $\pi^0 \to \gamma\gamma$ decay amplitude, which is proportiaonal to N_c. The measurement confirms that in QCD $N_c = 3$. When $(pq) = 0$ 1 of 3 form-factors is related to axial anomaly. In the case of coloured currents



$$G^{abc}_{\mu\nu\rho}(p,q) = i \int < 0|TA^a_\mu(x)V^b_\nu(y)V^c_\rho(0)|0 > e^{i(px+qy)}dxdy,$$

$V^a_\mu(x) = \bar{\psi}(x)\gamma_\mu t^a \psi(x)$ — vector non-singlet (NS) current,
$A^a_\mu(x) = \bar{\psi}(x)\gamma_\mu\gamma_5 t^a \psi(x)$ — axial NS current.

$$\int TV^a_\mu(x)V^b_\nu(0)e^{iqx}dx\Big|_{q^2\to\infty} \simeq d^{abc}\varepsilon_{\mu\nu\rho\lambda}\frac{q^\lambda}{Q^2}C_{Bjp}A^c_\rho(0) + \dots \ ,$$

$$i\int TA^a_\mu(x)A^b_\nu(0)e^{iqx}dx = \delta^{ab}(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2) \ .$$

The polarized Bjorken sum rule

The Bjorken function $C_{Bjp}(a_s)$ is a characteristic of deep inelastic scattering of charged leptons on polarized nucleons and is related to the corresponding Bjorken polarized sum rule:

$$\int\limits_0^1 \left(g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2) \right) dx = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C_{Bjp}(a_s(Q^2)) \ ,$$

where g_1^{lp} and g_1^{ln} are the structure functions of polarized lepton-proton and lepton-neutron deep inelastic scattering, characterizing the spin distribution of quarks inside nucleons. The ratio of axial and vector charge of the neutron β-decay is $g_A/g_V \approx -1.2723 \pm 0.0023$.

Bjorken function

Neglecting the mass dependence in the conditions of a large momentum transfer Q^2 in the $\overline{\text{MS}}$ renormalization scheme in the $\mathcal{O}(a_s^4)$ approximation of PT the Bjorken function can be represented in the form of two terms:

$$C_{Bjp}(a_s) = C_{Bjp}^{NS}(a_s) + \sum_f Q_f \; C_{Bjp}^{SI}(a_s) \ ,$$

where $C_{Bjp}^{NS}(a_s)$ and $C_{Bjp}^{SI}(a_s)$ — the flavor NS and SI contributions to the Bjorken function, $a_s = \alpha_s/\pi$, Q_f is the electric charge of the f-th quark.

$$C_{Bjp}^{NS}(a_s) = 1 + \sum_{k\geqslant 1} c_k a_s^k \ .$$

The coefficients of this series are known up to the fourth order:

$$c_1^{\overline{\text{MS}}} = -\frac{3}{4} C_F \ ,$$

$$c_2^{\overline{\text{MS}}} = \frac{21}{32} C_F^2 - \frac{23}{16} C_F C_A + \frac{1}{2} C_F T_F n_f \ , \ (\textit{Gorishny, Larin, 1986}) \ ,$$

$$c_3^{\overline{\text{MS}}} = -\frac{3}{128} C_F^3 + \left(\frac{1241}{576} - \frac{11}{12}\zeta_3\right) C_F^2 C_A + \left(-\frac{5437}{864} + \frac{55}{24}\zeta_5\right) C_F C_A^2$$
$$-\left(\frac{133}{576} + \frac{5}{12}\zeta_3\right) C_F^2 T_F n_f + \left(\frac{3535}{864} + \frac{3}{4}\zeta_3 - \frac{5}{6}\zeta_5\right) C_F C_A T_F n_f$$
$$-\frac{115}{216} C_F T_F^2 n_f^2 \ , \ (\textit{Larin, Vermaseren, 1991}) \ ,$$

where C_F and C_A are the Casimir operators, $(T^a T^a)_{ij} = C_F \delta_{ij}$, $f^{acd} f^{bcd} = C_A \delta^{ab}$, $T_F = 1/2$. For special case $SU_c(3)$ QCD $C_F = 4/3$, $C_A = 3$. ζ_n is the Riemann zeta-function. The coefficient $c_4^{\overline{\text{MS}}}$ was calculated in the work (*Baikov, Chetyrkin, Kühn, Phys.Rev.Lett. 104 (2010) 132004*) and contains additional color configurations $d_F^{abcd} d_A^{abcd}$ and $d_F^{abcd} d_F^{abcd}$.

