Hypergeometric approach in Feynman integral calculation

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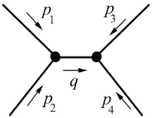
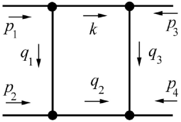
Feynman Diagrams: Basic Definitions





- Quantum field theory amplitudes are represented as a sum of **Feynman Diagrams**, graphs for which each line and vertex is represented by a factor in a term of the quantum amplitude.
- Integrating over all unconstrained momenta gives rise to a **Feynman Integral**. For L loops and n internal lines, and allowing the propagators to be raised to powers ν_j,

$$F_G = \int \prod_{r=1}^{L} d^d k_r \prod_{j=1}^{n} \frac{1}{(q_j^2 - m_j^2)^{\nu_j}}.$$

$$F(s,t,a_1,a_2,a_3,a_4;d) = \int \frac{d^d k}{(k^2)^{a_1}((k+p_3)^2)^{a_2}((k+p_3+p_4)^2)^{a_3}((k-p_1)^2)^{a_4}}.$$

IBP Relations and Master Integrals

- **Integration by parts** leads to a set of recurrence relations among diagrams of a given topology but different powers of the propagators.
- The full set of recurrence relations should be solved by finding how the integral with powers of propagators $(j_1 + j_2 + \cdots + j_k)$ reduced to integrals with powers $(j_1 + j_2 + \cdots + j_k - 1)$
- The method involves taking derivatives of each integral with respect to momenta and reducing it to the original integral.
- The relations found permit a **reduction** to a basis set of **master integrals** in terms of which the diagrams of this class may be expressed.

The IBP identity

$$\int d^d k \frac{\partial}{\partial k_\mu} \left[\frac{k_\mu}{(k^2 - m^2)^n} \right] = 0$$

leads to a recurrance relation

$$(d - 2n) I_n - 2nm^2 I_{n+1} = 0$$

Master Integral (MI) Calculation

- **Next step** is to express the MI in terms of well-known functions or estimate them numerically

$$MI(p_i, m_i) = \frac{1}{\varepsilon^n} f_1(p_i, m_i) + \frac{1}{\varepsilon^{n-1}} f_2(p_i, m_i) + ...$$

  - hypergeometric function (Gauss, Appell, Lauricella, etc.)
  - Harmonic numbers, polylogaritms, generalized polylogarithms, harmonic polylogarithms....
  - elliptic generalization of multiple polylogarithms
- Book "Evaluating Feynman Integrals", Smirnov, Vladimir A.
- some methods of MI evaluation:
  - differential equation
  - dimensional recurrence
  - asymptotic expansion in momenta and masses
  - evaluation by the Mellin-Barnes representation

Hypergeometric function definition

- Gauss hypergeometric function

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} $$

Pochhammer symbol:

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n\Gamma(n) = \Gamma(n+1) $$

  - differential equation (Fuchsian equation):

$$ z(z-1)\frac{d^2u}{dz^2} + (c-(a+b+1)z)\frac{du}{dz} - abu = 0 $$

  - ratio of series expansion coefficients:

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) = \sum_{n=0}^{\infty} C_n x^n $$

$$ \frac{c_{n+1}}{c_n} = \frac{(a+n)(b+n)}{(c+n)(n+1)} $$

- how to obtain new hypergeometric function:

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (b_2)_{m+n}}{(c)_n (e)_{2m-k+l}} \frac{x^n}{n!} \frac{y^m}{m!} \frac{z^l}{l!} $$

generalized Lauricella series

- Appell hypergeometric function:

$$F_c(a, b; c_1, c_2; z_1, z_2) = \sum_{k_1, k_2} \frac{(a)_{k_1+k_2}(b)_{k_1+k_2}}{(c_1)_{k_1}(c_2)_{k_2}} \frac{z_1^{k_1} z_2^{k_2}}{k_1! k_2!}$$

