On the one-loop calculations in Lipatov's EFT

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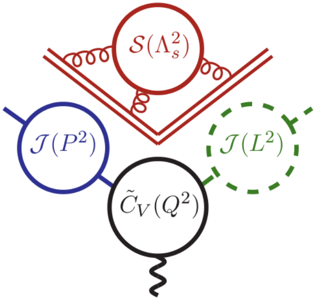
Motivation

- The gauge-invariant EFT for Multi-Regge processes in QCD, which includes *Reggeized gluons* [Lipatov; 1995] and *Reggeized quarks* [Lipatov, Vyazovsky; 2001] has been introduced as a systematic tool to *compute and resum* the higher-order corrections in QCD, enhanced by $\log(s/(-t))$, with the arbitrary $N^k LL$ accuracy.
- Another motivation is the *unitarization program* for high-energy scattering. The BFKL equation at the fixed logarithmic accuracy predicts *power-like* growth of the cross-section with s, which violates Froissart bound ($\Leftarrow$ Unitarity). The basic idea is to write-down the Hermitian effective Lagrangian for which Unitarity will hold automatically.

Motivation

- Currently, a number of approaches is developed with the aim of taking into account both DGLAP and BFKL effects. Many of them try to generalize the amplitudes from the Lipatov's EFT to the Soft and Collinear regions (e.g. PRA [M.N., V.A.S., *et. al.*] or HEJ [J. Andersen, *et. al.*] approaches, `KaTie` [A. van Hameren, *et. al.*] Monte-Carlo code) or incorporate BFKL effects into the framework of SCET (e.g. [I. Stewart, I. Rothstein, 2016]). Going beyond tree level is an important part of this activity.
- In the talk I would like to describe the one-loop structiure of Lipatov's EFT. The complete picture, similar to one in ordinary QCD, emerges.

What such EFTs as SCET or Lipatov's theory are (needed for)?



Picture is taken from [hep-ph/1410.1892].

- EFT = formalism which **explicitly implements** certain factorization properties of QCD amplitudes:
  - Factorization in the **soft** and **collinear** limits – *Soft-Collinear Effective Theory (SCET).*
  - Factorization in the **Multi-Regge limit** – *Lipatov's theory.*
- As a result of kinematic approximations, **artificial logarithmic divergences** arise in different factors, but they should cancel in order-by-order in PT.
- Factorization + cancellation of artificial divergences ⇒ **Renormalization group**. The latter allows to resum **large logarithms** of scale ratios.

Sudakov (light-cone) decomposition of momenta.

It is convinient to relate the basis vectors of Sudakov decomposition with (almost) light-like momenta of colliding highly energetic particles ($P_{1,2}^2 = 0$):

$$n_-^\mu = \frac{2P_1^\mu}{\sqrt{S}},\ n_+^\mu = \frac{2P_2^\mu}{\sqrt{S}},\ S = 2P_1P_2 \Rightarrow n_+n_- = 2.$$

Then for any four-vector k^μ one has:

$$\boxed{k^\mu = \frac{1}{2}\left(k_+n_-^\mu + k_-n_+^\mu\right) + k_T^\mu,}$$

where $k_\pm = k^\pm = n_\pm k$, $n_\pm k_T = 0$. For the dot-product one has:

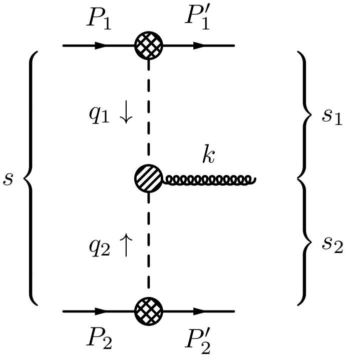
$$kq = \frac{1}{2}\left(k_+q_- + k_-q_+\right) - \mathbf{k}_T\mathbf{q}_T,\ \boxed{k^2 = k_+k_- - \mathbf{k}_T^2.}$$

Rapidity:

$$\boxed{\boxed{y = \frac{1}{2}\log\left(\frac{q^+}{q^-}\right).}}$$

Multi-Regge Kinematics.

At high energies, t-channel exchange diagrams with Multi-Regge(MRK) or Quasi-Multi Regge(QMRK) Kinematics of the final-state dominate in the $2 \to 2 + n$ amplitude.



Double Regge limit (MRK):

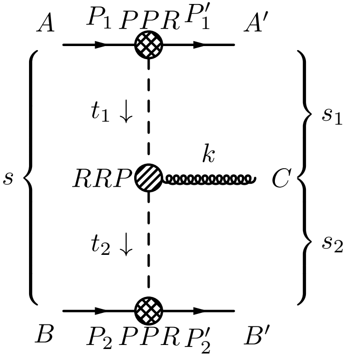
$$s_1 \gg -q_1^2,\ s_2 \gg -q_2^2,$$

momentum fractions $z_1 = q_1^+/P_1^+$, $z_2 = q_2^-/P_2^-$.

Properties of MRK:

- $y(P_1') \to +\infty$, $y(P_2') \to -\infty$, $y(k)$ – finite,
- $\boxed{z_1 \sim z_2 \sim z \ll 1}$, $|\mathbf{k}_T| \ll \sqrt{s}$,
- $q_1^+ \sim |\mathbf{q}_{T1}| \sim O(z) \gg q_1^- \sim O(z^2)$, $q_2^- \sim |\mathbf{q}_{T2}| \sim O(z) \gg q_2^+ \sim O(z^2)$.

Reggeization of amplitudes in QCD.



