All-loop conjecture for integrands of reggeon amplitudes in N=4 SYM.

Leonid Bork in collaboration with
Andrei Onishchenko hep-th
1607.02320, 1610.09693,
1704.04758 and
1802.03986 in collaboration with
Andrei Onishchenko and Art’em Bolshov

ITEP, BLTP, CFAR VNIIA

N=4 SYM theory

- **N=4 SYM - one may hope that this theory is <u>exactly solvable</u>.**
- **Physical content - resembles perturbative part of QCD (massless QED without running of the coupling). <u>Tree amplitudes identical to QCD</u>.**
- **The correlation functions in this theory can be studied in the weak and strong regimes (via AdS/CFT).**
- **The computation of anomalous dimensions of local operators in N=4 SYM in planar limit can be reduced to the problem of solving some integrable system.**
- **There are numerous results for perturbative expansions of amplitudes (S-matrix) and form factors/cor.functions with some results valid in all orders of PT (BDS ansatz for 4,5 points, collinear OPE).**
- **N=4 SYM is perfect theoretical laboratory development and tests of <u>new ideas, methods and representations</u> for D=4 gauge theories.**

On-shell variables

- **Why we know so much about N=4 SYM now ?**
- **Proper variables - helicity spinors:**

Rev. in
BernDixonKosower 96

$$p_\mu^{(i)} \mapsto (\sigma^\mu)_{\alpha\dot{\alpha}} p_\mu^{(i)} = \lambda_\alpha^{(i)} \tilde{\lambda}_{\dot{\alpha}}^{(i)}$$

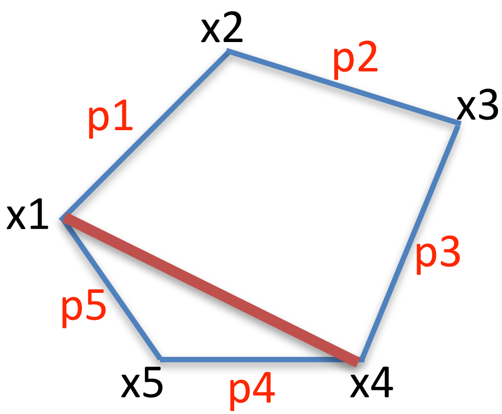
$$\lambda_\alpha \in SL(2, \mathbb{C})$$

$$\epsilon^{\alpha\beta} \lambda_\alpha^{(i)} \lambda_\beta^{(j)} \equiv \langle ij \rangle = \sqrt{(p_i + p_j)^2} e^{i\phi_{ij}} = \sqrt{s_{ij}} e^{i\phi_{ij}}, \ \phi_{ij} \in \mathbb{R} \quad (\langle ij \rangle)^* \equiv [ij]$$

- **and momentum twistors:**

$$Z_i^M = \begin{pmatrix} \lambda_i^\alpha \\ \mu_i^{\dot{\alpha}} \end{pmatrix}, \ \mu_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} \lambda_{\alpha i}$$

$$p_i^{\alpha\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i-1}^{\alpha\dot{\alpha}}$$



Hoges ~10

q=p5+p6

N=4 SYM theory. Amplitudes and form factors

Lots of explicit answers in weak and strong coupling for amplitudes:

$$A_{n,k}\left(\Omega_1,\ldots,\Omega_n\right)=\langle\Omega_1\ldots\Omega_m|0\rangle$$

Dixon et.al, *Basso, et al.*

k - labels overall helicity

Form factors:

$$F_n(\Omega_1,\ldots,\Omega_n;\mathcal{O})=\langle\Omega_1\ldots\Omega_m|\mathcal{O}|0\rangle$$

Maldacena, Zhiboedov, Brandhuber, et al. Bork, et al.

And correlation functions:

Gromov, et al. Korchemsky et al.

$$\mathcal{G}_n=\langle 0|\mathcal{O}_1\ldots\mathcal{O}_n|0\rangle$$

$$\Omega=g^{+}+\tilde{\eta}_A\psi^A+\frac{1}{2!}\tilde{\eta}_A\tilde{\eta}_B\phi^{AB}+\frac{1}{3!}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\epsilon^{ABCD}\bar{\psi}_D+\frac{1}{4!}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\tilde{\eta}_D\epsilon^{ABCD}g^{-}$$

The N=4 on-shell momentum superspace is often used to describe on-shell states of N=4 SYM. Grassmann variables encode helicity.

Examples of some form factors of gauge invariant operators

Wilson line operator (operator corresponding to reggeon state in *Lipatov et al. 95* approach to high energy scattering):

$$\mathcal{W}_p^c(k) = \int d^4x e^{ix\cdot k} \mathrm{Tr}\left\{ \frac{1}{\pi g} t^c\, \mathcal{P} \exp\left[\frac{ig}{\sqrt{2}} \int_{-\infty}^{\infty} ds\, p\cdot A_b(x+sp) t^b \right] \right\}$$

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) = \frac{q\cdot k}{q\cdot p} \quad \text{and} \quad q^2 = 0. \qquad k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle},\ \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}$$

$$A^*_{m+n}\left(\Omega_1,\ldots,\Omega_m, g^*_{m+1},\ldots,g^*_{n+m}\right) = \langle\Omega_1\ldots\Omega_m|\prod_{j=1}^{n}\mathcal{W}_{p_{m+j}}(k_{m+j})|0\rangle$$

I.e. in general Wilson line (W.L.) form factors are functions of the following arguments:

$$A^*_{k,m+n}\left(\Omega_1,\ldots,g^*_{m+n}\right) = A^*_{k,m+n}\left(\{\lambda_i,\tilde{\lambda}_i,\tilde{\eta}_i\}_{i=1}^{m};\{k_i,\lambda_{p,i},\tilde{\lambda}_{p,i}\}_{i=m+1}^{m+n}\right)$$

Also W.L.form factors are sometimes called **reggeon amplitudes** (W ~ reggeside gluon creation/ annihilation operator).

Examples of some form factors of gauge invariant operators

Wilson line operator (operator corresponding to reggeon state in *Lipatov et al. 95* approach to high energy scattering):

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For example in the simplest case (tree level):

$$A^*_{2,2+1}(\Omega_1,\Omega_2,g_3^*) \;=\; \prod_{A=1}^{4}\frac{\partial}{\partial\tilde{\eta}^A_{p_3}}\left[\frac{\delta^4(\lambda_1\tilde{\lambda}_1+\lambda_2\tilde{\lambda}_2+k_3)}{\kappa_3^*}\frac{\delta^8(\lambda_{p_3}\tilde{\eta}_{p_3}+\lambda_1\tilde{\eta}_1+\lambda_2\tilde{\eta}_2)}{\langle p_3\,1\rangle\langle 1\,2\rangle\langle 2\,p_3\rangle}\right]$$

BCFW recursion. Tree amplitudes

- **BCFW recursion:**

BrittoCachazoFengWitten 05

$$p_{(l)} = \lambda_{(l)}\tilde{\lambda}_{(l)} : \lambda_{(l)} \mapsto \lambda_{(l)} - z\lambda_{(k)}$$
$$p_{(k)} = \lambda_{(k)}\tilde{\lambda}_{(k)} : \tilde{\lambda}_{(k)} \mapsto \tilde{\lambda}_{(k)} + z\tilde{\lambda}_{(l)}$$