- To express the Feynman integral we need hypergeometric function called generalized Lauricella series:

$$\sum_{m_1, \ldots, m_l}^{\infty} \prod_{i,j} \frac{(a_j)_{\sum_k^l q_k m_k}}{(b_i)_{\sum_k^l q_k m_k}} \prod_{n=1}^{l} \frac{x_n^{m_n}}{m_n!}, \quad q_k \in \mathbb{Z},$$

-

$$F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}} \left(\begin{matrix} [(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^1):\phi^{(1)}];\ldots;[(b^n):\phi^{(n)}] \\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:[(d^1):\delta^{(1)}];\ldots;[(d^n):\delta^{(n)}] \end{matrix} \; x_1,\ldots,x_n \right)$$

$$= \sum_{s_1,\ldots,s_n=0}^{\infty} \Omega(s_1,\ldots,s_n) \frac{x_1^{s_1}}{s_1!} \frac{x_n^{s_n}}{s_n!},$$

$$\Omega(s_1,\ldots,s_n) = \frac{\prod_{j=1}^{A}(a_j)_{s_1\theta_j^{(1)}+\cdots+s_n\theta_j^{(n)}} \prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{s_1\phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{s_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{s_1\psi_j^{(1)}+\cdots+s_n\psi_j^{(n)}} \prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{s_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{s_n\delta_j^{(n)}}},$$

The Mellin-Barnes Representation

The Mellin-Barnes representation relies on the identity

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + \infty} dz \, \Gamma(-z) \, \Gamma(\lambda + z) \, \frac{B^z}{A^{\lambda + z}} .$$

The contour is chosen to separate the poles in $\Gamma(-z)$ from the poles in $\Gamma(\lambda + z)$.

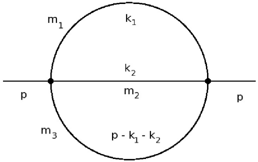
This relation is applied to the denominator in the Feynman Parametrization to break it up into monomials in the Feynman parameters x_i. The integration over the Feynman parameters can then be easily performed in terms of Γ functions,

Upon application of Cauchy's theorem, the Feynman integral can be converted into a linear combination of multiple series:

$$\Phi(n, \vec{x}\,) \sim \sum_{k_1, \cdots, k_{r+m}=0}^{\infty} \prod_{a,b} \frac{\Gamma(\sum_{i=1}^{m} A_{ai} k_i + B_a)}{\Gamma(\sum_{j=1}^{r} C_{bj} k_j + D_b)} x_1^{k_1} \cdots x_{r+m}^{k_{r+m}} \; ,$$

where x_i are some rational functions of Mandelstam variables and A_{ai}, B_a, C_{bj}, D_b are linear functions of the space-time dimension and the propagator powers.

Example: Sunset Diagram



$$F_G = \int \frac{d^d k_1 d^d k_2}{[(k_1-p)^2 - m_1^2][k_2^2 - m_2^2][(k_1-k_2)^2 - m_3^2]}$$

$$= \int_{-i\infty}^{i\infty} ds_1 ds_2 ds_3 \frac{m_1^{2s_1} m_2^{2s_2} m_3^{2s_3}}{(-p^2)^{s_1+s_2+s_3}} \Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)$$

$$\Gamma(3-d+s_1+s_2+s_3)\frac{\Gamma(d/2-1-s_1)\Gamma(d/2-1-s_2)\Gamma(d/2-1-s_3)}{\Gamma(3d/2-3-s_1-s_2-s_3)}$$

$$\sim z_1^{d/2-1} z_2^{d/2-1} F_C^{(3)}(1, d/2, d/2, d/2, d/2; z_1, z_2, z_3)$$
$$-z_1^{d/2-1}\Gamma^2(1-d/2)F_C^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3)$$
$$-z_2^{d/2-1}\Gamma^2(1-d/2)F_C^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3)$$
$$-\Gamma(\frac{d}{2}-1)\Gamma(1-\frac{d}{2})\Gamma(3-d)F_C^{(3)}(3-d, 2-d/2, 2-d/2, 2-d/2, d/2, z_1, z_2, z_3)$$