In MRK asymptotics, $2 \to 3$-amplitude factorizes (up to $O(z^{\#}_{1,2})$):

$$\mathcal{A}_{AB}^{A'B'C} = \gamma_{A'A}^{R_1} \cdot \left(\frac{s_1}{s_0}\right)^{\omega(t_1)} \frac{-i}{2t_1} \times$$

$$\Gamma^{C}_{R_1 R_2}(q_1, q_2) \cdot \frac{-i}{2t_2} \left(\frac{s_2}{s_0}\right)^{\omega(t_2)} \cdot \gamma^{R_2}_{B'B}$$

$\Gamma^{C}_{R_1 R_2}(q_1, q_2)$ – RRP production vertex,

$\gamma^{R}_{A'A}$ – PPR-scattering vertex,

$\omega(t)$ - Regge trajectory.

Two ways to obtain this asymptotics:

- BFKL-approach (Unitarity, renormalizability and gauge invariance), see. [Ioffe, Fadin, Lipatov, 2010].
- Effective action approach [Lipatov, 1995; Lipatov, Vyazovsky, 2001].

Structure of the EFT.

Light-cone derivatives:

$$\partial_\pm = n^\mu_\pm \partial_\mu = 2\frac{\partial}{\partial x^\mp}$$

EFT Lagrangian [Lipatov, 1995]:

$$L = L_{\rm kin} + \sum_i \left[L_{QCD}^{(y_i \leq y \leq y_{i+1})} + L_R^{(y_i \leq y \leq y_{i+1})} \right],$$

the separate copy of $L_{QCD}^{(y_i \leq y \leq y_{i+1})}$ lives in each interval in rapidity $y_i \leq y \leq y_{i+1}$. Different intervals interact via Reggeon exchanges ($R^a_\pm = R^a_\pm T_a$):

$$L_{\rm kin} = 2\partial_\mu R^a_+ \partial^\mu R^a_-,$$

kinematic constraints on Reggeon-fields ($\Leftrightarrow$ QMRK):

$$\partial_- R_+ = \partial_+ R_- = 0 \Rightarrow$$

$$R_+ \text{ carries } (k_+, \mathbf{k}_T) \text{ and } R_- \text{ carries } (k_-, \mathbf{k}_T).$$

(Semi-)Infinite light-like Wilson lines

Particles highly separated in rapidity "perceive" each-other as light-like Wilson lines.

$$W_{x^\mp}[A_\pm] = \quad P\exp\left[\frac{-ig_s}{2}\int_{-\infty}^{x_\mp} dx'_\mp A_\pm(x_\pm, x'_\mp, \mathbf{x}_T)\right] = \left(1 + ig_s\partial_\pm^{-1}A_\pm\right)^{-1},$$

$$W^\dagger_{x^\mp}[A_\pm] = \quad \overline{P}\exp\left[\frac{ig_s}{2}\int_{-\infty}^{x_\mp} dx'_\mp A_\pm(x_\pm, x'_\mp, \mathbf{x}_T)\right] = \overline{P}\left(1 - ig_s\partial_\pm^{-1}A_\pm\right)^{-1},$$

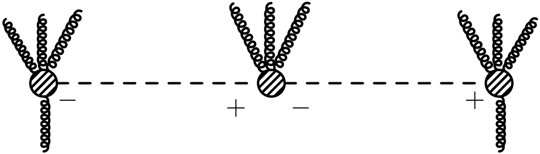
Notation for ordered integrals:

$$\frac{1}{2^n}\int_{-\infty}^{x^\mp} dx_1^\mp f_1(x_1^\mp)\int_{-\infty}^{x_1^\mp} dx_2^\mp f_2(x_2^\mp)\ldots\int_{-\infty}^{x_{n-1}^\mp} dx_n^\mp f_n(x_n^\mp) = \underbrace{\partial_\pm^{-1}f\ldots\partial_\pm^{-1}f}_{n}.$$

In the Feynman rules:

$$\boxed{\partial_\pm^{-1} \to \frac{-i}{k^\pm + i\varepsilon}.}$$

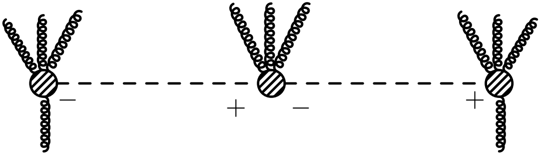
Basic structure of Induced interactions.



Induced interactions of particles and Reggeons [Lipatov, 1995]:

$$\begin{aligned} L_R^{(y_1<y<y_2)}(x) &\supset \frac{i}{g_s}\,\mathrm{tr}\left(R_+(x)\partial_\rho^2\partial_- W_x\left[A_-^{(y_1<y<y_2)}\right]\right. \\ &+ \left. R_-(x)\partial_\rho^2\partial_+ W_x\left[A_+^{(y_1<y<y_2)}\right]\right). \end{aligned}$$

Basic structure of Induced interactions.



Induced interactions of particles and Reggeons:

$$L_R^{(y_1<y<y_2)} \supset \frac{i}{g_s}\,\mathrm{tr}\left[R_+\partial_\rho^2\partial_- W\left[A_-^{(y_1<y<y_2)}\right] + R_-\partial_\rho^2\partial_+ W\left[A_+^{(y_1<y<y_2)}\right]\right],$$

expansion of P-exponent generaties induced vertices:

$$\begin{aligned} L_R \quad \supset \quad & \mathrm{tr}\Big[\big(R_+\partial_\sigma^2 A_- + R_-\partial_\sigma^2 A_+\big) + \\ & (-ig_s)(\partial_\sigma^2 R_+)(A_-\partial_-^{-1}A_-) + (-ig_s)^2(\partial_\sigma^2 R_+)(A_-\partial_-^{-1}A_-\partial_-^{-1}A_-) + \\ & (-ig_s)(\partial_\sigma^2 R_-)(A_+\partial_+^{-1}A_+) + (-ig_s)^2(\partial_\sigma^2 R_-)(A_+\partial_+^{-1}A_+\partial_+^{-1}A_+) \\ & +O(g_s^3)\Big]\,, \end{aligned}$$

but this structure is non-Hermitian: R_+ – Hermitian, W – Unitary!