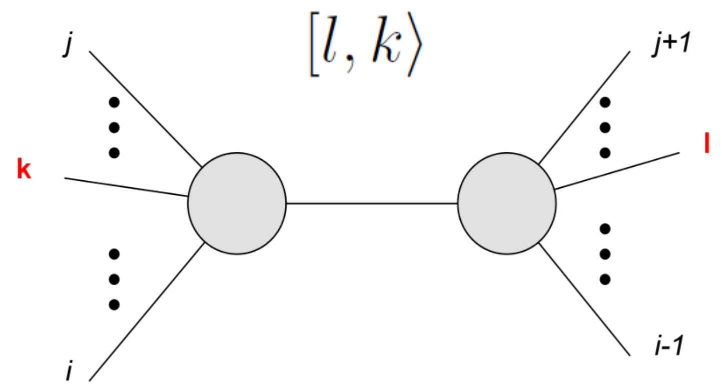
$$P^z_{ij} = p_i + ...p_j + z\lambda_{(k)}\tilde{\lambda}_{(l)}, \ (P^z_{ij})^2 = 0$$
$$z_{ij} : \ (p_i + ...p_j + z\lambda_{(k)}\tilde{\lambda}_{(l)})^2 = 0$$

$$A_n(z) \to 0 \text{ if } z \to \infty$$

$$z_{ij} = \frac{P^2_{ij}}{\langle k|P_{ij}|l]}$$

$$0 = \int_{C_\infty} \frac{dz}{z} A_n(z) = A_n(0) + \sum_{poles\ z_{ij}} Res[A_n(z)/z]$$

$$A_n = \sum_{z_{ij},pol} A_L(z_{ij}) \frac{1}{P^2_{ij}} A_R(z_{ij})$$



BCFW recursion. Amplitudes - momentum twistors

One can represent BCFW recursion in very compact form using momentum twistor variables:

$$\langle i,j,k,l\rangle = \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$$

$$(i,j)\cap(k,p,m) \equiv \mathcal{Z}_i\langle jkpm\rangle + \mathcal{Z}_j\langle ikpm\rangle$$

$$\mathcal{P}_{k,n}(\mathcal{Z}_1,...,\mathcal{Z}_n) = \mathcal{P}_{k,n-1}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n-1})$$

$$+\sum_{j=2}^{n-2}[j-1,j,n-1,n,1]\mathcal{P}_{k_1,n+2-j}(\mathcal{Z}_{I_j},\mathcal{Z}_j,\mathcal{Z}_{j+1},\ldots,\hat{\mathcal{Z}}_{n_j})\mathcal{P}_{k_2,j}(\mathcal{Z}_{I_j},\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_{j-1})$$

Where: $\mathcal{Z}_{n_j} = (n-1,n)\cap(1,j-1,j),\ \hat{\mathcal{Z}}_{I_j} = (j-1,j)\cap(1,n-1,n)$

$$[a,b,c,d,e] = \frac{\hat{\delta}^4(\langle a,b,c,d\rangle\chi_e + \text{cycl.})}{\langle a,b,c,d\rangle\langle b,c,d,e\rangle\langle c,d,e,a\rangle\langle d,e,a,b\rangle\langle e,a,b,c\rangle}$$

Example of solution:

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$$\frac{A_6^{NMHV}}{A_6^{MHV}} = [1,2,3,4,5]+[1,2,3,5,6]+[1,3,4,5,6]$$

Can be combined together into single "generatingfunction" …

Another representation for BCFW recursion solution. Grassmannian integral

One of such remarkable ideas is representations of amplitudes and leading singularities in N=4 SYM in terms of Grassmannian integral and development of on-shell diagram formalism and geometrical interpretation ("amplituhidron"):

On shell information about external particles

$$L_n^k[\Gamma] = \int_\Gamma \frac{d^{k\times n}C}{\mathrm{Vol}[GL(k)]} \frac{\delta^{k\times 2}\left(C\cdot\tilde{\lambda}\right)\delta^{k\times 4}\left(C\cdot\tilde{\eta}\right)\delta^{(n-k)\times 2}\left(C^\perp\cdot\lambda\right)}{(1\cdots k)\cdots(n-1\, n\cdots k-2)(n\, 1\cdots k-1)}$$

Hodges ~08

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$$A_{k,n}^{tree} = L_n^k[\Gamma_{tree}],\quad C\cdot C^\perp = \mathbf{1}$$

C - is nXk matrix - a point in Grassmannian

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This is multidimensional integral over multiple complex variables, which can be computed by residues. Different choices of integration contour gives different BCFW representations for <u>tree amplitudes</u> and <u>leading singularities</u> to all loop order (!). Also this is the most general form of rational Yangian invariant. And do not forget about <u>twistor strings</u>!

Another representation for BCFW recursion solution. Grassmannian integral

Similar expression can be obtained in momentum twistor representation:

On shell information about external particles

$$\mathcal{L}^k_{n+2} = \int \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[GL(k-2)]} \frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}.$$

Hodges ~08

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Mason Skiner~10

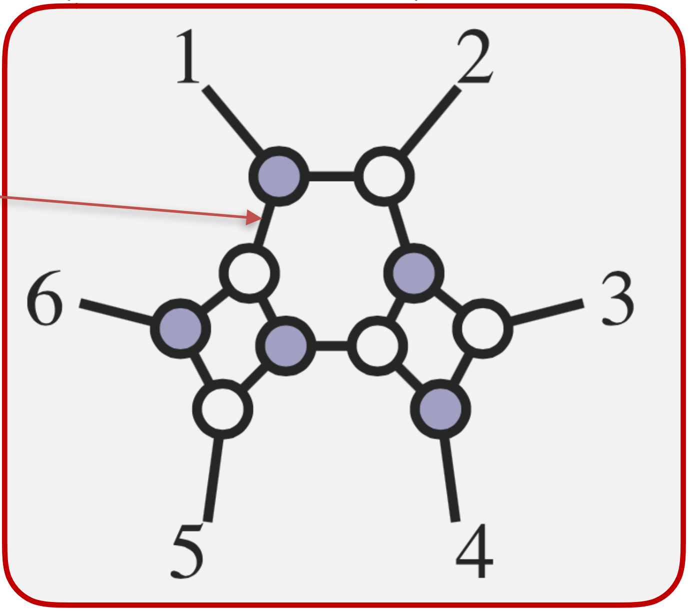
$$\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z}) = \prod_{a=1}^{k-2}\delta^{4|4}\left(\sum_{i=1}^{n+2} D_{ai}\mathcal{Z}_i\right)$$

Grassmannian integral, on shell diagrams and (decorated) permutations

$$A_{3,0}(\{\lambda_i,\tilde{\lambda}_i,\eta_i\}) = \int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\,\delta^2\left(\tilde{\lambda}_1+\alpha_1\tilde{\lambda}_3\right)\delta^2\left(\tilde{\lambda}_2+\alpha_2\tilde{\lambda}_3\right)\times\delta^2\left(\lambda_3+\alpha_1\lambda_1+\alpha_2\lambda_2\right)\times$$
$$\times\quad \hat{\delta}^4\left(\eta_1+\alpha_1\eta_3\right)\hat{\delta}^4\left(\eta_2+\alpha_2\eta_3\right),$$

$$A_{3,-1}(\{\lambda_i,\tilde{\lambda}_i,\eta_i\}) = \int \frac{d\beta_1}{\beta_1}\frac{d\beta_2}{\beta_2}\,\delta^2\left(\lambda_1+\beta_1\lambda_3\right)\delta^2\left(\lambda_2+\beta_2\lambda_3\right)\times\delta^2\left(\tilde{\lambda}_3+\beta_1\tilde{\lambda}_1+\beta_2\tilde{\lambda}_2\right)\times$$
$$\times\quad \hat{\delta}^4\left(\eta_3+\beta_1\eta_1+\beta_2\eta_2\right).$$

$$\frac{d^2\lambda_i d^2\tilde{\lambda}_i d^4\tilde{\eta}_i}{\mathrm{Vol}[GL(1)]}$$



picture
from th 1212.5605

One can show that integrating such vertexes in appropriate combination one can reproduce Grassmannian integral mentioned above.

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Grassmannian integral, on shell diagrams and (decorated) permutations

$$\int_\Gamma \frac{d^{n\times k}C_{al}}{Vol[GL(k)]}\frac{1}{M_1\ldots M_n}\prod_{a=1}^{k}\delta^2\left(\sum_{l=1}^{n}C_{al}\tilde{\lambda}_l\right)\delta^4\left(\sum_{l=1}^{n}C_{al}\eta_l\right)\times$$
$$\times\prod_{b=k+1}^{n}\delta^2\left(\sum_{l=1}^{n}\tilde{C}_{al}\lambda_l\right).$$

There is one to one correspondence between the following objects:

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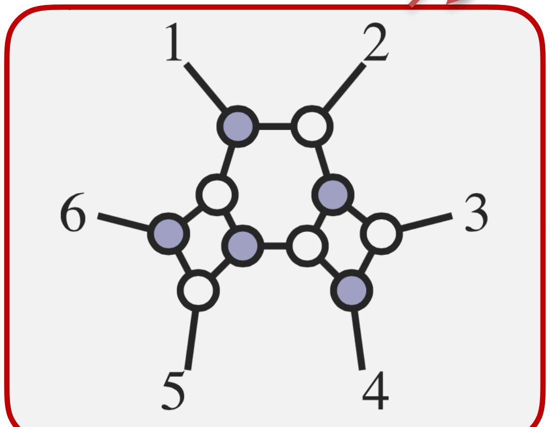
picture from th 1212.5605