in terms of the hypergeometric function (in the case $n = 3$)

$$F_C^{(n)}(a, b; c_1, \cdots, c_n; z_1, \cdots z_n) = \sum_{k_1,\ldots k_n} \frac{(a)_{k_1+\cdots+k_n}(b)_{k_1+\cdots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}} \frac{z_1^{k_1}\cdots z_n^{k_n}}{k_1!\cdots k_n!}$$

with arguments $z_1 = m_1^2/m_3^2, \quad z_2 = m_2^2/m_3^2, \quad z_3 = p^2/m_3^2.$

Differential Reduction

Consider the hypergeometric series

$$H(\vec{\gamma};\vec{\sigma};\vec{x}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\prod_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\prod_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \ .$$

The lists $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively.
Two functions with lists of parameters shifted by a unit, $\Phi(\vec{\gamma}+\vec{e}_c;\vec{\sigma};\vec{x})$ and $\Phi(\vec{\gamma};\vec{\sigma};\vec{x})$, are related by a linear differential operator:

$$\begin{aligned} H(\vec{\gamma}+\vec{e}_c;\vec{\sigma};\vec{x}) &= \left(\sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) H(\vec{\gamma};\vec{\sigma};\vec{x}) \\ H(\vec{\gamma};\vec{\sigma}-\vec{e}_c;\vec{x}) &= \left(\sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c - 1 \right) H(\vec{\gamma};\vec{\sigma};\vec{x}) \ . \end{aligned}$$

Example: Direct index-shifting operators

The generalized hypergeometric functions have the form

$$ {}_pF_q(\vec{a};\vec{b};z) \equiv {}_pF_q\left(\begin{array}{c} \vec{a} \\ \vec{b} \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \; , $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is called a *Pochhammer symbol*. The lists $\vec{a} = (a_1, \cdots, a_p)$ and $\vec{b} = (b_1, \cdots, b_q)$ are the upper and lower parameters of hypergeometric functions, respectively.
Direct index-shifting operators may be defined as follows:

$$ \begin{aligned} {}_pF_q(a_1+1,\vec{a};\vec{b};z) &= B^{+}_{a_1}\,{}_pF_q(a_1,\vec{a};\vec{b};z) \equiv \frac{1}{a_1}(\theta + a_1)\,{}_pF_q(a_1,\vec{a};\vec{b};z) \; , \\ {}_pF_q(\vec{a};b_1-1,\vec{b};z) &= H^{-}_{b_1}\,{}_pF_q(\vec{a};b_1,\vec{b};z) \equiv \frac{1}{b_1-1}(\theta + b_1 - 1)\,{}_pF_q(\vec{a};b_1,\vec{b};z) \; , \end{aligned} $$

where

$$ \theta = z\frac{d}{dz} \; . $$

Example: Inverse operators

For the special case ${}_{p+1}F_p$, inverse shifting operators satisfying

$$\begin{aligned} {}_{p+1}F_p(a_i - 1, \vec{a}; \vec{b}; z) &= B^-_{a_i} \; {}_{p+1}F_p(a_i, \vec{a}; \vec{b}; z) \,, \\ {}_{p+1}F_p(\vec{a}; b_i + 1, \vec{b}; z) &= H^+_{b_i} \; {}_{p+1}F_p(\vec{a}; b_1, \vec{b}; z) \,, \end{aligned}$$

are found to be given by

$$B^-_{a_i} = -\frac{a_i}{c_i}\left[t_i(\theta) - z \prod_{j \neq i} (\theta + a_j) \right]_- \quad , \quad H^+_{a_i} = \frac{b_i - 1}{d_i}\left[\frac{d}{dz} \prod_{j \neq i} (\theta + b_j - 1) - s_i(\theta) \right]_+$$