Hermitian effective action and pole prescription

Recently the new derivation of effective action has been proposed [Bondarenko, Zubkov, 2018] which fixes the **Hermitian** form of Reggeon-gluon interaction:

$$\frac{i}{g_s}\,\mathrm{tr}\left[R_+\partial_\rho^2\partial_-\left(W\left[A_-\right]-W^\dagger\left[A_-\right]\right)\right],$$

E.g. Rgg-vertex:

$$\frac{-ig_s}{2}\left(\partial_\sigma^2 R_+^a(x)\right)\left(A_-^{b_1}(x)\int\limits_{-\infty}^{x_-}dx_1^-\,A_-^{b_2}(x_1)\right)\mathrm{tr}\left[T^a\left[T^{b_1},T^{b_2}\right]\right],$$

$\Rightarrow$ Feynman rule:

$$g_s(-q^2)f^{ab_1b_2}(n_-^{\mu_1}n_-^{\mu_2})\frac{1}{2}\left[\frac{1}{k_1^-+i\varepsilon}+\frac{1}{k_1^--i\varepsilon}\right]=g_s(-q^2)(n_-^{\mu_1}n_-^{\mu_2})\frac{f^{ab_1b_2}}{[k_1^-]},$$

i.e. the PV-prescription for the $1/k^\pm$ poles for simplest induced vertices [Hentschinski, 2013].

Higher-order induced vertices

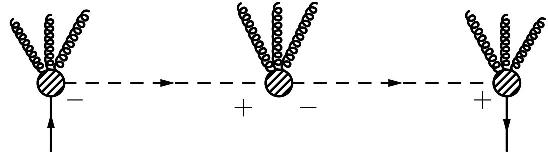
For higher-order induced vertices the $i(\partial^2 R_\pm)\partial_\mp \left(W[A_\mp] - W^\dagger[A_\mp]\right)$ interaction leads to the $i\varepsilon$ prescription proposed **independently** in [Hentschinski, 2013] (based on argments from Regge theory):

- The induced vertex is written according to $i(\partial^2 R_\pm)\partial_\mp W[A_\mp]$ interaction with $1/(k^\pm + i\varepsilon)$ prescription for all poles,
- The color structure $\mathrm{tr}\left(T^a T^{b_1} \ldots T^{b_n}\right)$ is projected on subspace, spanned by:

$$\mathrm{tr}\left(T^a \left[\left[\left[T^{b_{i_1}}, T^{b_{i_2}}\right], T^{b_{i_3}}\right], \ldots T^{b_{i_n}}\right]\right).$$

This pole prescription is very well tested: leads to the correct results for 1-loop amplitudes with Reggeized gluons and quarks and corrct 2-loop gluon Regge trajectory [Chachamis, Hentschinski, Sabio-Vera, 2012-2013; M.N., V.A.S., 2017].

The formalism is well-defined at all orders now!



EFT for Reggeized quarks [Lipatov, Vyazovsky, 2001]:

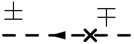
$$L_Q = \bar{Q}_- i\hat{\partial}\left(Q_+ - W^\dagger\left[A_+\right]\psi\right) + \bar{Q}_+ i\hat{\partial}\left(Q_- - W^\dagger\left[A_-\right]\psi\right) + \text{h.c.},$$

where $\hat{p} = p_\mu \gamma^\mu$, QMRK kinematic constraints:

$$\partial_\pm Q_\mp = \partial_\pm \bar{Q}_\mp = 0,$$
$$\hat{n}^\pm Q_\mp = 0,\ \bar{Q}_\mp \hat{n}^\pm = 0. \Rightarrow$$

Reggeized quark propagator ($\hat{P}_\pm = \hat{n}_\mp \hat{n}_\pm / 4$):

$$= \hat{P}_\pm \frac{i\hat{q}}{q^2},$$

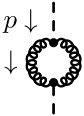


$$= \frac{i\hat{q}}{q^2}\hat{P}_\pm.$$

Rapidity divergences and regularization.

Due to the presence of the $1/q^{\pm}$-factors in the induced vertices, loop integrals in EFT contain the light-cone (Rapidity) divergences:

$$\Sigma^{(1)}_{ab} = q \downarrow$$



$$= g_s^2 C_A \delta_{ab} \int \frac{d^d q}{(2\pi)^D} \frac{\left(\mathbf{p}_T^2 (n_+ n_-)\right)^2}{q^2 (p-q)^2 q^+ q^-}$$

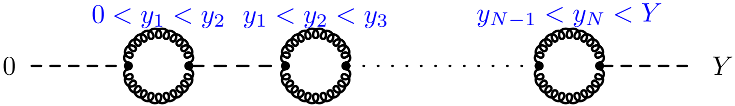
The regularization by explicit cutoff in rapidity was proposed by Lipatov [Lipatov, 1995] ($q^{\pm} = \sqrt{q^2 + \mathbf{q}_T^2} e^{\pm y}$):

$$\int \frac{dq^+ dq^-}{q^+ q^-} = \int_{y_1}^{y_2} dy \int \frac{dq^2}{q^2 + \mathbf{q}_T^2},$$

then

$$\Sigma^{(1)}_{ab} \sim \delta_{ab} \mathbf{p}_T^2 \times \underbrace{C_A g_s^2 \int \frac{\mathbf{p}_T^2 d^{D-2} \mathbf{q}_T}{\mathbf{q}_T^2 (\mathbf{p}_T - \mathbf{q}_T)^2}}_{\omega^{(1)}(\mathbf{p}_T^2)} \times (y_2 - y_1) + \text{finite terms}$$