The Adler function

The Adler function is related to the experimentally measured characteristic of the electron-positron annihilation process into hadrons, called the R-relation

$$R(s) = \frac{\sigma(e^+e^- \to \gamma \to hadrons)}{\sigma_{Born}(e^+e^- \to \gamma \to \mu^+\mu^-)} \ ,$$

by means of the dispersion relation:

$$D(Q^2) = Q^2 \int\limits_0^\infty ds \frac{R(s)}{(s+Q^2)^2} = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2) \ .$$

By analogy with the Bjorken function the following representation holds:

$$D(Q^2) = N_c \left(\sum_f Q_f^2 \ D^{NS}(a_s) + \left(\sum_f Q_f \right)^2 D^{SI}(a_s) \right) \ .$$

NS contribution to the Adler function

$$d_1^{\overline{\mathrm{MS}}} = \frac{3}{4} C_F \ ,$$

$$d_2^{\overline{\mathrm{MS}}} = -\frac{3}{32} C_F^2 + \left(\frac{123}{32} - \frac{11}{4}\zeta_3 \right) C_F C_A + \left(-\frac{11}{8} + \zeta_3 \right) C_F T_F n_f \ ,$$

(Chetyrkin, Kataev, Tkachov, 1979)

$$d_3^{\overline{\mathrm{MS}}} = -\frac{69}{128} C_F^3 + \left(-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5 \right) C_F^2 C_A +$$
$$+ \left(\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5 \right) C_F C_A^2 + \left(-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5 \right) C_F^2 T_F n_f$$
$$+ \left(-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5 \right) C_F C_A T_F n_f + \left(\frac{151}{54} - \frac{19}{9}\zeta_3 \right) C_F T_F^2 n_f^2$$

(Gorishny, Kataev, Larin, 1991) .

The coefficient $d_4^{\overline{\mathrm{MS}}}$ was computed in work of (*Baikov, Chetyrkin, Kühn, Phys.Rev.Lett. 104 (2010) 132004*) and comfirmed later in work (*Herzog, Ruijl, Ueda, Vermaseren, Vogt, JHEP 1708 (2017) 113*).

On the other hand the dressed NS AVV three-point Green's function which do not contain fermion and gluon loop contributions inside main triangle diagram is proportional to the triangular one-loop fermionic loop that determines the $\pi^0 \to \gamma\gamma$ decay:

$$G^{abc}_{\mu\nu\rho}(p,q)\bigg|_{conf-inv} = d^{abc}\Delta^{1-loop}_{\mu\nu\rho}(p,q) \ .$$

In this limit in the massless QCD the derived by Crewther relation (*Crewther, 1972*) is :

$$D^{NS}C^{NS}_{Bjp}\bigg|_{c-i \text{ limit}} = 1 \ .$$

However, when the charge renormalization is taken into account, the conformal symmetry is violated. This circumstance leads to a modification of the Crewther relation:

$$D^{NS}(a_s)C^{NS}_{Bjp}(a_s) = 1 + \Delta_{csb}(a_s) \ ,$$

where term $\Delta_{csb}(a_s)$ can be presented in the **gauge-invariant** $\overline{\text{MS}}$-scheme in the following factorized form (*Broadhurst, Kataev, Phys.Lett. B315 (1993)*):

$$\Delta_{csb} = \left(\frac{\beta(a_s)}{a_s}\right)\sum_{i\geqslant 1} K_i a_s^i \ , \qquad \beta(a_s) = \mu^2 \frac{da_s}{d\mu^2} = -\sum_{i\geqslant 0} \beta_i a_s^{i+2} \ .$$

This statement was confirmed explicitly at the $\mathcal{O}(a_s^4)$ level by *(Baikov, Chetyrkin, Kühn, Phys.Rev.Lett. 104 (2010) 132004)*. Theoretical indications on validity in all orders of PT were given in works of *(Crewther (1997); Gabadadze, Kataev (1995))* and others.