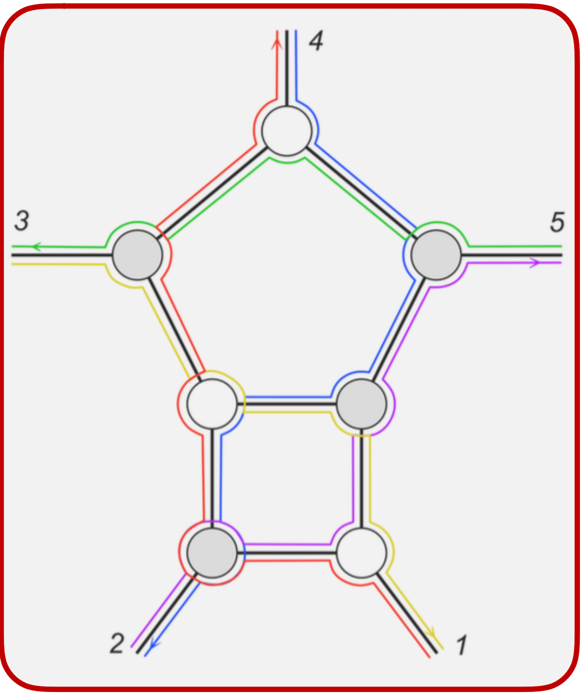
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 3 & 2 & 6 & 1 & 4 \end{pmatrix}$$

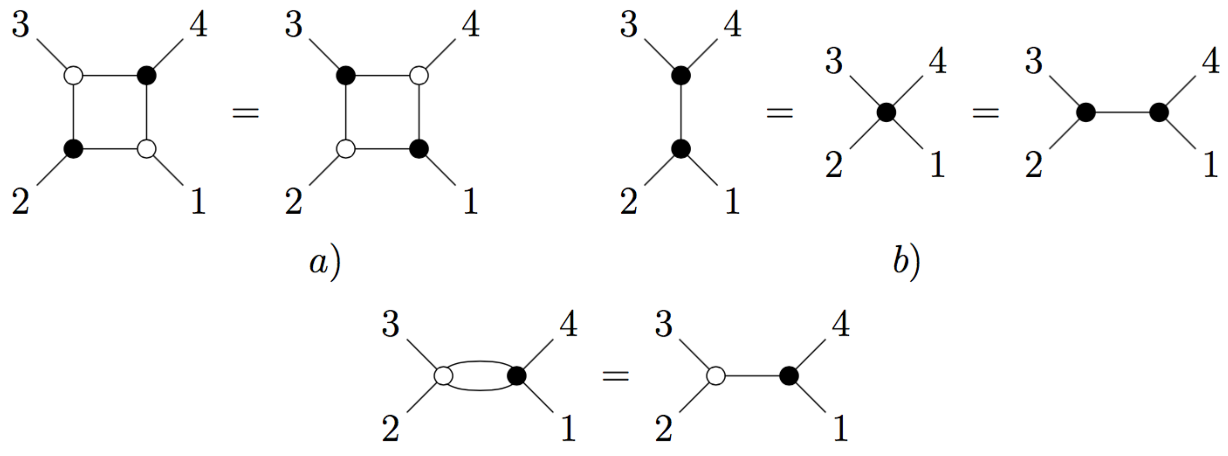
Grassmannian integral, on shell top-cell diagram

- **Every Grassmannian integral can be reduced to:**

$$\oint \frac{df_1}{f_1} \oint \frac{df_2}{f_2} \cdots \oint \frac{df_d}{f_d}\, \delta^{k\times 2}(C(f_i)\cdot\tilde{\lambda})\delta^{k\times 4}(C(f_i)\cdot\tilde{\eta})\delta^{(n-k)\times 2}(C^{\perp}(f_i)\cdot\lambda)$$

- **On shell diagrams are equivalent after:**





Top-cell for NMHV_5

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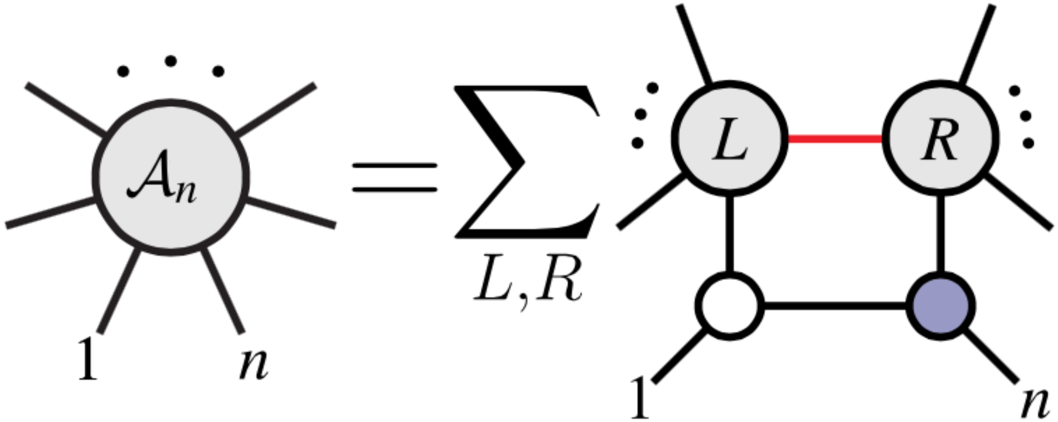
Grassmannian integral, on shell top-cell diagram

- **BCFW recursion in terms of on-shell diagrams:**



$$A_n = \sum_{z_{ij},pol} A_L(z_{ij}) \frac{1}{P_{ij}^2} A_R(z_{ij})$$

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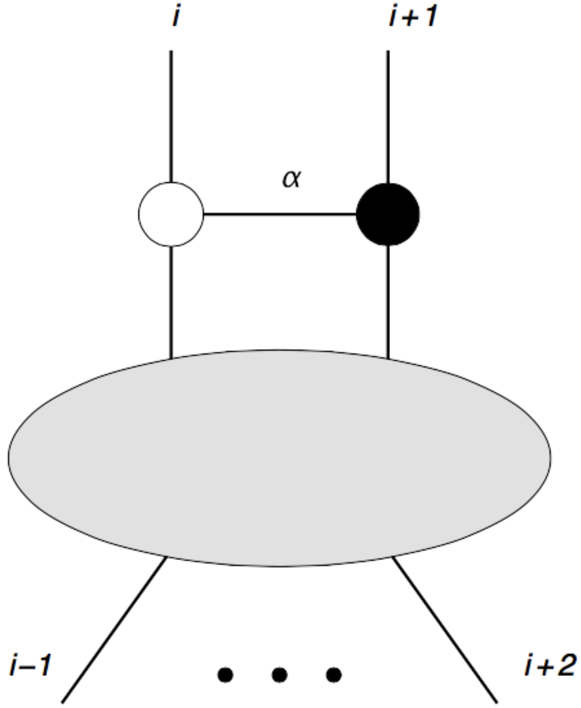
Grassmannian integral, on shell top-cell diagram



$$Br(i,i+1)\left[f(\ldots,\lambda_i,\tilde{\lambda}_i\tilde{\eta}_i,\ldots,\lambda_j,\tilde{\lambda}_j,\tilde{\eta}_j,\ldots)\right] = \int \frac{\mathrm{d}\alpha}{\alpha} f(\ldots,\lambda_i,\hat{\tilde{\lambda}}_i,\hat{\tilde{\eta}}_i,\ldots,\hat{\lambda}_j,\tilde{\lambda}_j,\tilde{\eta}_j,\ldots) =$$

$$\int \frac{\mathrm{d}\alpha}{\alpha} f(\ldots,\lambda_i,\tilde{\lambda}_i-\alpha\tilde{\lambda}_j,\tilde{\eta}_i-\alpha\tilde{\eta}_j,\ldots,\lambda_j+\alpha\lambda_i,\tilde{\lambda}_j,\tilde{\eta}_j,\ldots)$$