Gluon Reggeization in the EFT (cutoff regularization)



$$= \int\limits_0^Y dy_1\, \omega^{(1)}(\mathbf{p}_T^2) \cdot \int\limits_{y_1}^Y dy_2\, \omega^{(1)}(\mathbf{p}_T^2) \cdot \ldots \int\limits_{y_{N-1}}^Y dy_N\, \omega^{(1)}(\mathbf{p}_T^2) = \frac{\left(Y\omega^{(1)}(\mathbf{p}_T^2)\right)^N}{N!}$$

Sum of such diagrams will give the factor

$$\exp\left[Y\omega^{(1)}(\mathbf{p}_T^2)\right].$$

"**Reggeization**" of the particle in t-channel.
In the EFT, this exponentiation can be proven at all orders.

Covariant regularization.

The regularization and **pole prescription** was introduced in a series of papers [Hentschinski, Sabio Vera, Chachamis *et. al.*, 2012-2013], also known in TMD factorization as “tilted Wilson lines” [Collins, 2011].
Regularization of the light-cone divergences is achieved by shifting $n^{\pm}$ vectors from the light-cone:

$$\boxed{\tilde{n}^{\pm} = n^{\pm} + r \cdot n^{\mp},\ \tilde{k}^{\pm} = k^{\pm} + r \cdot k^{\mp},\ r \to 0,}$$

and for the lowest-order(Rgg, Qqg) induced vertices the PV prescription is at work:

$$I^{[\pm]}:\ \frac{1}{[\tilde{k}^{\pm}]} = \frac{1}{2}\left(\frac{1}{\tilde{k}^{\pm} + i\varepsilon} + \frac{1}{\tilde{k}^{\pm} - i\varepsilon}\right),$$

Regularization and gauge-invariance

Regularization should preserve the gauge-invariance of Reggeon-gluon interactions:

$$S^{(-)}_{Rg} = \int d^2\mathbf{x}_T \int\limits_{-\infty}^{+\infty} \frac{dx_+ dx_-}{2} \,\mathrm{tr}\left[R^- \tilde{\partial}_+ \partial^2_\sigma W_{\tilde{x}_-}[\tilde{A}_+]\right]$$

$$= \int d^2\mathbf{x}_T \int\limits_{-\infty}^{+\infty} \frac{d\tilde{x}_+ d\tilde{x}_-}{1-r^2} \,\mathrm{tr}\left[R^- \frac{\partial}{\partial \tilde{x}_-} \partial^2_\sigma W_{\tilde{x}_-}[\tilde{A}_+]\right] =$$

$$= \int d^2\mathbf{x}_T \int\limits_{-\infty}^{+\infty} \frac{d\tilde{x}_+ d\tilde{x}_-}{1-r^2} \left\{ \frac{\partial}{\partial \tilde{x}_-} \,\mathrm{tr}\left[R^- \partial^2_\sigma W_{\tilde{x}_-}[\tilde{A}_+]\right] - \frac{1}{2}\,\mathrm{tr}\left[\left(\tilde{\partial}_+ R_-\right) \partial^2_\sigma W_{\tilde{x}_-}[\tilde{A}_+]\right] \right\}.$$

First term – *infinite Wilson line* is gauge invariant (w.r.t. gauge transformations trivial at ∞) $\Rightarrow$ **new kinematic constraint**:

$$\boxed{\tilde{\partial}_+ R_- = \tilde{\partial}_- R_+ = 0,}$$

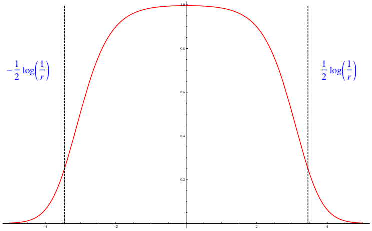
or $\tilde{p}^+ = 0$ for R_- and $\tilde{p}^- = 0$ for R_+.

Rapidity divergences in real corrections

New constraint allows to use same regularization for RDs in *virtual and real corrections*. Without it, e.g., the RDs in Lipatov's vertex ($k = q_1 + q_2$, $k^2 = 0$):

$$\Gamma_{+\mu-} = 2\left[(q_2 - q_1)_\mu + \left(\frac{q_1^2}{\tilde{k}_-} + \tilde{q}_1^+\right)\tilde{n}_\mu^- - \left(\frac{q_2^2}{\tilde{k}_+} + \tilde{q}_2^-\right)\tilde{n}_\mu^+\right],$$

are **not regularized at all** and the Slavnov-Taylor identity $\boxed{k^\mu \Gamma_{+\mu-} = 0}$ is broken by terms $O(r)$.



The square of regularized LV:

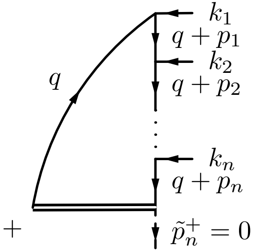
$$\Gamma_{+\mu-}\Gamma_{+\nu-}P^{\mu\nu} = \frac{16\mathbf{q}_{T1}^2\mathbf{q}_{T2}^2}{\mathbf{k}_T^2} f(y),$$

$$\longleftarrow f(y) = \frac{1}{(re^{-y} + e^{y})^2(re^{y} + e^{-y})^2},$$

$$\int\limits_{-\infty}^{+\infty} dy\ f(y) = -1 - \log r + O(r)$$

RDs in 1-loop, 1-Reggeon amplitude

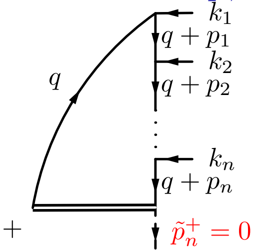
$$p_i = \sum_{j=1}^{i} k_j, \quad p_0 = 0, \quad d = 4 - 2\epsilon.$$