Consideration in the mMOM-*scheme*

Is the $\beta(a_s)$-factorization possible in the **gauge-dependent renormalization schemes** such as the mMOM-scheme?

$$A^a_{0,\,\mu} = \sqrt{Z_A} A^a_\mu, \quad c^a_0 = \sqrt{Z_c} c^a, \quad g_0 = \mu^\varepsilon Z_g g, \quad \xi_0 = Z_A Z_\xi^{-1} \xi \ ,$$

where A^a_μ, c^a are fields of gluons and ghosts correspondingly, ξ is the gauge parameter, included in the Lagrangian in the form $(\partial_\mu A^a_\mu)^2/2\xi$. The gauge-dependent mMOM-scheme is determined by the requirement of equality of the renormalization constant of the gluon-ghost-antighost vertex $Z_{cg} = Z_g Z_A^{1/2} Z_c$ to its analogue, defined in the $\overline{\mathrm{MS}}$-scheme (*Smekal, Maltman, Sternbeck, Phys.Lett. B681 (2009)*):

$$Z^{\mathrm{mMOM}}_{cg}(a^{\mathrm{mMOM}}_s) = Z^{\overline{\mathrm{MS}}}_{cg}(a^{\overline{\mathrm{MS}}}_s)$$

In this case the relation between coupling constants in these two schemes will look like as

$$a^{\mathrm{mMOM}}_s(\mu^2) = \frac{Z^{\mathrm{mMOM}}_A}{Z^{\overline{\mathrm{MS}}}_A} \left(\frac{Z^{\mathrm{mMOM}}_c}{Z^{\overline{\mathrm{MS}}}_c} \right)^2 a^{\overline{\mathrm{MS}}}_s(\mu^2) \ .$$

Taking into account the renormalization conditions on the polarization operators of gluons and ghosts, we obtain the following relations between the coupling constants and the gauge parameters
(Gracey, J.Phys. A46 (2013) 225403),
(Ruijl, Ueda, Vermaseren, Vogt, JHEP 1706 (2017) 040):

$$a_s^{\mathrm{mMOM}}(\mu^2) = \left(1 + \Pi^{\overline{\mathrm{MS}}}(\mu^2)\right)^{-1} \left(1 + \tilde{\Pi}^{\overline{\mathrm{MS}}}(\mu^2)\right)^{-2} a_s^{\overline{\mathrm{MS}}}(\mu^2)\ ,$$

$$\xi^{\mathrm{mMOM}}(\mu^2) = \left(1 + \Pi^{\overline{\mathrm{MS}}}(\mu^2)\right) \xi^{\overline{\mathrm{MS}}}\ ,$$

where $\Pi^{\overline{\mathrm{MS}}}$ and $\tilde{\Pi}^{\overline{\mathrm{MS}}}$ are the self-energy operators of the gluons and ghosts correspondingly, which depend on $\xi^{\overline{\mathrm{MS}}}$.

Further we find the following expansions of the quantities defined in the $\overline{\mathrm{MS}}$-scheme in terms of the quantities computed in the mMOM-scheme:

$$\xi^{\overline{\mathrm{MS}}} = \xi\Bigg(1 + \left[\left(\frac{97}{144} + \frac{1}{8}\xi + \frac{1}{16}\xi^2\right)C_A - \frac{5}{9}T_F n_f\right]a_s +$$

$$+\left[\left(\frac{5591}{4608} - \frac{3}{16}\zeta_3 + \left(-\frac{121}{1536} + \frac{1}{8}\zeta_3\right)\xi + \frac{7}{256}\xi^2 + \frac{7}{256}\xi^3 + \frac{1}{256}\xi^4\right)C_A^2 +\right.$$

$$\left. +\left(-\frac{371}{576} - \frac{1}{2}\zeta_3\right)C_A T_F n_f + \left(-\frac{55}{48} + \zeta_3\right)C_F T_F n_f\right]a_s^2 + \ldots\Bigg) ,$$

$$a_s^{\overline{\mathrm{MS}}} = a_s + b_1 a_s^2 + b_2 a_s^3 + \ldots ,$$

$$b_1 = \left(-\frac{169}{144} - \frac{1}{8}\xi - \frac{1}{16}\xi^2\right)C_A + \frac{5}{9}T_F n_f \ ,$$

$$b_2 = \left(-\frac{18941}{20736} + \frac{39}{128}\zeta_3 + \left(\frac{889}{2304} - \frac{11}{64}\zeta_3\right)\xi + \left(\frac{203}{2304} + \frac{3}{128}\zeta_3\right)\xi^2 - \frac{3}{256}\xi^3\right)C_A^2$$

$$+\left(-\frac{107}{648} + \frac{\zeta_3}{2} - \frac{5}{36}\xi - \frac{5}{72}\xi^2\right)C_A T_F n_f + \left(\frac{55}{48} - \zeta_3\right)C_F T_F n_f + \frac{25}{81}T_F^2 n_f^2 \ .$$

where $a_s^{\mathrm{mMOM}} = a_s$ и $\xi^{\mathrm{mMOM}} = \xi$.