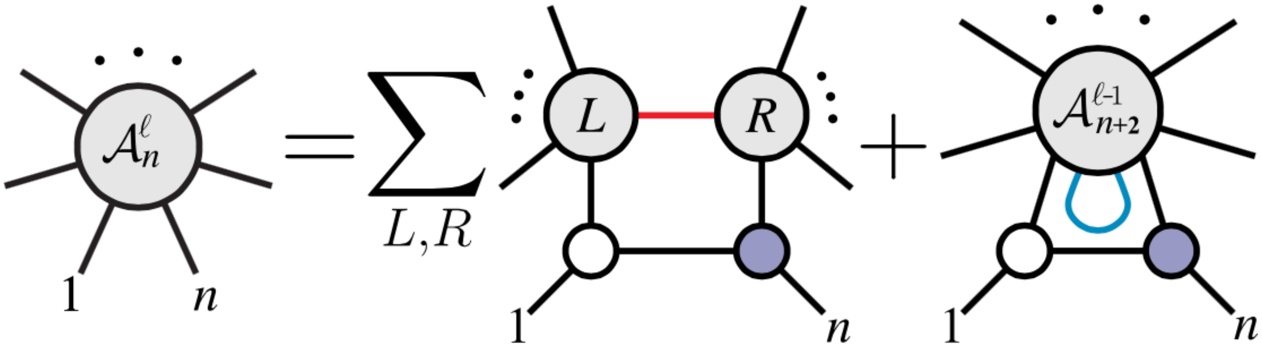
But what about loops …

But what about loops ? BCFW for loop integrands

Arkani Hammed *et-al* 09-10

$$I^{(L)}_{k,n} = I^{(L)}_{k,n-1}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n-1})$$
$$+\sum_{j=2}^{n-2}[j-1,j,n-1,n,1]I^{(L_1)}_{k_1,n+2-j}(\mathcal{Z}_{I_j},\mathcal{Z}_j,\mathcal{Z}_{j+1},\ldots,\hat{\mathcal{Z}}_{n_j})I^{(L_2)}_{k_2,j}(\mathcal{Z}_{I_j},\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_{j-1})$$
$$+\int\frac{d^{4|4}\mathcal{Z}_A d^{4|4}\mathcal{Z}_B}{\mathrm{Vol}[GL(2)]}\int_{GL(2)}[A,B,n-1,n,1]I^{(L-1)}_{k+1,n+2}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\hat{\mathcal{Z}}_{n_{AB}},\mathcal{Z}_A,\mathcal{Z}_B)$$

where $\hat{\mathcal{Z}}_{n_j}=(n-1,n)\cap(1,j-1,j)$, $\mathcal{Z}_{I_j}=(j-1,j)\cap(1,n-1,n)$, $\hat{\mathcal{Z}}_{n_{AB}}=(n-1,n)\cap(A,B,1)$ and $k_1+k_2+1=k$.



But what about loops ? BCFW for loop integrands

$$I^{(L)}_{k,n} = I^{(L)}_{k,n-1}(\mathcal{Z}_1,\ldots,\mathcal{Z}_{n-1})$$

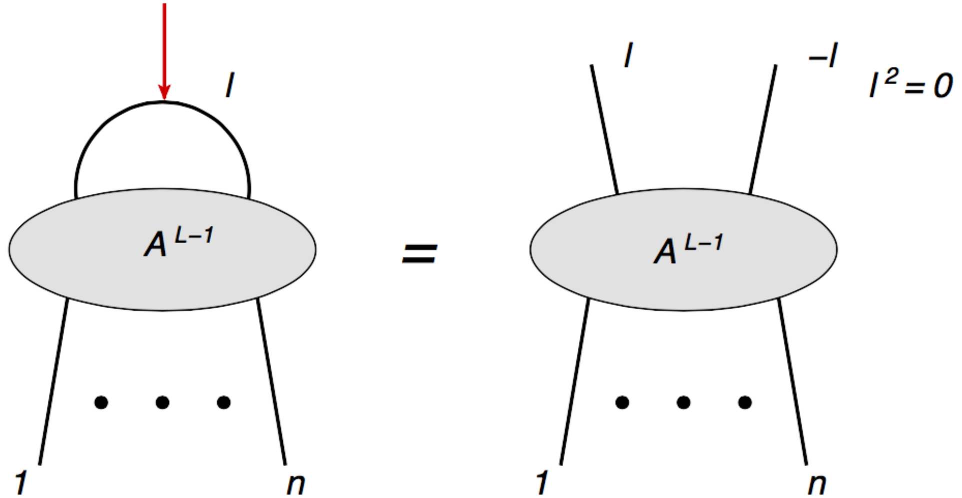
$$\hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$$

$$+\sum_{j=2}^{n-2}[j-1,j,n-1,n,1]\,I^{(L_1)}_{k_1,n+2-j}(\mathcal{Z}_{I_j},\mathcal{Z}_j,\mathcal{Z}_{j+1},\ldots,\hat{\mathcal{Z}}_{n_j})\,I^{(L_2)}_{k_2,j}(\mathcal{Z}_{I_j},\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\mathcal{Z}_{j-1})$$

$$+\int\frac{d^{4|4}\mathcal{Z}_A d^{4|4}\mathcal{Z}_B}{\mathrm{Vol}[GL(2)]}\int_{GL(2)}[A,B,n-1,n,1]\,I^{(L-1)}_{k+1,n+2}(\mathcal{Z}_1,\mathcal{Z}_2,\ldots,\hat{\mathcal{Z}}_{n_{AB}},\mathcal{Z}_A,\mathcal{Z}_B)$$

$$d^4l = \langle ABd^2A\rangle\langle ABd^2B\rangle$$

$$= \frac{d^4Z_Ad^4Z_B}{\mathrm{Vol}[GL(2)]}$$

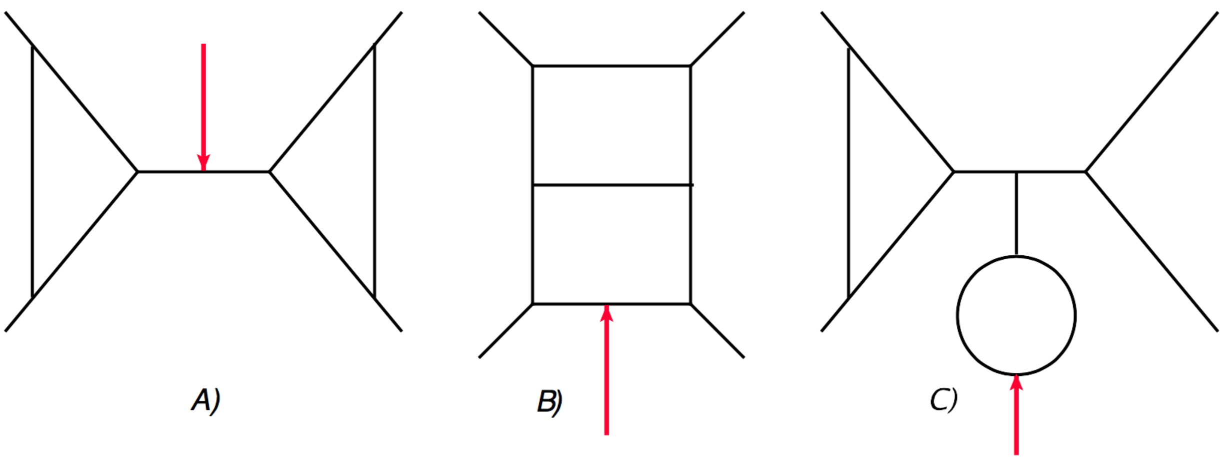


But what about loops ? BCFW for loop integrands

$$I_{k,n}^{(L)} = I_{k,n-1}^{(L)}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}) \qquad \hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$$

$$+ \sum_{j=2}^{n-2} [j-1, j, n-1, n, 1] I_{k_1, n+2-j}^{(L_1)}(\mathcal{Z}_{I_j}, \mathcal{Z}_j, \mathcal{Z}_{j+1}, \ldots, \hat{\mathcal{Z}}_{n_j}) I_{k_2, j}^{(L_2)}(\mathcal{Z}_{I_j}, \mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_{j-1})$$

$$+ \int \frac{d^{4|4}\mathcal{Z}_A d^{4|4}\mathcal{Z}_B}{\mathrm{Vol}[GL(2)]} \int_{GL(2)} [A, B, n-1, n, 1] I_{k+1, n+2}^{(L-1)}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \hat{\mathcal{Z}}_{n_{AB}}, \mathcal{Z}_A, \mathcal{Z}_B)$$