"Mixed" Feynman parametrization:

$$\begin{aligned} I &= \int \frac{d^d q}{q^2 (q+p_1)^2 \ldots (q+p_n)^2 (\tilde{n}^+ q)} \\ &\sim \int_0^1 da_1 \ldots da_{n+1} \int_0^\infty dx_1 \; \delta\left(1 - \sum_{j=1}^{n+1} a_j\right) \\ &\times \int d^d q \left[x_1 (\tilde{n}^+ q) + \sum_{i=1}^{n+1} a_i (q + p_{i-1})^2 \right]^{n+2} \end{aligned}$$

RDs in 1-loop, 1-Reggeon amplitude



$$I \sim \int_0^1 da_1 \dots da_{n+1} \int_0^\infty dx_1 \; \delta\left(1 - \sum_{j=1}^{n+1} a_j\right) \times \left[\mathcal{D} + x_1 \sum_{j=1}^{n-1} \tilde{p}_j^+ a_{j+1} + r x_1^2\right]^{-n-\epsilon},$$

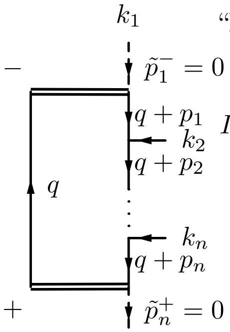
where $\mathcal{D} = -\frac{1}{2} \sum_{i,j=0}^{n} a_{i+1} a_{j+1} (p_i - p_j)^2$.

Let's put $r = 0 \Rightarrow$ after integration over x_1:

$$I \sim \int_0^1 da_1 \dots da_{n+1} \; \delta(\dots) \left(\sum_{j=1}^{n-1} p_j^+ a_{j+1}\right)^{-1} \mathcal{D}^{-n-\epsilon+1},$$

- **Log-divergent** for $\boxed{n = 2}$ as $\int_0 \frac{da_2}{a_2}$, for $n > 2$ – **finite**.
- For $n = 2$, divergence can be **removed** by differentiating $\partial I / \partial k_1^2$ or $\partial I / \partial k_2^2$.

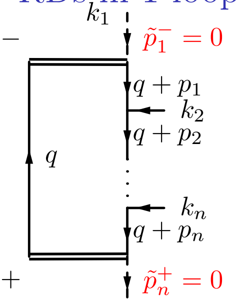
RDs in 1-loop, 2-Reggeon amplitude



"Mixed" Feynman parametrization:

$$
\begin{aligned}
I &= \int \frac{d^d q}{q^2 (q+p_1)^2 \dots (q+p_n)^2 (\tilde{n}^+ q)(\tilde{n}^- q)} \\
&\sim \int_0^1 da_1 \dots da_{n+1} \int_0^\infty dx_1 dx_2 \; \delta\left(1 - \sum_{j=1}^{n+1} a_j\right) \\
&\times \int d^d q \left[x_1 (\tilde{n}^+ q) + x_2 (\tilde{n}^- q) + \sum_{i=1}^{n+1} a_i (q + p_{i-1})^2 \right]^{n+3}
\end{aligned}
$$

RDs in 1-loop, 2-Reggeon amplitude



$$I \sim \int_0^1 da_1 \dots da_{n+1} \int_0^\infty dx_1 dx_2 \; \delta\left(1 - \sum_{j=1}^{n+1} a_j\right)$$

$$\times \left[\mathcal{D} + \sum_{j=1}^{n} a_{j+1}\left(x_1 \tilde{p}_j^+ + x_2 \tilde{p}_j^-\right) + x_1 x_2 + r(x_1^2 + x_2^2)\right]^{-n-1-\epsilon}$$

For $r = 0$, after integration over x_2:

$$I \sim \int_0^1 da_1 \dots da_{n+1} \int_0^\infty dx_1 \; \delta(\dots) \left(x_1 + \sum_{j=2}^{n} a_{j+1} p_j^-\right)^{-1} \left[\mathcal{D} + x_1 \sum_{j=1}^{n-1} a_{j+1} p_j^+\right]^{-n-\epsilon}$$

- **Log-divergence** for $\boxed{n = 1}$ $(p_1^- = p_1^+ = 0)$ as $\int_0 \frac{dx_1}{x_1}$.
- For $n > 1$ – **no divergence** if $p_2^- \neq 0, \dots, p_n^- \neq 0$ and $p_1^+ \neq 0, \dots, p_{n-1}^+ \neq 0$.

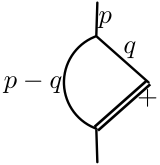
“Tadpoles” and “Bubbles”.

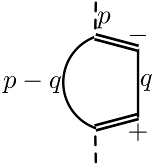
“Tadpoles” (one quadratic propagator):

$$A_0^{[+]}(p) = \int \frac{[d^d q]}{(p-q)^2[\tilde{q}^+]}, \; A_0^{[+-]}(p) = \int \frac{[d^d q]}{(p-q)^2[\tilde{q}^+][\tilde{q}^-]}$$

where $[d^D q] = \frac{(\mu^2)^\epsilon d^d q}{i\pi^{D/2} r_\Gamma}$, $r_\Gamma = \Gamma^2(1-\epsilon)\Gamma(1+\epsilon)/\Gamma(1-2\epsilon)$.