RG β-function in the mMOM*-scheme*

$$\beta^{\mathrm{mMOM}} = \beta^{\overline{\mathrm{MS}}}\frac{\partial a_s^{\mathrm{mMOM}}}{a_s^{\overline{\mathrm{MS}}}} + \xi^{\overline{\mathrm{MS}}}\gamma_\xi^{\overline{\mathrm{MS}}}\frac{\partial a_s^{\mathrm{mMOM}}}{\partial\xi^{\overline{\mathrm{MS}}}} \;,$$

$$\beta_0^{\mathrm{mMOM}} = \frac{11}{12}C_A - \frac{1}{3}T_F n_f \;,$$

$$\beta_1^{\mathrm{mMOM}} = \left[\frac{17}{24} - \frac{13}{192}\xi - \frac{5}{96}\xi^2 + \frac{1}{64}\xi^3\right]C_A^2 + \left[-\frac{5}{12} + \frac{1}{24}\xi + \frac{1}{24}\xi^2\right]C_A T_F n_f - $$
$$-\frac{1}{4}C_F T_F n_f \;,$$

$$\beta_2^{\mathrm{mMOM}} = \left[\frac{9655}{4608} - \frac{143}{512}\zeta_3 + \left(-\frac{1097}{6144} + \frac{33}{512}\zeta_3\right)\xi + \left(-\frac{725}{6144} + \frac{13}{512}\zeta_3\right)\xi^2 + \right.$$
$$\left. + \left(\frac{21}{2048} - \frac{3}{512}\right)\xi^3 + \frac{55}{6144}\xi^4\right]C_A^3 + \left[-\frac{2009}{1152} - \frac{137}{384}\zeta_3 + \frac{37}{384}\xi + \left(\frac{23}{256} - \frac{\zeta_3}{128}\right)\xi^2 + \right.$$
$$\left. + \frac{1}{128}\xi^3\right]C_A^2 T_F n_f + \left[-\frac{641}{576} + \frac{11}{12}\zeta_3 + \frac{1}{16}\xi + \frac{3}{64}\xi^2\right]C_A C_F T_F n_f + \frac{1}{32}C_F^2 T_F n_f +$$
$$\left[\frac{23}{96} + \frac{\zeta_3}{6}\right]C_A T_F^2 n_f^2 + \left[\frac{23}{72} - \frac{\zeta_3}{3}\right]C_F T_F^2 n_f^2$$

NS contribution to the Bjorken function in the mMOM*-scheme*

Using the renormalization invariance of the $C^{NS}_{Bjp}(a_s)$ function and the obtained relation $a_s^{\overline{\text{MS}}}(\xi^{\text{mMOM}}, a_s^{\text{mMOM}})$ we find:

$$c_1^{\text{mMOM}} = -\frac{3}{4} C_F \ ,$$

$$c_2^{\text{mMOM}} = \frac{21}{32} C_F^2 + \left(-\frac{107}{192} + \frac{3}{32}\xi + \frac{3}{64}\xi^2 \right) C_F C_A + \frac{1}{12} C_F T_F n_f \ ,$$

$$c_3^{\text{mMOM}} = -\frac{3}{128} C_F^3 + \left[\frac{13}{9} + \frac{3}{8}\zeta_3 - \frac{5}{6}\zeta_5 - \frac{1}{48}\xi - \frac{1}{96}\xi^2 \right] C_F C_A T_F n_f +$$
$$+ \left[-\frac{13}{36} + \frac{\zeta_3}{3} \right] C_F^2 T_F n_f + \left[\frac{1415}{2304} - \frac{11}{12}\zeta_3 - \frac{21}{128}\xi - \frac{21}{256}\xi^2 \right] C_F^2 C_A -$$
$$- \frac{5}{24} C_F T_F^2 n_f^2 + \left[-\frac{20585}{9216} - \frac{117}{512}\zeta_3 + \frac{55}{24}\zeta_5 + \left(\frac{215}{3072} + \frac{33}{256}\zeta_3 \right)\xi + \right.$$
$$\left. + \left(\frac{349}{3072} - \frac{9}{512}\zeta_3 \right)\xi^2 + \frac{9}{1024}\xi^3 \right] C_F C_A^2 \ .$$