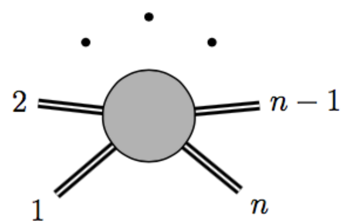


BCFW recursion for W.L. form factors

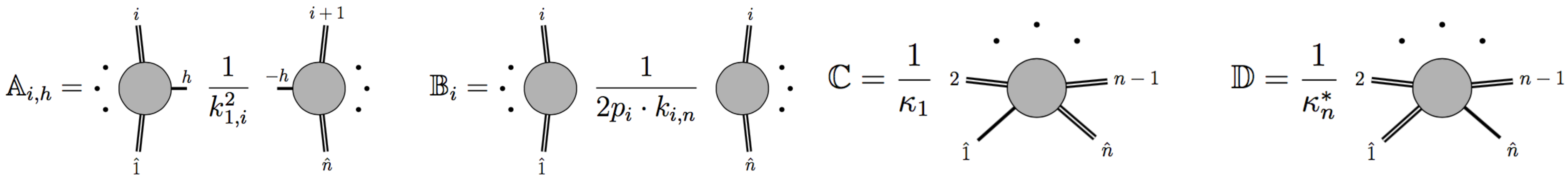
- **Also version of BCFW recursion exists for W.L. form factors:**

$$\hat{k}_i^{\mu}(z) \equiv k_i^{\mu} + z e^{\mu} = x_i(p_j) p_i^{\mu} - \frac{\kappa_i - [ij]z}{2} \frac{\langle i|\gamma^{\mu}|j]}{[ij]} - \frac{\kappa_i^*}{2} \frac{\langle j|\gamma^{\mu}|i]}{\langle ji \rangle},$$

$$\hat{k}_j^{\mu}(z) \equiv k_j^{\mu} - z e^{\mu} = x_j(p_i) p_j^{\mu} - \frac{\kappa_j}{2} \frac{\langle j|\gamma^{\mu}|i]}{[ji]} - \frac{\kappa_j^* + \langle ij \rangle z}{2} \frac{\langle i|\gamma^{\mu}|j]}{\langle ij \rangle}.$$



$$= \sum_{i=2}^{n-2} \sum_{h} \mathbb{A}_{i,h} + \sum_{i=2}^{n-1} \mathbb{B}_i + \mathbb{C} + \mathbb{D},$$



Kotko, Hammeren ~14

Gluing operator

$$\hat{A}_{n+1,n+2}[f] \equiv \int \prod_{i=n+1}^{n+2} \frac{d^2\lambda_i d^2\tilde{\lambda}_i d^4\tilde{\eta}_i}{\text{Vol}[GL(1)]} A^*_{2,2+1}(g^*, \Omega_{n+1}, \Omega_{n+2}) \times f\left(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}\right)$$

$$\hat{A}_{n+1,n+2}[f] = \frac{\langle p_{n+1}\xi_{n+1}\rangle}{\kappa^*_{n+1}} \int \frac{d\beta_1}{\beta_1} \wedge \frac{d\beta_2}{\beta_2} \frac{1}{\beta_1^2\beta_2} f\left(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}_{i=1}^{n+2}\right)\Big|_*$$

where $\big|_*$ denotes substitutions $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}_{i=n+1}^{n+2} \mapsto \{\lambda_i(\beta), \tilde{\lambda}_i(\beta), \tilde{\eta}_i(\beta)\}_{i=n+1}^{n+2}$ with

$$\lambda_{n+1}(\beta) = \underline{\underline{\lambda}}_{n+1} + \beta_2\underline{\underline{\lambda}}_{n+2}\,, \qquad \tilde{\lambda}_{n+1}(\beta) = \beta_1\underline{\underline{\tilde{\lambda}}}_{n+1} + \frac{(1+\beta_1)}{\beta_2}\underline{\underline{\tilde{\lambda}}}_{n+2}\,, \quad \tilde{\eta}_{n+1}(\beta) = -\beta_1\underline{\underline{\tilde{\eta}}}_{n+1}$$

$$\lambda_{n+2}(\beta) = \underline{\underline{\lambda}}_{n+2} + \frac{(1+\beta_1)}{\beta_1\beta_2}\underline{\underline{\lambda}}_{n+1}\,, \quad \tilde{\lambda}_{n+2}(\beta) = -\beta_1\underline{\underline{\tilde{\lambda}}}_{n+2} - \beta_1\beta_2\underline{\underline{\tilde{\lambda}}}_{n+1}\,, \quad \tilde{\eta}_{n+2}(\beta) = \beta_1\beta_2\underline{\underline{\tilde{\eta}}}_{n+1}$$

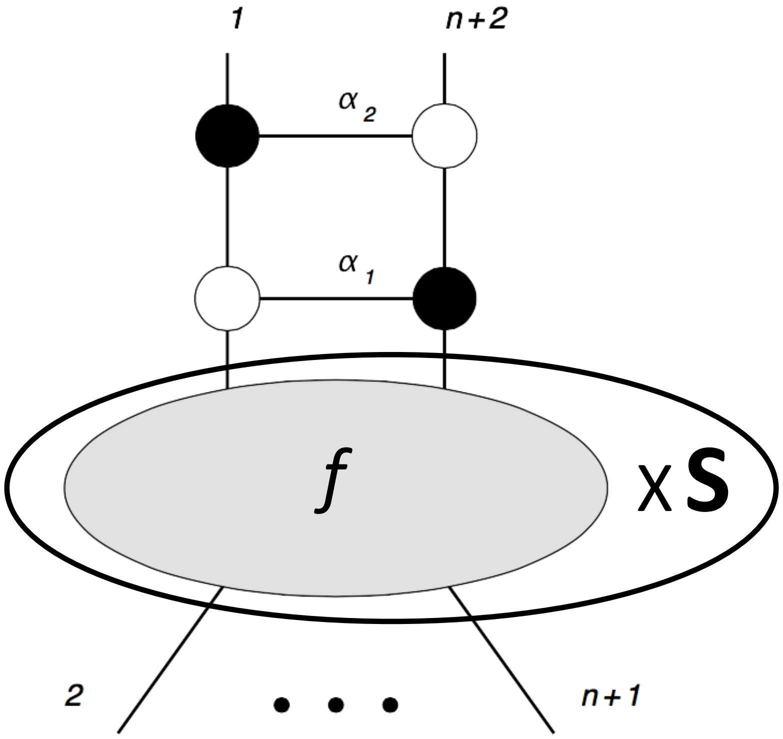
and

Bork, Onishchenko ~14, 16

$$\underline{\underline{\lambda}}_{n+1} = \lambda_p,\ \underline{\underline{\tilde{\lambda}}}_{n+1} = \frac{\langle\xi|k}{\langle\xi p\rangle},\ \underline{\underline{\tilde{\eta}}}_n = \tilde{\eta}_p;\ \ \underline{\underline{\lambda}}_{n+2} = \lambda_\xi,\ \underline{\underline{\tilde{\lambda}}}_{n+2} = \frac{\langle p|k}{\langle\xi p\rangle},\ \underline{\underline{\tilde{\eta}}}_{n+2} = 0.$$

Gluing operator in terms of on-shell diagrams

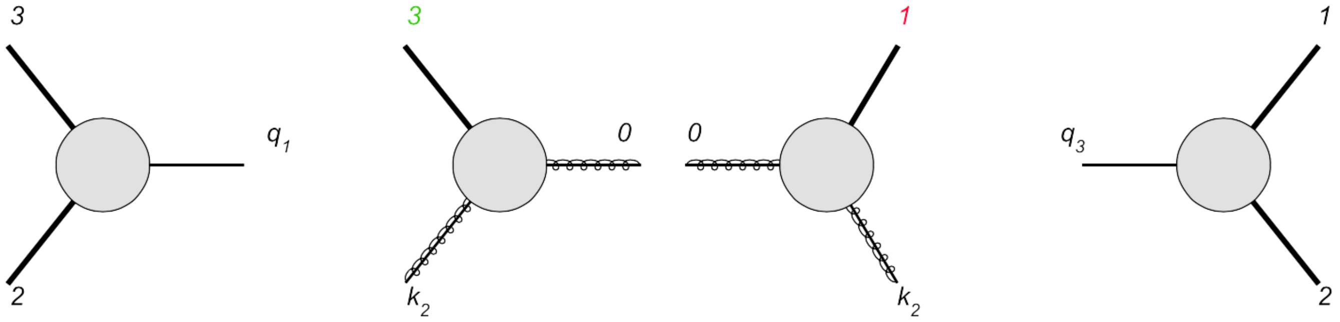
$$\hat{A}_{n+1,n+2}[f] =$$



Bork, Onishchenko ~14, 16

$$S(1, n+2, n+1) = \frac{\kappa^*_{n+1}\langle 1 n+1\rangle}{\langle 1 n+2\rangle\langle n+2 n+1\rangle}$$