“Bubbles” (two quadratic propagators):





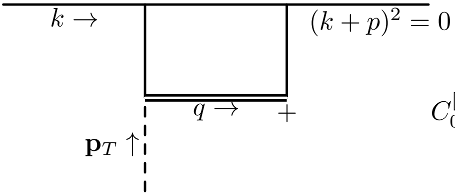
$$B_0^{[+]}(p) = \int \frac{[d^d q]}{q^2(p-q)^2[\tilde{q}^+]},$$

$$B_0^{[+-]}(\mathbf{p}_T) = \int \frac{[d^d q]}{q^2(p-q)^2[\tilde{q}^+][\tilde{q}^-]},$$

where $p^+ = p^- = 0$ for the last integral.

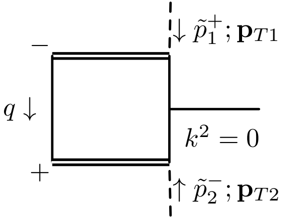
"Triangle" integrals

One light-cone propagator:



$$C_0^{[+]} = \int \frac{[d^D q]}{q^2 (p-q)^2 (p+k-q)^2 [\tilde{q}^+]}.$$

Two light-cone propagators:



$$C_0^{[+-]} = \int \frac{[d^D q]}{q^2 (p_1-q)^2 (p_2+q)^2 [\tilde{q}^+][\tilde{q}^-]}.$$

Rapidity divergences at one loop

Only log-divergence $\sim \log r$ (Blue cells in the table) is related with Reggeization of particles in t-channel.

Integrals which **do not** have log-divergence may still contain the power-dependence on r:

- $r^{-\epsilon} \to 0$ for $r \to 0$ and $\epsilon < 0$.
- $r^{+\epsilon} \to \infty$ for $r \to 0$ and $\epsilon < 0$ – **weak-power divergence** (Pink cells in the table)
- $r^{-1+\epsilon} \to \infty$ – **power divergence.** (Red)

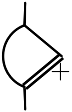
(# LC prop.) \ (# quadr. prop.)	1	2	3	4
1	$A_0^{[+]}$	$B_0^{[+]}$	$C_0^{[+]}$	$\ldots$
2	$A_0^{[+-]}$	$B_0^{[+-]}$	$C_0^{[+-]}$	$\ldots$
3	$\ldots$	$\ldots$	$\ldots$	$\ldots$

The **weak-power** and **power-divergences** cancel between Feynman diagrams describing one region in rapidity, so only log-divergences are left.

Results for scalar integrals.

Notation: $\left\{\frac{\mu}{k}\right\}^{2\epsilon} = \frac{1}{2}\left[\left(\frac{\mu}{k-i\varepsilon}\right)^{2\epsilon} + \left(\frac{\mu}{-k-i\varepsilon}\right)^{2\epsilon}\right].$

- [+]-bubble in general kinematics (leading term of the Mellin-Barnes representation):

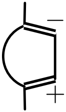


$$B_0^{[+]}(p) = \frac{1}{\tilde{p}^+}\frac{r^{\epsilon}}{\cos(\pi\epsilon)}\frac{1}{2\epsilon^2}\left\{\frac{\mu}{\tilde{p}^+}\right\}^{2\epsilon} + O(r^{1/2}),$$

- Tadpoles (direct integration):

$$\begin{aligned} A_0^{[+]}(p) &= \frac{\epsilon\tilde{p}_+^2 r^{-1}}{(1-2\epsilon)} B_0^{[+]}(p), \\ A_0^{[+-]}(p) &= \tilde{p}^+ B_0^{[+]}(p) + \tilde{p}^- B_0^{[-]}(p) \\ &\quad - \left\{\frac{\mu}{\tilde{p}^+}\right\}^{\epsilon}\left\{\frac{\mu}{\tilde{p}^-}\right\}^{\epsilon}\frac{1}{\epsilon^2}\frac{\sin(\pi\epsilon)\Gamma(1-2\epsilon)\Gamma^2(1+\epsilon)}{\pi\epsilon} \end{aligned}$$

- $[+-]$-bubble in transverse kinematics $p^- = p^+ = 0$ (direct integration):



$$B_0^{[+-]}(\mathbf{p}_T) = \frac{1}{\mathbf{p}_T^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \frac{i\pi + 2\log r}{\epsilon},$$

- $[+-]$-bubble in $p^- = 0$ kinematics (leading term of MB expansion):

$$\begin{aligned} B_0^{[+-]}(\mathbf{p}_T, p^+) &= \frac{1}{\mathbf{p}_T^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \frac{\Gamma^2(1+\epsilon)\Gamma(2+\epsilon)\sin(\pi\epsilon)}{\pi\epsilon^2} \\ &\times \left[i\pi + \log r - \log\frac{p_+^2}{\mathbf{p}_T^2} - \psi(1+\epsilon) + \psi(1)\right] + O(r^{1/2}) \end{aligned}$$

- $[+-]$-bubble in light-like kinematics $p^2 = 0$:

$$B_0^{[+-]}(\mathbf{p}_T^2, p^2 = 0) = \int \frac{[d^d q]}{q^2(q-p)^2[q^+][q^- - p^-]} = \frac{-2\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\mathbf{p}_T^2\epsilon^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon}.$$

Single-scale triangle.

Calculation of the single-scale triangle integral:

$$C_0^{(+)}(\mathbf{p}_T^2, k^+) = \int \frac{[d^D q]}{q^2(p-q)^2(p+k-q)^2(-\tilde{q}^+ + i\varepsilon)},$$

is *significantly simplified by the new kinematic constraint* $\tilde{p}^+ = 0$.