with

$$\begin{aligned} c_i &= -a_i \prod_{j=1}^{p} (b_j - 1 - a_i) \,, \quad t_i(x) = \frac{x \prod_{j=1}^{p} (x + b_j - 1) - c_i}{x + a_i} \\ d_i &= \prod_{j=1}^{p+1} (1 + a_j - b_i) \,, \quad s_i(x) = \frac{\prod_{j=1}^{p+1} (x + a_j) - d_i}{x + b_i - 1} , \end{aligned}$$

and the $\pm$ subscripts on the brackets are shorthand indicating that $a_i \to a_i - 1$, $b_i \to b_i + 1$, inside the respective brackets.

Horn-type Hypergeometric Functions: Takayama

The inverse differential operators can be constructed:

$$\begin{aligned} H(\vec{\gamma}-\vec{e}_c;\vec{\sigma};\vec{x}) &= \sum_a S_a(\vec{x},\vec{\partial}_x)H(\vec{\gamma};\vec{\sigma};\vec{x}) \\ H(\vec{\gamma};\vec{\sigma}+\vec{e}_c;\vec{x}) &= \sum_b L_b(\vec{x},\vec{\partial}_x)H(\vec{\gamma};\vec{\sigma};\vec{x}) \; . \end{aligned}$$

In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts:

$$P_0(\vec{x})H(\vec{\gamma}+\vec{k};\vec{\sigma}+\vec{l};\vec{x}) = \sum_{m_1,\cdots,m_p=0} P_{m_1,\cdots,m_r}(\vec{x}) \left(\frac{\partial}{\partial\vec{x}}\right)^{\vec{m}} H(\vec{\gamma};\vec{\sigma};\vec{x}) \; ,$$

where $P_0(\vec{x})$ and $P_{m_1,\cdots,m_p}(\vec{x})$ are polynomials with respect to $\vec{\gamma},\vec{\sigma}$ and $\vec{x}$ and $\vec{k},\vec{l}$ are lists of integers.

special value parameters

Let us write explicit expressions for the inverse operators for several hypergeometric functions. For the Gauss hypergeometric function ${}_2F_1$, we have:

$$ {}_2F_1\left(\begin{array}{c} a_1-1, a_2 \\ b_1 \end{array} \middle| z \right) = \frac{1}{b_1-a_1}\left[(1-z)\theta + b_1 - a_1 - a_2 z\right] {}_2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1 \end{array} \middle| z \right) \;, $$

$$ {}_2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1+1 \end{array} \middle| z \right) = \frac{b_1((1-z)\frac{d}{dz} + b_1 - a_1 - a_2)}{(b_1-a_1)(b_1-a_2)} {}_2F_1\left(\begin{array}{c} a_1, a_2 \\ b_1 \end{array} \middle| z \right) \;. $$

From direct and inverse differential operators we could find additional equations over hypergeometric function for special values of the parameters.

criterion of reducibility

$$
{}_pF_q\left(\begin{array}{c} b_1+m_1,\cdots,b_n+m_n,a_{n+1},\cdots,a_p \\ b_1,\cdots,b_n,b_{n+1},\cdots,b_q \end{array}\middle| z\right)
$$

$$
=\sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n} A(j_1,\cdots j_n) z^{J_n}{}_{p-n}F_{q-n}\left(\begin{array}{c} a_{n+1}+J_n,\cdots a_p+J_n \\ b_{n+1}+J_n,\cdots,b_q+J_n \end{array}\middle| z\right) ,
$$

- The hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ which has pairs of parameters satisfying $a_i=b_i+m_i$ with m_i being positive integers is expressible in terms of functions of lower order.
- The hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ which has two or more pairs of parameters satisfying $b_i=a_i+m_i+1$ with m_i being positive integers is expressible in terms of functions of lower order
- If one of the upper parameters of a hypergeometric function is an integer, the result of the differential reduction of this hypergeometric function has one less derivative