Gluing operator and BCFW for on-shell amplitudes

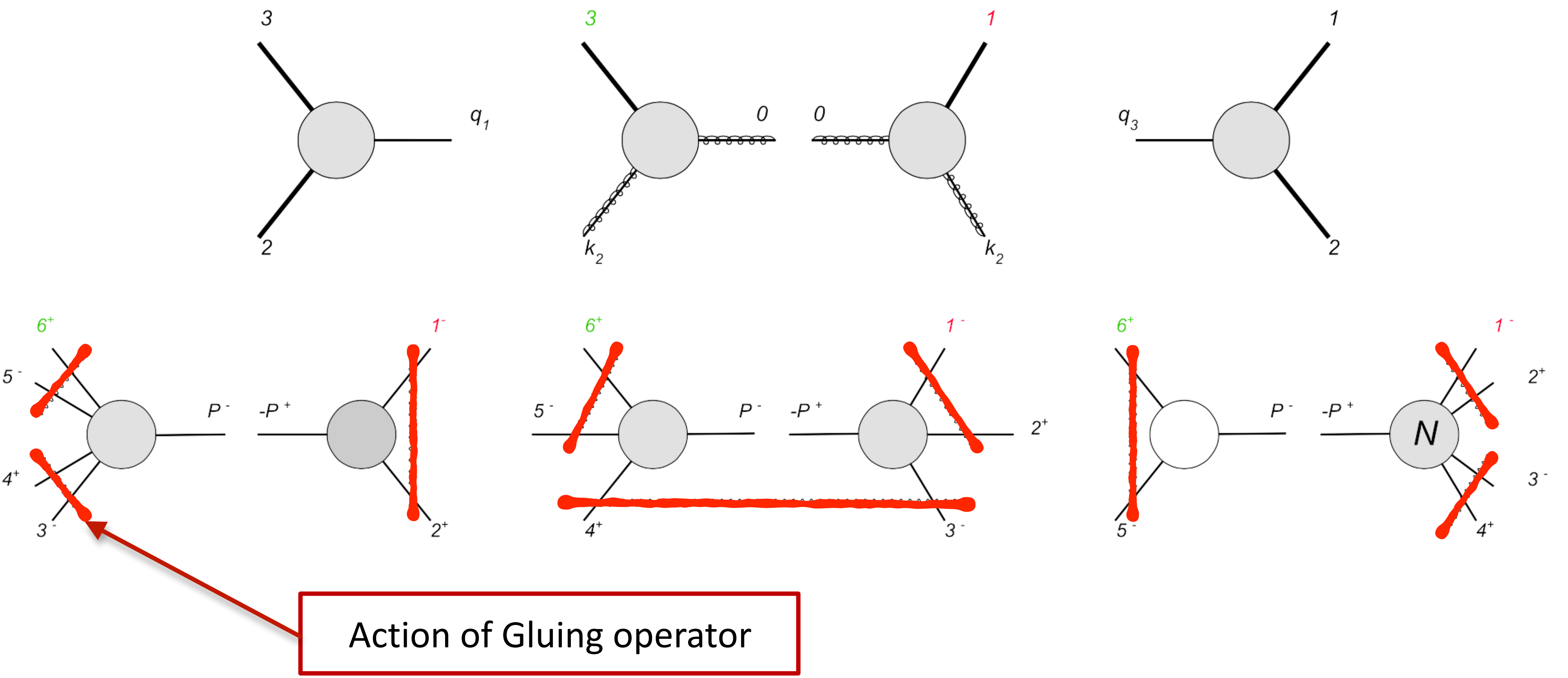


$$\prod_{j=1,2,3}^{3} \frac{\partial^4}{\partial \tilde{\eta}_{p_j}^4}\, \Omega_{0+6}^{3}[\Gamma_{135}] = \{1\} + \{3\} + \{5\} = A_{3,0+3}^{*}(g_1^*, g_2^*, g_3^*)$$

$$\begin{aligned}\{1\} &= \frac{\langle p_1\xi_1\rangle\langle p_2\xi_2\rangle\langle p_3\xi_3\rangle}{\kappa_1^*\kappa_2^*\kappa_3^*} \frac{\langle p_1p_2\rangle^3[p_2p_3]^3\kappa_2^{*3}\kappa_3^{*3}\langle p_2\xi_2\rangle^{-3}\langle p_3\xi_3\rangle^{-3}}{\kappa_3\kappa_3^*\langle p_3\xi_3\rangle^{-2}\langle p_1\xi_1\rangle\kappa_3^*\kappa_2^{*2}\langle p_2\xi_2\rangle^{-2}\langle p_2|k_1|p_2]\langle p_2|k_1|p_3]\langle p_1|k_3|p_2]} \\ &= \frac{1}{\kappa_1^*\kappa_3} \frac{\langle p_1p_2\rangle^3[p_2p_3]^3}{\langle p_2|k_1|p_2]\langle p_2|k_1|p_3]\langle p_1|k_3|p_2]}.\end{aligned}$$

Gluing operator and BCFW for on-shell amplitudes

One can transform BCFW recursion for on-shell amplitudes into BCFW for W.L. form factors (term-to-term correspondence!). n=3 Example:



Gluing operator and BCFW for on-shell amplitudes

One can transform BCFW recursion for on-shell amplitudes into BCFW for W.L. form factors (term-to-term correspondence!). It is not hard to see that actually:

$$\hat{A}^{m.twistor}_{n+1,n+2}\left[\mathcal{L}^k_{n+2}\right] = \omega^k_{n+2}$$

Bork, Onishchenko 18

$$\omega^k_{n+2} = \int \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[GL(k-2)]}\, Reg.\, \frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}$$

$$Reg. = \frac{1}{1+\frac{\langle p_{n+1}\,\xi_{n+1}\rangle}{\langle p_{n+1}\,1\rangle}\frac{(n+2\;2\;\ldots\;k-2)}{(1\;\ldots\;k-2)}},$$

Grassmannian integral for W.L. form factor

$$\mathcal{L}^k_{n+2} = \int \frac{d^{(k-2)\times(n+2)}D}{\mathrm{Vol}[GL(k-2)]}\frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}.$$

Gluing operator and BCFW for on-shell amplitudes

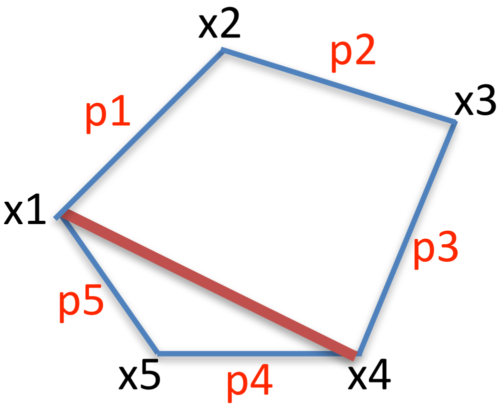
$$\omega^k_{n+2} = \int \frac{d^{(k-2)\times(n+2)}D}{\text{Vol}[GL(k-2)]} \, Reg. \, \frac{\delta^{4(k-2)|4(k-2)}(D\cdot\mathcal{Z})}{(1\ \ldots\ k-2)\ \ldots\ (n+2\ \ldots\ k-3)}$$

$$Reg. = \frac{1}{1+\frac{\langle p_{n+1}\,\xi_{n+1}\rangle}{\langle p_{n+1}\,1\rangle}\frac{(n+2\;2\;\ldots\;k-2)}{(1\;\ldots\;k-2)}},$$

Grassmannian integral for W.L. form factor

$$Z^M_i = \begin{pmatrix} \lambda^\alpha_i \\ \mu^{\dot\alpha}_i \end{pmatrix}, \; \mu^{\dot\alpha}_i = x^{\alpha\dot\alpha}_i \lambda_{\alpha i}$$

$$p^{\alpha\dot\alpha}_i = x^{\alpha\dot\alpha}_i - x^{\alpha\dot\alpha}_{i-1}$$



n=3 example
Dual variables for 3 point W.L. form factor <0|W|123>

q=p5+p6

Gluing operator and BCFW for on-shell amplitudes. Examples

$$\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = \frac{1}{1+\frac{\langle p_5\xi_5\rangle}{\langle p_5 1\rangle}\frac{\langle 1345\rangle}{\langle 3456\rangle}}[13456] + \frac{1}{1+\frac{\langle p_5\xi_5\rangle}{\langle p_5 1\rangle}\frac{\langle 1235\rangle}{\langle 2356\rangle}}[12356] + [12345],$$

$$\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = \frac{A^*_{3,4+1}}{A^*_{2,4+1}}\left(\Omega_1,\ldots,\Omega_4,g_5^*\right),$$

$$\mathcal{P}_6^4 = [12345] + [13456] + [12356]$$

and

$$\hat{A}_{3,4}\circ\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = c_{35}[12345] + c_{36}[12356] + c_{46}[13456],$$