Solution to the constraints:

$$p^- = \frac{\mathbf{p}_T^2}{k^+} + \frac{\mathbf{p}_T^4 r}{k_+^3} + O(r^2),\ p^+ = -rp^-.$$

After Feynman parametrization:

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int\limits_0^\infty \frac{x_1^{n_1-1} dx}{(xa_1+a_2)^{n_1+n_2}},$$

we get:

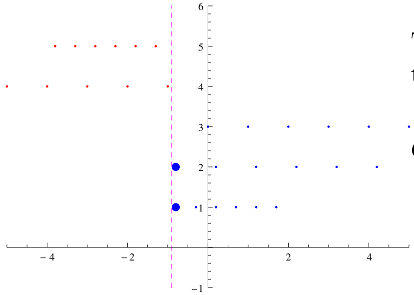
$$C_0^{(+)} \sim \int\limits_0^\infty dx_1 dx_2 dx_3 (1+x_1+x_2)^{2\epsilon} \left[\left(\mathbf{p}_T^2 + O(r)\right) x_1 \oplus x_3 \left(k_+ x_2 + r x_3\right)\right]^{-2-\epsilon}.$$

Mellin-Barnes representation for binomial:

$$\frac{1}{(X \oplus Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \Gamma(-z)\Gamma(z+\lambda)\; Y^z X^{-z-\lambda}.$$

Applying it to the square bracket and taking integrals over $x_{1,2,3}$ we get:

$$\begin{aligned} C_0^{(+)} \quad &\sim \quad \int dz\, r^{z+\epsilon+1} \left(\frac{\mathbf{p}_T^2}{k_+^2}\right)^z \Gamma(-z)\Gamma^2(1+z)\Gamma(-2z-2\epsilon-2) \\ &\times \quad \Gamma(-z-\epsilon-1)\Gamma(2z+2\epsilon+3) \end{aligned}$$



Taking residue in the "right" pole closest to $z = -1$ we obtain:

$$C_0^{(+)} = \frac{(-1)}{k^+ \mathbf{p}_T^2} \left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \frac{1}{\epsilon} \left[-\log r + \log \frac{(k^+)^2}{\mathbf{p}_T^2} + \psi(1+\epsilon) + \psi(1) - 2\psi(-\epsilon)\right] + O(r^{1/2})$$

For the case of *single* light-cone propagator, the result with PV-prescription:

$$\frac{1}{[\tilde{q}^+]} = \frac{1}{2}\left(\frac{1}{\tilde{q}^+ + i\varepsilon} + \frac{1}{\tilde{q}^+ - i\varepsilon}\right)$$

is obtained by analytic continuation in k^+:

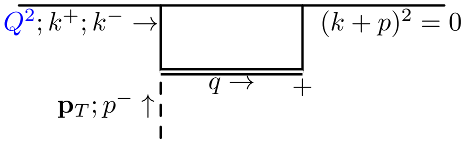
$$I^{[+]}(k) = \frac{1}{2}\left[I^{(+)}(-k^+ - i\varepsilon) - I^{(+)}(k^+ - i\varepsilon)\right].$$

The final result:

$$C_0^{[+]} = \frac{1}{k^+ \mathbf{p}_T^2}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon}\frac{1}{\epsilon}\left[-\log r - i\pi + \log\frac{(k^+)^2}{\mathbf{p}_T^2} + \psi(1+\epsilon) + \psi(1) - 2\psi(-\epsilon)\right],$$

coincides with the result of [G. Chachamis, *et. al.*, 2012].

Triangle with two scales.



where now $k^2 = k^+ k^- = -Q^2$, $X = Q^2/\mathbf{p}_T^2$.
Apply “Rudimentary DE-method”. The integral:

$$\left.\frac{\partial C_0^{(+)}}{\partial X}\right|_{r=0} = -\frac{\mathbf{p}_T^2 \mu^{2\epsilon} \Gamma(3+\epsilon)}{r_\Gamma} \int\limits_0^\infty dx_1 dx_2 dx_3 \;\, x_1 (1 + x_1 + x_2)^{2\epsilon}$$

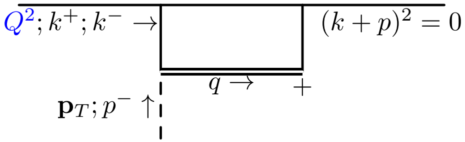
$$\times \left[\mathbf{p}_T^2 x_1 (x_2 + X) + k_+ x_3\right]^{-3-\epsilon},$$

is **finite** and can be calculated analitically. The answer is:

$$\left.\frac{\partial I}{\partial X}\right|_{r=0} = \frac{2X^{-1-\epsilon}}{\epsilon} - \frac{2}{\epsilon} \frac{1 - X^{-\epsilon}}{1 - X},$$

where $I(X) = \mathbf{p}_T^2 k_+ \left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{-\epsilon} \left[C_0^{[+]}(X) - C_0^{[+]}(X = 0)\right]_{r=0}$.

Triangle with two scales.



The final answer:

$$C_0^{[+]}(X) = C_0^{[+]}(X=0) + \left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^\epsilon \frac{I(X)}{k_+ \mathbf{p}_T^2},$$

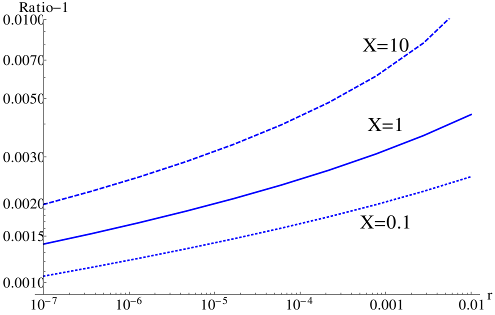
where

$$\begin{aligned} I(X) &= -\frac{2X^{-\epsilon}}{\epsilon^2} - \frac{2}{\epsilon}\int_0^X \frac{(1-x^{-\epsilon})dx}{1-x} \\ &= -\frac{2X^{-\epsilon}}{\epsilon^2} + 2\left[\mathrm{Li}_2(X) + \log(1-X)\log X\right] + O(\epsilon). \end{aligned}$$

Numerical cross-check

The results for $C_0^{(+)}$ integrals with 1 and 2 scales has been cross-checked numerically, using **sector decomposition** algorithm.