Invariants of Hypergeometric Representation

- This analysis demonstrates that there is a very simple relation between the number h of nontrivial master integrals found from IBP (which are not expressible in terms of Gamma functions) and the maximal power v of θ generated by the differential reduction, namely $h = v + 1$. This relation does not depend on the number k of hypergeometric functions entering original equation.
- it was considered all Horn-type hypergeometric function of two variables, namely 34, including specially considered case of four Appell functions F_1, F_2, F_3, F_4 of two variables.
- Laurichella functions F_C,F_S of three variables and F_D of multiple variable cases
- direct an inverse differential operator was found, conditions of complete integrability and complete system of independent differential equations and number of independent solutions in different cases of parameter values.
- The package called HYPERDIRE (HYPERgeometric DIFFerential REduction), based on language of program Mathematica
- Key feature is that the product of non-commutative step-up and step-down operators of differential reduction turn into product of special 2-dimensional matrices and vectors which greatly simplify and reduce the time of calculation

Example, sunset type diagram

$$J_{22}^{q}(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \left[i^{1-n}\pi^{n/2}\right]^q \frac{(-m^2)^{\frac{n}{2}q-\alpha_{1,2}-\sigma}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\{\Pi_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$$

$$\times \frac{\Gamma\left(\alpha_1+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\alpha_2+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\sigma-\frac{n}{2}(q-2)\right)\Gamma\left(\alpha_{1,2}+\sigma-\frac{n}{2}q\right)}{\Gamma\left(\alpha_{1,2}+2\sigma-n(q-1)\right)\Gamma\left(\frac{n}{2}\right)}$$

$$ {}_4F_3\left(\begin{array}{c} \alpha_1+\sigma-\frac{n}{2}(q-1),\alpha_2+\sigma-\frac{n}{2}(q-1),\sigma-\frac{n}{2}(q-2),\alpha_{1,2}+\sigma-\frac{n}{2}q \\ \frac{n}{2},\frac{1}{2}(\alpha_{1,2}-n(q-1))+\sigma,\frac{1}{2}(1+\alpha_{1,2}-n(q-1))+\sigma, \end{array}\middle| \frac{p^2}{4m^2}\right)$$

- $q=1$

$$(1)\times\ {}_2F_1\left(\begin{array}{c} 1,I_1-\frac{n}{2} \\ I_2 \end{array}\middle| z\right).$$

- $q=2$

$$(1,\theta)\times\ {}_3F_2\left(\begin{array}{c} 1,I_1-\frac{n}{2},I_2-n \\ I_3+\frac{n}{2},I_4+\frac{1}{2}-\frac{n}{2} \end{array}\middle| z\right).$$

- $q=3$

$$(1,\theta,\theta^2)\times\ {}_3F_2\left(\begin{array}{c} I_1-\frac{n}{2}(q-1),I_2-\frac{n}{2}(q-2),I_3-\frac{n}{2}q \\ \frac{n}{2},I_4+\frac{1}{2}-\frac{n}{2}(q-1) \end{array}\middle| z\right).$$

derivative of hypergeometric function

- To evaluate FI we have to made expansion over parameter of dimensional regularization ε, namely calculate derivatives over parameter of hypergeometric function

$$\sum_{m_1,\ldots,m_l}^{\infty} \prod_{i,j} \frac{(a_j)_{\sum_k^l q_k m_k}}{(b_i)_{\sum_k^l q_k m_k}} \prod_{n=1}^{l} \frac{x_n^{m_n}}{m_n!} = \frac{1}{\varepsilon^n} f_1(x_1, \ldots) + \frac{1}{\varepsilon^{n-1}} f_2(x_1, \ldots) + \ldots$$

- We have to derivate the Pochhammer symbol:

$$\frac{\mathrm{d}(a)_n}{\mathrm{d}a} = (a)_n \Big[\Psi(a+n) - \Psi(a) \Big] = (a)_n \sum_{k=0}^{n-1} \frac{1}{a+k} = (a)_n \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k}$$