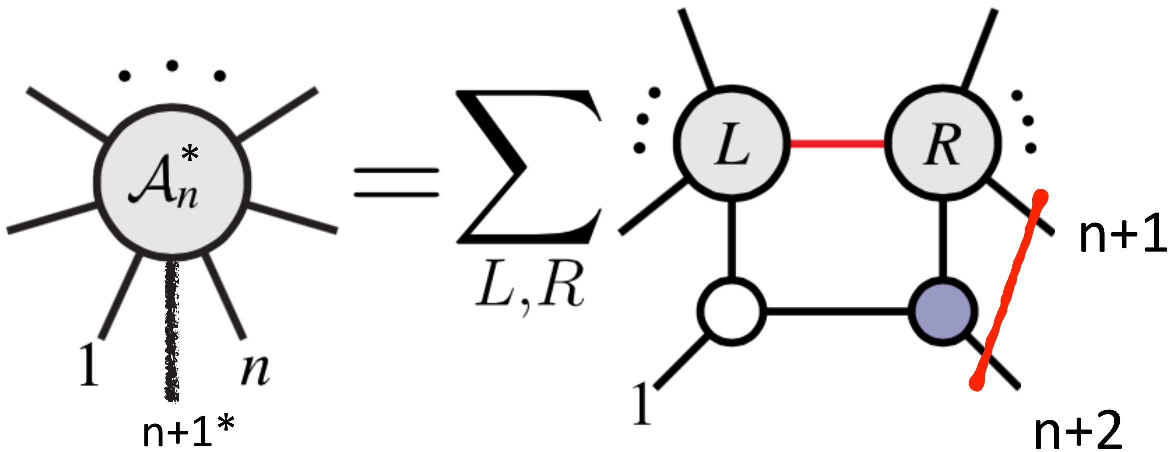
$$\hat{A}_{3,4}\circ\hat{A}_{5,6}\left[\mathcal{P}_6^4\right] = \frac{A^*_{3,2+2}}{A^*_{2,2+2}}\left(\Omega_1,\Omega_2,g_3^*,g_4^*\right),$$

with

$$c_{35} = \frac{1}{1+\frac{\langle p_3\xi_3\rangle\langle 1235\rangle}{\langle p_3p_4\rangle\langle 1234\rangle}},\quad c_{36} = \frac{1}{1+\frac{\langle p_4\xi_4\rangle}{\langle p_4 1\rangle}\frac{\langle 1235\rangle}{\langle 2356\rangle}},\quad c_{46} = \frac{1}{1+\frac{\langle p_3\xi_3\rangle}{\langle p_3p_4\rangle}\frac{\langle 1356\rangle}{\langle 1346\rangle}}\frac{1}{1+\frac{\langle p_4\xi_4\rangle}{\langle p_4 1\rangle}\frac{\langle 1345\rangle}{\langle 3456\rangle}}.$$

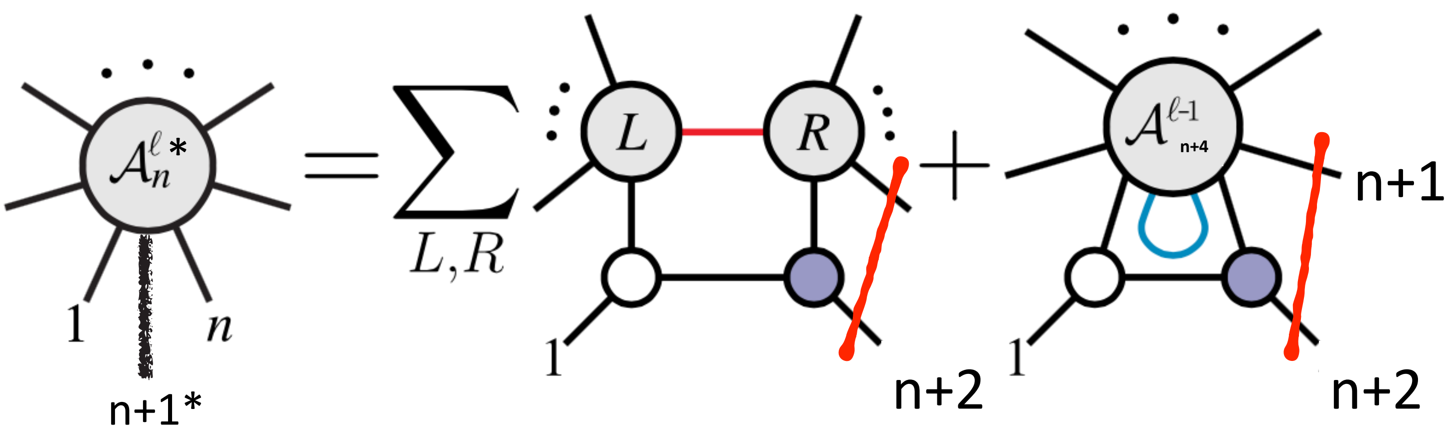
Gluing operator and BCFW for on-shell amplitudes

BCFW recursion for W.L. form factors in terms of on-shell diagrams at tree level:



Gluing operator at loop level

$$I^{*(L\quad)}_{k,(n-2)+1} = \hat{A}_{n-1n}[I^{(L\quad)}_{k,n}]$$



Gluing operator at loop level

$$I^{*(L\quad)}_{k,(n-2)+1} = \hat{A}_{n-1n}[I^{(L\quad)}_{k,n}]$$

$$\hat{\mathcal{Z}}_n = \mathcal{Z}_n + w\mathcal{Z}_{n-1}$$

$$\begin{aligned} I^{*(L)}_{k,(n-2)+1} &= I^{(L)}_{k,n-1}(\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}) \\ &+ \sum_{j=2}^{n-2} [j-1, j, n-1, n^*, 1] I^{(L_1)}_{k_1,n+2-j}(\mathcal{Z}_{I_j}, \mathcal{Z}_j, \mathcal{Z}_{j+1}, \ldots, \hat{\mathcal{Z}}^*_{n_j}) I^{(L_2)}_{k_2,j}(\mathcal{Z}_{I_j}, \mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_{j-1}) \\ &+ \int \frac{d^{4|4}\mathcal{Z}_A d^{4|4}\mathcal{Z}_B}{\mathrm{Vol}[GL(2)]} \int_{GL(2)} [A, B, n-1, n^*, 1] I^{(L-1)}_{k+1,n+2}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \hat{\mathcal{Z}}^*_{n_{AB}}, \mathcal{Z}_A, \mathcal{Z}_B)\,, \qquad (5.119) \end{aligned}$$