The r-dependence of the ratio, including $\Delta I(X, r, \epsilon = -0.01)$:



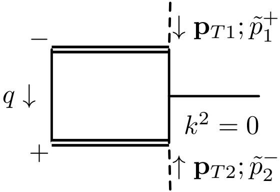
Remaining r-dependence is $O(r^{\sim 0.1})$.

- For $C_0^{(+)}$ with 2 scales: 3D integral (`cuhre` algorithm of `CUBA` was used), 8 sectors, up to 4 subsectors in some of them.
- For numerical comparison, the $1/\epsilon^2$ pole is subtracted.
- The leading r-dependent term can be identified from numerical data:

$$\Delta I(X, r, \epsilon) = \frac{r^{-\epsilon}}{2\epsilon^2} \frac{X^{-2\epsilon}}{\cos(\pi\epsilon)}.$$

Triangle with two light-cone propagators

Usual one-loop Feynman integrals with more than 4 propagators are reducible to more simple integrals up to terms $O(\epsilon)$. We apply method of [Bern, Dixon, Kosower, 1992].



$$I^d[1] \sim \int_0^1 da_1 da_2 da_3 \int_0^\infty dx_1 dx_2 \;\delta(...) \cdot 1 \cdot D^{d/2-5}$$

$$D = a_1 a_2 \mathbf{p}_{T1}^2 + a_1 a_3 \mathbf{p}_{T2}^2 + p_1^+ a_2 x_1 + (-p_2^-) a_3 x_2 + (x_1 + r x_2)(x_2 + r x_1),$$

After the shift $q \to q + q_\star$, propagators can be expressed through D:

$$\left.\begin{array}{rcl} q^2\big|_{q\to q+q_\star} & = & q^2 + D - a_2 \mathbf{p}_{T1}^2 - a_3 \mathbf{p}_{T2}^2 \\ (q-p_1)^2\big|_{q\to q+q_\star} & = & q^2 + D - a_1 \mathbf{p}_{T1}^2 - k_+ x_1 \\ (q+p_2)^2\big|_{q\to q+q_\star} & = & q^2 + D - a_1 \mathbf{p}_{T2}^2 + k_- x_2 + O(r) \\ \tilde{q}^+\big|_{q\to q+q_\star} & = & a_2 k_+ + x_2 + O(r) \\ \tilde{q}^-\big|_{q\to q+q_\star} & = & -a_3 k_- + x_1 + O(r) \end{array}\right\} \Rightarrow \begin{array}{l}\text{Linear} \\ \text{system for} \\ a_1, a_2, a_3, x_1, x_2\end{array}$$

Integral with $q^2 + D$ *reduces to* $(d-4) I^{d+2}[1] = O(\epsilon)$. $I^6[1]$ *– finite!*

Triangle with two light-cone propagators

From the above linear system one can express $I^d[a_1]$, $I^d[a_2]$, $I^d[a_3]$ through integrals **with one cancelled propagator** and $(d-4)I^{d+2}[1] = O(\epsilon)$, which we put to zero.
Taking into account that: $I^d[1] = I^d[a_1] + I^d[a_2] + I^d[a_3]$, one obtains:

$$C_0^{[+-]}(\mathbf{p}_{T1}^2, \mathbf{p}_{T2}^2, p_1^+, -p_2^-) = \frac{1}{2\mathbf{p}_{T1}^2\mathbf{p}_{T2}^2\mathbf{k}_T^2} \times$$
$$\Big\{\mathbf{p}_{T1}^2(\mathbf{p}_{T2}^2 - \mathbf{k}_T^2 - \mathbf{p}_{T1}^2)\Big[B_0^{[+-]}(\mathbf{p}_{T1}^2, p_1^+) + (-p_2^-)C_0^{[-]}(\mathbf{p}_{T1}^2, \mathbf{p}_{T2}^2, -p_2^-)\Big]$$
$$+\mathbf{p}_{T2}^2(\mathbf{p}_{T1}^2 - \mathbf{k}_T^2 - \mathbf{p}_{T2}^2)\Big[B_0^{[+-]}(\mathbf{p}_{T2}^2, p_2^-) + p_1^+ C_0^{[+]}(\mathbf{p}_{T2}^2, \mathbf{p}_{T1}^2, p_1^+)\Big]$$
$$+\mathbf{k}_T^2(\mathbf{k}_T^2 + \mathbf{p}_{T1}^2 + \mathbf{p}_{T2}^2)B_0^{[+-]}(\mathbf{k}_T^2, k^2 = 0)\Big\},$$

where $\mathbf{k}_T^2 = p_1^+(-p_2^-)$.
(Euclidean region: $p_1^+ > 0$, $-p_2^- > 0$, $\mathbf{p}_{T1,2}^2 > 0$).
The $\log r$-divergence cancels within square brackets, as expected.

Conclusions

- The consistent procedure of rapidity regularization is proposed. One shoud modify not only Wilson lines, but also kinematic constraints.
- One-loop integrals with log-RDs are identified. The power-RDs seem to be contained just in a few simplest integrals.
- Triangle integrals with 1 and 2 scales are calculated.
- Reduction of one-loop integrals with more than four propagators (quadratic or light-cone) seems to work similar to the case of ordinary loop integrals.
- Possible applications: DIS@NLO in PRA, re-derivation of NLO BFKL kernel, calculation of NLO corrections to BFKL equation with quark in t-channel (Fadin-Sherman equation), calculation of NLO impact-factors for different processes.

Thank you for your attention!