NS contribution to the Adler function in the mMOM-*scheme*

$$d_1^{\mathrm{mMOM}} = \frac{3}{4} C_F \; ,$$

$$d_2^{\mathrm{mMOM}} = -\frac{3}{32} C_F^2 + \left[\frac{569}{192} - \frac{11}{4}\zeta_3 - \frac{3}{32}\xi - \frac{3}{64}\xi^2 \right] C_F C_A + \left[\zeta_3 - \frac{23}{24} \right] C_F T_F n_f \; ,$$

$$d_3^{\mathrm{mMOM}} = -\frac{69}{128} C_F^3 + \left[-\frac{1355}{768} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5 + \frac{3}{128}\xi + \frac{3}{256}\xi^2 \right] C_F^2 C_A +$$

$$+ \left[-\frac{2033}{192} + \frac{89}{12}\zeta_3 + \frac{5}{6}\zeta_5 + \left(\frac{23}{96} - \frac{\zeta_3}{4} \right)\xi + \left(\frac{23}{192} - \frac{\zeta_3}{8} \right)\xi^2 \right] C_F C_A T_F n_f +$$

$$+ \left[\frac{50575}{3072} - \frac{18929}{1536}\zeta_3 - \frac{55}{24}\zeta_5 + \left(-\frac{2063}{3072} + \frac{143}{256}\zeta_3 \right)\xi + \right.$$

$$\left. + \left(-\frac{1273}{3072} + \frac{185}{512}\zeta_3 \right)\xi^2 - \frac{9}{1024}\xi^3 \right] C_F C_A^2 +$$

$$+ \left[\frac{29}{96} + 4\zeta_3 - 5\zeta_5 \right] C_F^2 T_F n_f + \left[\frac{3}{2} - \zeta_3 \right] C_F T_F^2 n_f^2 \; .$$

Is the factorization of RG β-function possible in the generalized Crewther relation in the gauge-dependent schemes?

Using the obtained expressions for the Bjorken and Adler functions, we find that in the $\mathcal{O}(a_s^2)$ approximation the factorization of the β-function is possible for any value of gauge parameter ξ:

$$K_1^{\rm mMOM} = K_1^{\overline{\rm MS}} = \Big(-\frac{21}{8} + 3\zeta_3 \Big) C_F \ ,$$

In the $\mathcal{O}(a_s^3)$ order of PT this property holds for three values of the gauge parameter only, namely

$$\xi = -3, \quad -1, \quad 0 \ .$$

The β-factorization in the mMOM*-scheme in the* $\mathcal{O}(a_s^3)$ *approxiamation*

$$\textit{Landau gauge} \quad \xi = 0:$$

$$K_2^{\rm mMOM} = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(-\frac{2591}{192} + \frac{91}{8}\zeta_3\right)C_F C_A + \\ + \left(\frac{31}{8} - 3\zeta_3\right)C_F T_F n_f \ ,$$

$$\textit{anti-Feynman gauge} \quad \xi = -1:$$

$$K_2^{\rm mMOM} = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(-\frac{1327}{96} + \frac{47}{4}\zeta_3\right)C_F C_A + \\ + \left(\frac{31}{8} - 3\zeta_3\right)C_F T_F n_f \ ,$$

$$\textit{Stefanis–Mikhailov gauge} \quad \xi = -3:$$

$$K_2^{\rm mMOM} = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(-\frac{695}{48} + \frac{25}{2}\zeta_3\right)C_F C_A + \\ + \left(\frac{31}{8} - 3\zeta_3\right)C_F T_F n_f \ .$$

The β-factorization in the mMOM-*scheme in the* $\mathcal{O}(a_s^4)$ *approxiamation*