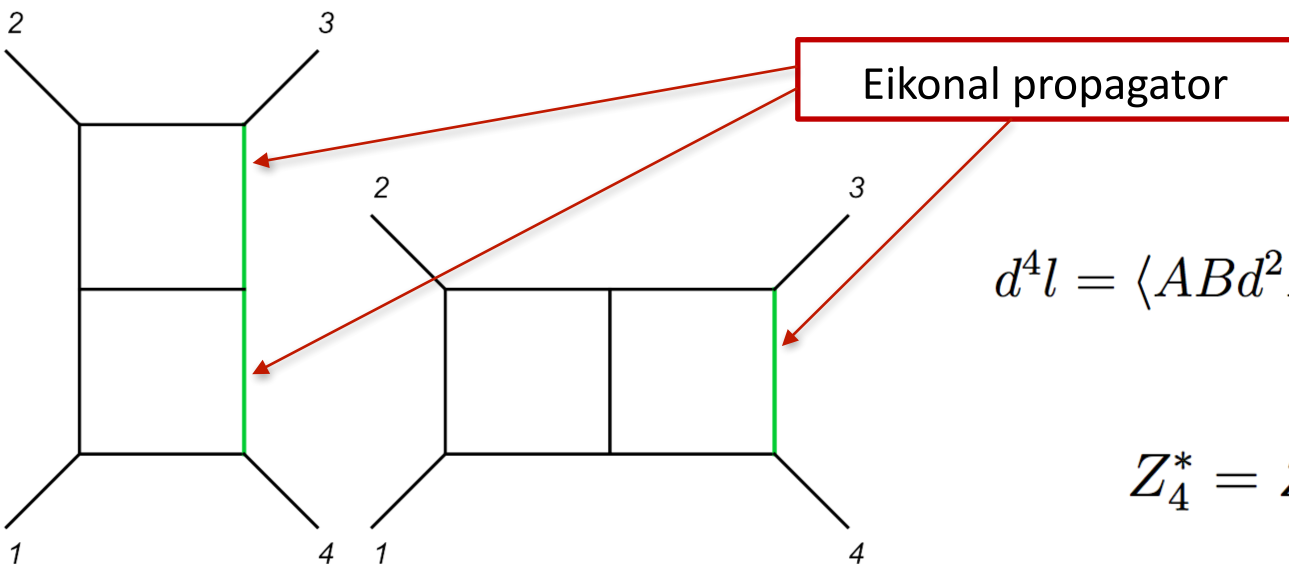
where $\hat{\mathcal{Z}}_{n_j} = (n-1, n^*) \cap (1, j-1, j)$, $\mathcal{Z}_{I_j} = (j-1, j) \cap (1, n-1, n)$, $\hat{\mathcal{Z}}_{n_{AB}} = (n-1, n^*) \cap (A, B, 1)$ and $k_1 + k_2 + 1 = k$. $\mathcal{Z}^*_n$ is given by

$$\mathcal{Z}^*_i = \mathcal{Z}_i - \frac{\langle i-1\, i\rangle}{\langle i-1\, i+1\rangle} \mathcal{Z}_{i+1}$$

Gluing operator at loop level. Examples

$$A^{*(2)}_{2,2+1}(\Omega_1,\Omega_2,g^*_3)$$

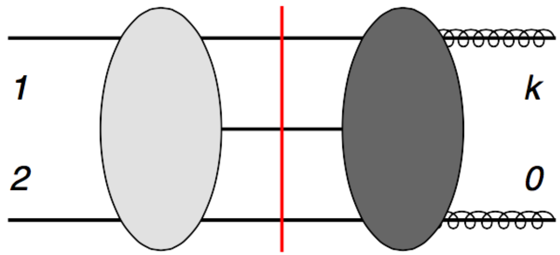
$$I^{*(2)}_{2,2+1} = \frac{\langle 2341\rangle\langle 3412\rangle\langle 4123\rangle}{\langle AB41\rangle\langle AB12\rangle\langle AB23\rangle\langle CD23\rangle\langle CD34^*\rangle\langle CD41\rangle\langle ABCD\rangle} + \frac{\langle 3412\rangle\langle 4123\rangle\langle 1234\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34^*\rangle\langle CD34^*\rangle\langle CD41\rangle\langle CD12\rangle\langle ABCD\rangle}$$

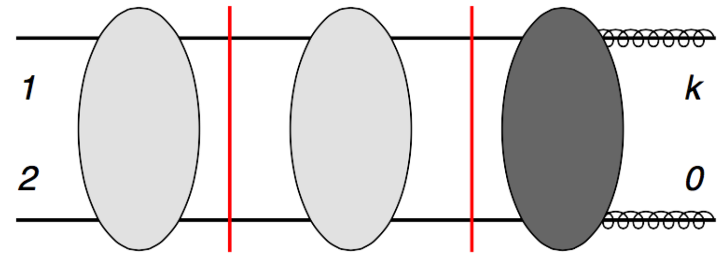


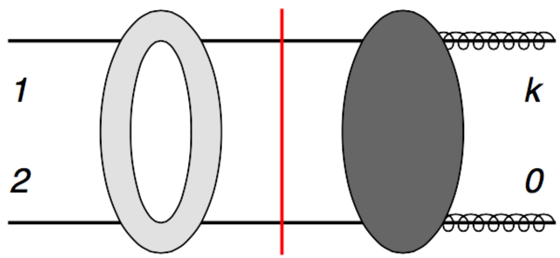
$$d^4l = \langle ABd^2A\rangle\langle ABd^2B\rangle = \frac{d^4Z_Ad^4Z_B}{\mathrm{Vol}[GL(2)]}$$

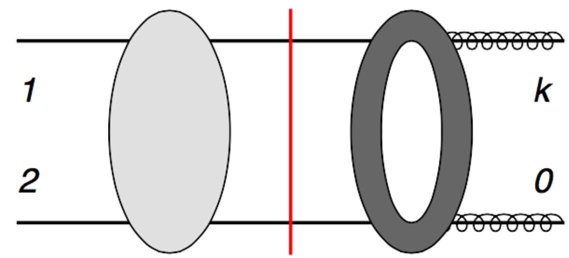
$$Z^*_4 = Z_4 - \frac{\langle p\xi\rangle}{\langle p1\rangle}Z_1$$

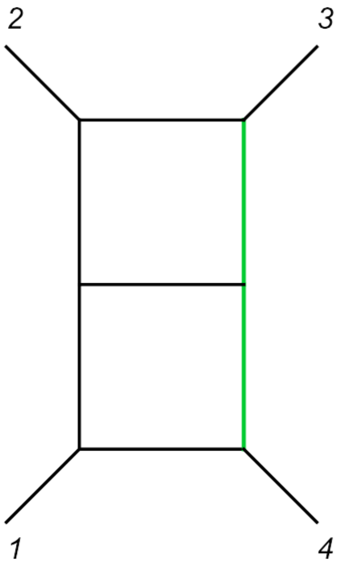
Gluing operator at loop level. Examples

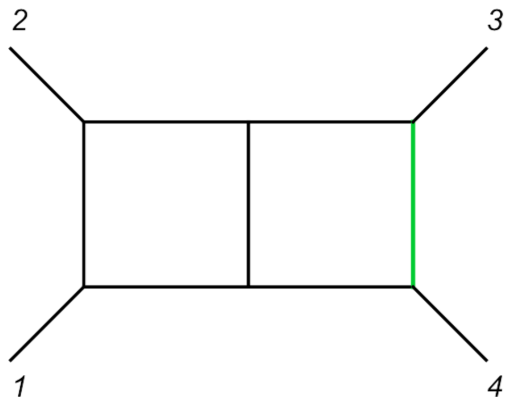












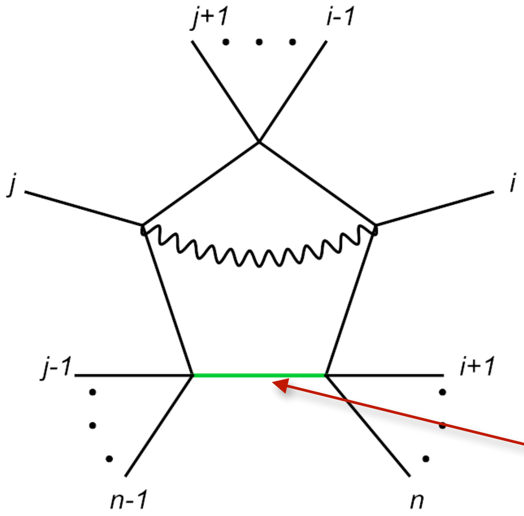
Gluing operator at loop level. Examples

$$A^{*(1)}_{2,(n-2)+1}(\Omega_1,\ldots,\Omega_{n-2},g^*_{n-1})$$

$$I^{*(1)}_{2,(n-2)+1}=\sum_{i<j}\frac{\langle AB(i-1\,i\,i+1)\cap(j-1\,j\,j+1)\rangle\langle n-1n^*ij\rangle}{\langle AB\,n-1\,n^*\rangle\langle AB\,i-1\,i\rangle\langle AB\,i\,i+1\rangle\langle AB\,j-1\,j\rangle\langle AB\,j\,j+1\rangle}$$

where Z_n^* is given by:

$$Z_n^*=Z_n-\frac{\langle p\xi\rangle}{\langle p1\rangle}Z_1.$$



NMHV, NNMHV, et.c. form factors with arbitrary number of states can be considered among the same lines (1802.03986).

Eikonal propagator <p|l|p]

Conclusions

- **The set of variables and methods which initially was associated with on-shell scattering amplitudes can be successfully applied to all type of gauge invariant objects in N=4 SYM at tree an loop level**
- **<u>Similar construction also holds for other type of operators ?</u>**
- **Proper description in terms of integrable system.**
- **Chern-Simons theory generalisation.**
- **…**