- After derivation we have unknown function, that is not of hypergeometric type.
- There exist some algorithms and packages that gives possibility to express some type of hypergeometric function for some special values of parameter in terms of known special functions (XSummer (S. Moch, P. Uwer, S. Weinzierl), HypExp (T. Huber and D. Maitre), M. Kalmykov), etc.

derivative of hypergeometric function

- hypergeometric function with parameter for derivation:

$$F(a) = \sum_{n=0}^{\infty} B(n)(a)_n \frac{x^n}{n!}$$

- By using the resummation formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+k)$$

- we could obtain the derivative of hypergeometric function with one summation index:

$$\frac{\mathrm{d}F(a)}{\mathrm{d}a} = x \sum_{n,k=0}^{\infty} B(n+k+1) \frac{x^n}{n!} \frac{x^k}{k!} \frac{(1)_k (1)_n}{(2)_{n+k}} \frac{(a+1)_{n+k}(a)_k}{(a+1)_k}$$

- in this form we could see that derivative of hypergeometric function is hypergeometric function.

derivative of Gauss hypergeometric function

- For the Gauss hypergeometric function derivative ${}_2F_1(a,b,c,x)$:

$$\frac{\mathrm{d}}{\mathrm{d}a}\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| x\right) = \frac{bx}{c}\sum_{k=0}^{\infty}\frac{(a)_k}{(a+1)_k}\sum_{n=0}^{\infty}\frac{(a+1)_{n+k}(b+1)_{n+k}}{(c+1)_{n+k}}\frac{x^{n+k}}{(n+k+1)!}$$

$$= \frac{bx}{c}\sum_{k=0}^{\infty}\frac{(1)_k(a)_k}{(a+1)_k}\sum_{n=0}^{\infty}\frac{(1)_n(a+1)_{n+k}(b+1)_{n+k}}{(2)_{n+k}(c+1)_{n+k}}\frac{x^n x^k}{n!k!}$$

- This hypergeometric series can be understood as a generalized Kampé de Fériet hypergeometric function

$$\frac{\mathrm{d}}{\mathrm{d}a}\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| x\right) = \frac{bx}{c}F^{2:2;1}_{2:1;0}\left[\begin{array}{c}(a+1,b+1):(1,a);(1) \\ (c+1,2):(a+1);(-)\end{array}\; x,x\right].$$

- In establishing the region of convergence we can utilize the parameter cancellation theorem. By that we could find that region of convergence for function and its derivate the same

derivative of any hypergeometric function

- the derivative of a hypergeometric function in one of its upper parameters related to a summation index with any integer coefficient:

$$
\begin{aligned}
\frac{\mathrm{d}F(a)}{\mathrm{d}a} &= \sum_{k,n_1,\ldots,n_\phi=0}^{\infty} \sum_{\xi=1}^{\phi} \sum_{\gamma=0}^{|q_\xi|-1} x_\xi B(n_1,\ldots,n_\xi+k+1,\ldots,n_\phi) \frac{(1)_k (1)_{n_\xi}}{(2)_{n_\xi+k}} \prod_{r=1}^{\phi} \frac{x_r^{n_r} x_\xi^k}{n_r! k!} \\
&\quad \times \frac{\Gamma(a+q_\xi)}{\Gamma(a)} \prod_{r=1}^{\xi-1} \frac{(a)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda}}{(a)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda}} \prod_{r=\xi+1}^{\phi} \frac{(a+q_\xi)_{\sum_{\lambda=1}^{r} q_\lambda n_\lambda + q_\xi k}}{(a+q_\xi)_{\sum_{\lambda=1}^{r-1} q_\lambda n_\lambda + q_\xi k}} \\
&\quad \times \frac{(a+q_\xi)_{\sum_{\lambda=1}^{\xi} q_\lambda n_\lambda + q_\xi k}}{(a)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda}} \beta \,, \\
\beta &= \frac{1}{a+\gamma} \frac{(a+\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}{(a+\gamma+1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}} \,, \qquad q_\xi > 0 \,, \\
\beta &= -\frac{1}{a-\gamma-1} \frac{(a-\gamma-1)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}}{(a-\gamma)_{\sum_{\lambda=1}^{\xi-1} q_\lambda n_\lambda + q_\xi k}} \,, \qquad q_\xi < 0 \,.
\end{aligned}
$$