In the $\mathcal{O}(a_s^4)$ order of PT the factorization property of the RG β-function in the generalized Crewther relation remains valid for the Landau gauge $\xi = 0$ only, namely

$$K_{3,\,\xi=0}^{\mathrm{mMOM}} = \left(\frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7\right)C_F^3 + \left(\frac{132421}{4608} + \frac{451}{8}\zeta_3 - \right.$$
$$\left. - \frac{3685}{48}\zeta_5 - \frac{105}{8}\zeta_7\right)C_F^2 C_A +$$
$$+ \left(-\frac{1840145}{18432} + \frac{152329}{3072}\zeta_3 + \frac{2975}{48}\zeta_5 - \frac{2113}{128}\zeta_3^2\right)C_F C_A^2 +$$
$$+ \left(-\frac{1273}{144} - \frac{599}{24}\zeta_3 + \frac{75}{2}\zeta_5\right)C_F^2 T_F n_f + \left(-\frac{49}{6} + \frac{7}{2}\zeta_3 + 5\zeta_5\right)C_F T_F^2 n_f^2 +$$
$$\left(\frac{71251}{1152} - \frac{539}{24}\zeta_3 - \frac{125}{3}\zeta_5 + \frac{5}{2}\zeta_3^2\right)C_F C_A T_F n_f \ .$$

Consideration in the MOMgggg-scheme

It is interesting to find out whether there are other MOM-schemes in QCD, which respect the property of the β-function factorization in the GCR for concrete choice of the gauge parameter. We consider the *MOMgggg*-scheme, determined by renormalization of the quartic gluon vertex through subtractions of UV divergences in the symmetric point *(Gracey, Phys. Rev. D 90 (2014) 025011)*. For Landau gauge in QCD with $SU(3)$ color group we find:

$$\beta_1^{\mathrm{MOMgggg}}\Big|_{\xi=0}^{N_c=3} = \frac{51}{8} - \frac{19}{24}n_f \ ,$$

$$K_2^{\mathrm{MOMgggg}}\Big|_{\xi=0}^{N_c=3} = -\frac{280073}{8640} + \frac{3017}{100}\log\left(\frac{4}{3}\right) - \frac{595}{256}\Phi_1 - \frac{50533}{51200}\Phi_2$$
$$+\zeta_3\left(\frac{15973}{360} - \frac{862}{25}\log\left(\frac{4}{3}\right) + \frac{85}{32}\Phi_1 + \frac{7219}{6400}\Phi_2\right) - \frac{80}{3}\zeta_5 +$$
$$\left[\frac{65}{36} - \frac{49}{24}\log\left(\frac{4}{3}\right) - \frac{49}{96}\Phi_1 + \frac{7}{96}\Phi_2 + \zeta_3\left(-\frac{10}{9} + \frac{7}{3}\log\left(\frac{4}{3}\right) + \frac{7}{12}\Phi_1 + \frac{1}{12}\Phi_2\right)\right]n_f$$

The special functions Φ_1 and Φ_2 are expressed through the Clausen function $\mathrm{Cl}_2(\theta)$ and have the following form

$$\Phi_1 = \sqrt{2}\left[2\mathrm{Cl}_2\left(2\arccos\left(\frac{1}{\sqrt{3}}\right)\right) + \mathrm{Cl}_2\left(2\arccos\left(\frac{1}{3}\right)\right)\right] ,$$

$$\Phi_2 = \frac{4}{\sqrt{5}}\left[2\mathrm{Cl}_2\left(2\arccos\left(\frac{2}{3}\right)\right) + \mathrm{Cl}_2\left(2\arccos\left(\frac{1}{9}\right)\right)\right] ,$$

$$\mathrm{Cl}_2(\theta) = -\int\limits_0^\theta dx \log\left|2\sin\frac{x}{2}\right| ,$$

and numerically $\Phi_1 \approx 2.832045$ and $\Phi_2 \approx 3.403614$ correspondingly.

At $\xi = -3$ we have:

$$\frac{\beta_1^{\overline{\text{MS}}} - \beta_{1,\ \xi=-3}^{\text{MOMgggg}}}{\beta_0} = -\frac{333}{20} - \frac{3537}{200}\log\left(\frac{4}{3}\right) + \frac{9}{4}\Phi_1 + \frac{33993}{6400}\Phi_2\ ,$$

$$K_2^{\text{MOMgggg}}\Big|_{\xi=-3}^{N_c=3} = -\frac{9337}{270} + \frac{13769}{400}\log\left(\frac{4}{3}\right) - \frac{35}{32}\Phi_1 - \frac{2191}{12800}\Phi_2$$
$$+\zeta_3\left(\frac{2108}{45} - \frac{1967}{50}\log\left(\frac{4}{3}\right) + \frac{5}{4}\Phi_1 + \frac{313}{1600}\Phi_2\right) - \frac{80}{3}\zeta_5$$
$$+\Bigg[\frac{65}{36} - \frac{49}{24}\log\left(\frac{4}{3}\right) - \frac{49}{96}\Phi_1 + \frac{7}{96}\Phi_2$$
$$+\zeta_3\left(-\frac{10}{9} + \frac{7}{3}\log\left(\frac{4}{3}\right) + \frac{7}{12}\Phi_1 + \frac{1}{12}\Phi_2\right)\Bigg]n_f\ .$$

Thus, we come to conclusion that in the $\mathcal{O}(a_s^3)$ order of PT in the *MOMgggg*-scheme the factorization property holds in Landau and Stefanis–Mikhailov gauges and is not satisfied in anti-Feynman gauge (the gauge $\xi = -1$ is the feature of the mMOM-scheme since $\beta_1^{\text{mMOM}} = \beta_1^{\overline{\text{MS}}}$ in this gauge).

Are the values of $\xi = -3$ and $\xi = 0$ distinguished in all gauge-dependent schemes?

Using relation $a_s^{\overline{\mathrm{MS}}} = a_s^{AS} + \sum\limits_{k=1} b_k^{AS} (a_s^{AS})^{k+1}$ and explicit form of the term which breaks the conformal symmetry, we arrive at equation (AS denotes any scheme with linear covariant gauge):

$$\frac{\beta^{\overline{\mathrm{MS}}}(a_s^{\overline{\mathrm{MS}}}(a_s^{AS}))}{a_s^{\overline{\mathrm{MS}}}(a_s^{AS})} K^{\overline{\mathrm{MS}}}(a_s^{\overline{\mathrm{MS}}}(a_s^{AS})) = \frac{\beta^{AS}(a_s^{AS})}{a_s^{AS}} K^{AS}(a_s^{AS}) \ ,$$

which allows us to obtain the following relations:

$$K_1^{AS} = K_1^{\overline{\mathrm{MS}}} \ ,$$

$$K_2^{AS} = K_2^{\overline{\mathrm{MS}}} + \left(\frac{\beta_1^{\overline{\mathrm{MS}}} - \beta_1^{AS}}{\beta_0} + 2 b_1^{AS} \right) K_1^{\overline{\mathrm{MS}}} \ ,$$

$$K_3^{AS} = K_3^{\overline{\mathrm{MS}}} + \left(\frac{\beta_1^{\overline{\mathrm{MS}}} - \beta_1^{AS}}{\beta_0} + 3 b_1^{AS} \right) K_2^{\overline{\mathrm{MS}}} + \left(2 b_2^{AS} + (b_1^{AS})^2 \right.$$
$$\left. + \frac{\beta_2^{\overline{\mathrm{MS}}} - \beta_2^{AS}}{\beta_0} + \frac{(3\beta_1^{\overline{\mathrm{MS}}} - 2\beta_1^{AS}) b_1^{AS}}{\beta_0} + \frac{\beta_1^{AS}(\beta_1^{AS} - \beta_1^{\overline{\mathrm{MS}}})}{\beta_0^2} \right) K_1^{\overline{\mathrm{MS}}} \ .$$

Conditions for factorization

Thus, we come to the conclusion that the question of the factorization of the β-function reduces to the conditions of division without remainder of terms of type $(\beta_1^{\overline{\text{MS}}} - \beta_1^{AS})/\beta_0$, $(\beta_2^{\overline{\text{MS}}} - \beta_2^{AS})/\beta_0$, $\beta_1^{AS}(\beta_1^{AS} - \beta_1^{\overline{\text{MS}}})/\beta_0^2$ etc. (hence we conclude that there is the factorization in QED in all popular schemes, such as MS–like, MOM and OS-schemes).
Further we find the relation of the $\mathcal{O}(a_s^2)$ coefficients of the β-functions:

$$\beta_1^{\overline{\text{MS}}} - \beta_1^{AS} = \xi\gamma_0^{\overline{\text{MS}}}(\xi)\frac{\partial b_1(\xi)}{\partial\xi} ,$$

where $\gamma_0^{\overline{\text{MS}}} = (-13/24 + \xi^{\overline{\text{MS}}}/8)C_A + T_F n_f/3$. From this relation we obtain, that at $\xi = 0$ $\beta_1^{AS} = \beta_1^{\overline{\text{MS}}}$, and division by β_0 is performed in the $\mathcal{O}(a_s^3)$ approximation.
At $\xi = -3$: $\gamma_0^{\overline{\text{MS}}} = -\beta_0$ and division without remainder also carried out.