derivative of any hypergeometric function and Feynman integral

- Derivatives of the generalized Lauricella series, i.e., of Horn-type hypergeometric series with summation coefficients $q_k \in N$, in one of their (upper or lower) parameters can be expressed as a finite sum of the generalized Lauricella series
- Feynman integrals and the ε expansion of Feynman integrals at any order are expressible in terms of **generalized Lauricella series**
- the n-th term of the ε series can be expressed as a Horn-type hypergeometric function in $n + m$ variables, where m is the number of summations in the Horn-type representation of the Feynman integral
- The region of convergence of any of these parameter derivatives, i.e., the coefficients in the ε expansion, and the initial Feynman integral are the same
- By expressing the ε expansion of a Feynman diagram in terms of Horn-type hypergeometric functions and applying the above-mentioned method of differential reduction, one can reduce the corresponding integrals to some subset of basic hypergeometric functions and express them as series with the least number of infinite summations

differential system

- The key feature that for derivatives of hypergeometric function (Feynman Integral) we could construct full differential system (Fuchsian type)
- From that system it is possible to obtain full system of differential equations (Fuchsian type) with lower number of variables- we could move the variables to the parameters of function.

$$\sum_{m_1,\ldots,m_l}^{\infty} C(a_1, \ldots..a_j)\frac{x_n^{m_n}}{m_n!} \rightarrow \sum_{m_1,\ldots,m_{l-1}}^{\infty} C(a_1, \ldots..a_j, x_k)\frac{x_n^{m_n}}{m_n!}$$

- New system is of Fuchsian type, that gives us possibility to obtain the answer by Frobenius method- in terms of power series solution.
- New power series are not hypergeometric.

Gauss hypergeometric function transformation

- Derivative of Gauss hypergeometric function (series of two variables):

$$\frac{\mathrm{d}}{\mathrm{d}a}\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| x\right) = \frac{bx}{c}\sum_{k=0}^{\infty}\frac{(1)_k(a)_k}{(a+1)_k}\sum_{n=0}^{\infty}\frac{(1)_n(a+1)_{n+k}(b+1)_{n+k}}{(2)_{n+k}(c+1)_{n+k}}\frac{x^n x^k}{n!k!}$$

- After full differential system construction, we could find the nonhomogeneous differential equation over one variable:

$$x^2(x-1)f'' + x(-2-c+(3+a+b)x)f' + (-c+(1+a)(1+b)x)f$$
$$= -c\ {}_2F_1(a,b+1,c,x)$$

- It is a Fuchsian type equation, so we could construct convergent power series in one variable by Frobenius method.
- the function f is not hypergeometric one
- we could construct homogeneous differential equation of fourth order.

thank you for attention

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hypergeometric function of one variable, special cases

ToGroebnerBasis [{{3+b₁,1+a₂,1+a₃},{2+b₁,2+b₂},x}];

IntegerPart={3,1,1,2,2} changeVector={-1,-1,-1}

$$\left\{\left\{-\frac{b_2+1}{(x-1)(b_1+2)}, -\frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)}\right\}, \{\{a_2,a_3\},\{b_2+1\},x\},1\right\}$$

The upper implemented function finds the equation:

$$ {}_3F_2\left(\begin{matrix} 3+b_1, 1+a_2, 1+a_3 \\ 2+b_1, 2+b_2 \end{matrix}\middle| x\right) = \left[-\frac{b_2+1}{(x-1)(b_1+2)} - \frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)}\theta\right] {}_2F_1\left(\begin{matrix} a_2, a_3 \\ b_2+1 \end{matrix}\middle| x\right) $$