Conditions for factorization: Landau gauge

Similarly, we obtain the following formulas at $\xi = 0$:

$$
\begin{aligned}
K_2^{AS} &= K_2^{\overline{\mathrm{MS}}} + 2b_1^{AS}(0)K_1^{\overline{\mathrm{MS}}} \, , \\
K_3^{AS} &= K_3^{\overline{\mathrm{MS}}} + 3b_1(0)K_2^{\overline{\mathrm{MS}}} + 3b_2(0)K_1^{\overline{\mathrm{MS}}} \, , \\
K_4^{AS} &= K_4^{\overline{\mathrm{MS}}} + 4b_1(0)K_3^{\overline{\mathrm{MS}}} + \Big(4b_2(0) + 2b_1^2(0)\Big)K_2^{\overline{\mathrm{MS}}} + 4b_3(0)K_1^{\overline{\mathrm{MS}}} \, , \\
K_5^{AS} &= K_5^{\overline{\mathrm{MS}}} + 5b_1(0)K_4^{\overline{\mathrm{MS}}} + 5\Big(b_2(0) + b_1^2(0)\Big)K_3^{\overline{\mathrm{MS}}} \\
&+ 5\Big(b_3(0) + b_1(0)b_2(0)\Big)K_2^{\overline{\mathrm{MS}}} + 5b_4(0)K_1^{\overline{\mathrm{MS}}} \quad \ldots
\end{aligned}
$$

Thus, we come to the conclusion that the gauge invariance of the renormalization schemes is a sufficient condition for the factorization of the RG β-function in the GCR, but is not a necessary condition.

Conclusions

- Initially, it was assumed that the true cause of the β-factorization lies in the gauge invariance of the renormalization schemes. In a more detailed analysis, it was unexpectedly found that a similar property is possible in gauge-dependent schemes.
- On the example of the mMOM-scheme we explained that gauges $\xi = 0, -3$ are highlighted among the remaining values of ξ (the gauge $\xi = -1$ is a specific of the mMOM-scheme).
- It is shown that for $\xi = -3$ the factorization of the β-function takes place in all schemes with linear covariant gauge in the $\mathcal{O}(a_s^3)$ approximation.
- The factorization in the Landau gauge occurs in all orders of PT (if such is observed in the $\overline{\mathrm{MS}}$-scheme; there are theoretical grounds for believing this).
- The gauge invariance of renormalization schemes is a sufficient but not necessary condition for factorization.
- The question of the theoretical reason for the factorization of the β-function still remains open.

The generalized Crewther relation: is it valid in the proposed recently C-scheme ?

(Boito, Jamin and Miravitlas, PRL 117 (2016) 152001) : the C-scheme coupling $a_s^C = \alpha_s^C/\pi$ is defined as

$$\frac{1}{a_s^C} + \frac{\beta_1}{\beta_0} ln(a_s^C) - \frac{\beta_0}{2} C = \beta_0 ln(Q^2/\Lambda^2_{\overline{MS}})$$

where C -is the free parameter (the similar free parameter was introduced by *A.A. Vladimirov, Sov. J. Nucl. Phys. 31 (1980) 558* and the RG β-function in fixed as

$$\beta(a_s^C) = -\frac{\beta_0 (a_s^C)^2}{(1 - \frac{\beta_1}{\beta_0} a_s^C)}$$

In this scheme the coefficients of definite Eucledian two-point RG -invariant Green functions at 5-loop order (related to Higgs decay to $\overline{q}q$ pair correlatot of scalar quark currents) are free from Riemann ζ-functions from even arguments. Was used *(Baikov and Chetyrkin (2018)* to get predictions for contributions of these ζ-functions to six-loop coefficients of the QCD β-function.

Result and Hope

However, like in the t' Hooft scheme - in the C-scheme the factorization of the β -function in the generalized Crewther relation IS VIOLATED
In case one will be able to undesratnd whether the factorization of the β-function in the generalized Crewther relation is the fundamental property, related to violation of conformal symmetry in QCD (or to conformal anomaly) - the indications do exist *(Crewther, 1997)*
- this may allow to understand whether possible theoretical restrictions on the class of renormalization schemes to be used in the theoretical (and phenomenological) studies do exist (Personal point of view).

Thank you for your attention!